

On Instability for the Quintic Nonlinear Schrödinger Equation of Some Approximate Periodic Solutions

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ABSTRACT. Using the Fermi Golden Rule analysis developed in [CM], we prove asymptotic stability of asymmetric nonlinear bound states bifurcating from linear bound states for a quintic nonlinear Schrödinger operator with symmetric potential. This goes in the direction of proving that the approximate periodic solutions for the cubic Nonlinear Schrödinger Equation (NLSE) with symmetric potential in [MW] do not persist in the comparable quintic NLSE.

1. INTRODUCTION

We consider the Nonlinear Schrödinger Equation (NLSE) for $p = 5$:

$$(1.1) \quad \begin{aligned} iu_t &= H_L u - |u|^{p-1}u, \quad u(0, x) = u_0(x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \\ H_L &= -\partial_x^2 u + V_L u. \end{aligned}$$

Here, $V_L(x) = V(x - L) + V(x + L)$ with an even, smooth, and rapidly decreasing to 0 potential $V(x)$ such that $-\partial_x^2 + V$ admits exactly one eigenvalue, $-\Omega$, corresponding to an L^2 normalized eigenfunction φ . For $L \gg 1$, H_L has two eigenvalues, $-\omega_0 < -\omega_1$, both arbitrarily close to $-\Omega$ as $L \nearrow \infty$; see [H]. We will consider real-valued eigenfunctions $\psi_j \in \ker(H_L + \omega_j)$ for $j = 0, 1$ with $\|\psi_j\|_{L^2} = 1$. The number of eigenvalues of H_L is exactly two when V_0 is also compactly supported (see [KI]). It is well known that (1.1) has a family of nonlinear ground states $e^{it\omega} \phi_\omega(x)$ defined for $\omega > \omega_0$ that bifurcates out of the linear ground state. For $L \gg 1$, first [KKS] for $p = 3$ and then [KKP] for $p \geq 2$ prove there is a critical value ω_* such that for $\omega_0 < \omega < \omega_*$, the $e^{it\omega} \phi_\omega(x)$ are even in x and orbitally stable for (1.1). At the critical value $\omega = \omega_*$, the ground states

bifurcate for $\omega > \omega_*$ in two families, one formed by even functions, which are orbitally unstable, and the other formed by nonsymmetric functions, which are orbitally stable for $2 \leq p < 4 + \sqrt{13} \approx 7.606$ and unstable for $p > 4 + \sqrt{13}$.

In [MW], the existence of more complex long-time patterns for (1.1) for $p = 3$ is analyzed by studying the dynamics of a finite dimensional truncation of the NLSE. Over long times, this is shown to be a good approximation of the full NLSE. In particular, this finite dimensional approximation admits a larger class of time periodic solutions than just the standing waves. The question then becomes whether or not these new periodic solutions persist also for the full NLSE equation. In [MW] it is conjectured they do not persist.

In this paper, we do not address the conjecture in [MW], but nonetheless, for an easier problem, we provide the mechanism by which the full NLSE disrupts periodic solutions of a simplified system similar to that in [MW]. Indeed, we prove that the branch of asymmetric bound states above the bifurcation point serves as a local attractor for the infinite dimensional dynamics. We recall that [MW] simplifies the NLSE by first considering the coordinates associated to the spectral decomposition of H_L , and by then setting equal to 0 the components related to the essential spectrum. Here, we consider instead a natural representation of a tubular neighborhood in $H^1(\mathbb{R})$ of the surface of the stable asymmetric ground states. Within this neighborhood, there are then natural finite dimensional approximations of equation (1.1) still admitting periodic solutions. Although these finite dimensional periodic solutions will give approximations to equation (1.1) (as in [MW]), we do not focus on their local in-time dynamics here, but instead on their global in-time behavior. Our solutions are relatively easy because they live arbitrarily close to the surface of asymmetric ground states. Even though we change coordinates in the course of the proof, all these changes occur in the domain of a fixed coordinate chart.

When the full NLSE (1.1) is turned on, the approximate periodic solutions do not persist, because the ground states are asymptotically stable. Hence, the solution will split into a finite dimensional part converging in $H^1(\mathbb{R})$ to the ground state manifold, and another part which scatters like free radiation (see Theorem 1.1). This part of the paper fits easily in the framework of the literature of asymptotic stability of ground states initiated in [SW1, SW2, BP1, BP2]. We recall that the most general results are in [Cu1], which contains a quite general proof of the so-called Fermi Golden Rule. In the present paper, though, we treat a situation that is quite special because of the hypothesis that $\sigma_d(H_L)$ consists of just two eigenvalues. Hence, it is enough to use the simpler framework of [CM, Cu4] (we recall that [Cu4] is a revision and a simplification of [Cu3], which contains some errors).

We consider the quintic NLSE rather than the cubic of [KKS, MW]. There are various reasons why we consider the quintic NLSE. First, we need our non-linearity $|t|^{p-1}t$ to be smooth for our normal forms argument in Section 5, and this rules out exponents $3 < p < 5$. We are not able to treat the case $p = 3$, because the cubic NLSE is a *long-range* perturbation of the flat linear Schrödinger

equation. This means that scattering involves phase corrections with respect to linear approximations. Proof of scattering for small (in an appropriate sense) solutions of the cubic NLSE was an open problem for a long time, solved for the translation-invariant NLSE only in [HN] (it is still an open problem for non-translation-invariant NLSEs). In particular, we know of no general results proving that ground states are asymptotically stable in the presence of a potential for the cubic NLSE. (An interesting attempt to prove this result is in [MP].) Finally, a related issue is that Sobolev embeddings do not provide global results using Strichartz estimates for subcritical problems. We note here that techniques in higher dimensions have been derived in [KiMi, KiZa], but these have not been extended to $1d$. The bottom line is that we do not know how to treat cubic NLSE to get results similar to those proved here for the quintic NLSE. As for possible problems of the quintic NLSE, which is well known to have solutions which blow up, we note that since we work with solutions close to 0 in $H^1(\mathbb{R})$, the initial value problem is globally well posed; in any case, since we will work near stable standing waves, obtaining global well-posedness does not even require appealing to the small size of the solutions.

In Appendix A, we present evidence that, for the quintic NLSE, dynamical solutions exist that are similar to those in [MW] for the cubic NLSE. It is plausible that, generically, they should scatter via an asymptotic stability analysis to an asymmetric nonlinear bound state or to the 0 solution. Proving this can be a challenging problem, because even if the solutions in [MW] scatter to a ground state, they may go through intermediate phases which are not within the reach of the local analysis near a ground state such as that in [CM, Cu1]. The issues then seem in some sense more “global”, and closer in spirit to the problems addressed in [TY, SW3].

We will prove the following theorem.

Theorem 1.1. *Consider equation (1.1) for $p = 5$, with V a smooth, compactly supported function such that $-\partial_x^2 + V$ has exactly one eigenvalue $-\Omega$, and 0 is not a resonance. Let $L \gg 1$ sufficiently large. Then $\sigma(H_L)$ contains exactly two eigenvalues $\{-\omega_0, -\omega_1\}$, both arbitrarily close to $-\Omega$, and 0 is not a resonance. Consider the first branch point $\omega_* > \omega_0$ and the asymmetric stable ground states $e^{it\omega} \phi_\omega$ provided by [KKP]. Assume the Fermi Golden Rule $\Gamma(\omega, \omega) \neq 0$ (see Section 6). Then, given any $\omega \in (\omega_*, \omega_* + \delta_0)$ for $\delta_0 > 0$ sufficiently small, there exist an $\varepsilon_0 > 0$ and a $C > 0$ so that if $\|u_0 - e^{iy} \phi_\omega\|_{H^1} < \varepsilon < \varepsilon_0$, then*

$$(1.2) \quad \lim_{t \rightarrow \pm\infty} \|u(t, \cdot) - e^{i\theta(t)} \phi_{\omega_\pm} - e^{it\partial_x^2} h_\pm\|_{H^1} = 0,$$

where $\omega_\pm \in (\omega^*, \omega^* + \delta_0)$, $\theta \in C^1(\mathbb{R}; \mathbb{R})$, and $h_\pm \in H^1$ with $\|h_\pm\|_{H^1} + |\omega_\pm - \omega_1| \leq C\varepsilon$. It is possible to write

$$u(t, x) = e^{i\theta(t)} \phi_{\omega(t)} + A(t, x) + \tilde{u}(t, x),$$

with $|A(t, x)| \leq C_N(t) \langle x \rangle^{-N}$ for any N , with $\lim_{|t| \rightarrow \infty} C_N(t) = 0$, with $\omega(t) \in C^1$, $\lim_{t \rightarrow \pm\infty} \omega(t) = \omega_{\pm}$, and such that the following Strichartz estimates are satisfied:

$$(1.3) \quad \|\tilde{u}\|_{L_t^\infty(\mathbb{R}, H_x^1(\mathbb{R})) \cap L_t^2(\mathbb{R}, W_x^{1,10}(\mathbb{R})) \cap L_t^4(\mathbb{R}, L_x^\infty(\mathbb{R}))} \leq C\varepsilon.$$

Here, $\langle x \rangle = \sqrt{1 + x^2}$.

Remark 1.2. We note that, in this paper, we focus only on small solutions $u(t)$ of (1.1) that are close to the stable ground states $e^{it\omega} \phi_\omega$, and that our solutions possibly do not include all of those described in [MW]—which is what would be required to address the conjectured asymptotic stability there, and which would be more in the spirit of the analysis in [TY, SW3]. The solutions analyzed here are formed from initial data in a conical neighborhood of the asymmetric ground states with vertex at the bifurcation point.

Once the necessary spectral hypotheses in [CM, Cu4] are proved in Section 3, Theorem 1.1 is a direct consequence of [CM, Cu4]. Nonetheless, we give a sketch of the main steps in the proof. In particular, we review in Section 4 the material on dispersion of linear operators needed in the proof. In particular, we recall that the absence of the endpoint Strichartz estimate on \mathbb{R} requires some surrogates. The surrogates were found by Mizumachi [M]. However, [M] can be substantially simplified. Indeed the smoothing estimates contained in [M], while interesting *per se*, are not necessary in the proof of the main result in [M], and can be replaced by the classical smoothing estimates introduced by Kato in [K]. This is discussed in [CT, Cu4], and reviewed in Section 4. See also the recent results of [DMW] to allow singular potentials in our analysis.

2. NONLINEAR GROUND STATES

In this section, we review some of the results in [KKP] needed later.

We introduce the space $H^{k,s}$ with norm

$$(2.1) \quad \|u\|_{H^{k,s}} := \|\langle x \rangle^s (1 - \partial_x^2)^{k/2} u\|_{L^2}.$$

We recall that 0 is not a *resonance* for a Schrödinger operator $-\partial_x^2 + U$ if $(-\partial_x^2 + U)u = 0$ and $u \in L^\infty$ together imply that $u = 0$.

We quote from [KI] the following result.

Lemma 2.1. *Let $V(x)$ be smooth, compactly supported in \mathbb{R} , such that*

$$\sigma_d(-\partial_x^2 + V) = \{-\Omega\}$$

and 0 is not a resonance. Then there exists $L_ > 0$ such that for $L > L_*$, we have $\sigma_d(H_L) = \{-\omega_0(L), -\omega_1(L)\}$ where $\omega_1(L) < \omega_0(L)$. Furthermore, there are fixed constants $C_0 > 0$ and $\alpha_0 > 0$ such that $|\Omega - \omega_j(L)| \leq C_0 e^{-\alpha_0 L}$.*

Somewhat less specific information is in [H]. See also [Goodman] for some examples of operators with exactly two eigenvalues, obtained by inverse scattering.

For $\varphi(x)$ a normalized ground state for $-\partial_x^2 + V$, it is shown in [KKSJ] that there is a unitary Ψ_{jL} generator of $\ker(H_L + \omega_j(L))$ such that

$$(2.2) \quad \lim_{L \rightarrow +\infty} \left(2^{-1/2} (\varphi(x-L) + (-1)^j \varphi(x+L)) - \Psi_{jL} \right) = 0 \quad \text{in } H^2.$$

The following result holds true.

Lemma 2.2. *There exist $L_* > 0$ and $\varepsilon_1 > 0$ such that for any $L > L_*$, there exists a function $\omega \rightarrow \psi_{\omega,L}$ in $C^\infty((\omega_0(L), \omega_0(L) + \varepsilon_1), H^{k,s}(\mathbb{R}, \mathbb{R}))$ for any (k, s) such that*

$$(2.3) \quad H_L \psi_{\omega,L} - \psi_{\omega,L}^5 = -\omega \psi_{\omega,L}.$$

Furthermore, there exists a fixed $C_{ks} > 0$ such that

$$(2.4) \quad \left\| \psi_{\omega,L} - (\omega - \omega_0(L))^{1/4} \|\Psi_{0L}\|_{L^6}^{-3/2} \Psi_{0L} \right\|_{H^{k,s}} \leq C_{ks} (\omega - \omega_0(L))^{5/4}.$$

We furthermore have $\psi_{\omega,L}(-x) \equiv \psi_{\omega,L}(x)$.

The above result is standard and classical, except for the uniformity of the constants in terms of $L > L_*$, which can be proved as in [KKP].

We recall that a standing wave $e^{itE} \mathcal{G}_E(x)$ of (1.1) is *orbitally stable* in H^1 if, for any $\varepsilon > 0$, there is $\delta > 0$ such that for $u_0 \in H^1$ with

$$\sup_{y \in \mathbb{R}} \|u_0 - e^{iy} \mathcal{G}_E\|_{H^1} < \delta,$$

we have that $u(t)$ is globally defined as a solution to (1.1) with $u(0) = u_0$ and

$$\sup_{t, y \in \mathbb{R}} \|u(t) - e^{iy} \mathcal{G}_E\|_{H^1} < \varepsilon.$$

We consider the pair of operators

$$(2.5) \quad \begin{aligned} L_+(\mathcal{G}_E) &:= H_L + E - 5\mathcal{G}_E^4, \\ L_-(\mathcal{G}_E) &:= H_L + E - \mathcal{G}_E^4. \end{aligned}$$

The following result is due to [KKP] (see equations (3.52) and (3.53)).

Lemma 2.3. *Let $L > L_*$ for $L_* > 0$ sufficiently large, and let ε_1 be as in Lemma 2.2. Then, we have the following:*

- (1) *There is a critical value $0 < \omega_*(L) < \varepsilon_1$ such that for $0 < \omega < \omega_*(L)$, the $e^{it\omega} \psi_{\omega,L}$ are orbitally stable, while for $0 < \omega - \omega_*(L) \ll 1$, they are orbitally unstable. We have*

$$\lim_{L \rightarrow +\infty} \frac{\omega_*(L) - \omega_0(L)}{\omega_0(L) - \omega_1(L)} = \frac{1}{4}.$$

(2) We have $\ker L_+(\psi_{\omega_*(L),L}) \neq 0$. Let ϕ_{*L} be a unitary generator.

Then there exists $\varepsilon_L > 0$ and a second branch of standing waves $\rho \rightarrow e^{it\omega_L(\rho)}\phi_{L,\rho}$ with $\omega_L(\rho) \in C^0([0, \varepsilon_L], \mathbb{R}) \cap C^\infty((0, \varepsilon_L), \mathbb{R})$ and $\phi_{L,\rho} \in C^0([0, \varepsilon_L], H^{k,s}(\mathbb{R}, \mathbb{R})) \cap C^\infty((0, \varepsilon_L), H^{k,s}(\mathbb{R}, \mathbb{R}))$ for any (k, s) . Furthermore, we have

$$\begin{aligned}\omega_L(\rho) &= \omega_*(L) + \frac{Q_L}{2}\rho^2 + o(\rho^2), \\ \phi_{L,\rho} &= \psi_{\omega_*(L),L_*} + \rho\phi_{*L} + O(\rho^2),\end{aligned}$$

with

$$\lim_{L \rightarrow +\infty} \frac{Q_L}{\sqrt{\omega_0(L) - \omega_1(L)}} = 5\|\varphi\|_{L^6}^3,$$

and with Q_L a constant determined by the rate of change of the second eigenvalue of L_+ with respect to ω at ω_* and the inner products of powers of ψ_{ω_*} and ϕ_{*L} . (For more information, see equation (3.35) in Theorem 4 of [KKP].)

- (3) The standing waves $e^{it\omega_L(\rho)}\phi_{L,\rho}$ are orbitally stable for $0 < \rho < \varepsilon_L$ if ε_L is sufficiently small.
- (4) We have $\sigma_d(L_+(\psi_{L,\rho})) = \{-\mu_0(L, \rho), \mu_1(L, \rho)\}$ with $-\mu_0(L, \rho) < 0 < \mu_1(L, \rho)$ with $\mu_1(L, \rho) = -\mu'(L)Q_L\rho^2 + o(\rho^2)$ for a constant $\mu'(L)$ such that $\lim_{L \rightarrow +\infty} \mu'(L) = -4$.

Notice that, in particular, for $\lim_{L \rightarrow +\infty} o_L(1) = 0$,

$$(2.6) \quad \mu_1(L, \rho) = 20\|\varphi\|_{L^6}^3\sqrt{\omega_0(L) - \omega_1(L)}(1 + o_L(1))\rho^2 + o(\rho^2).$$

By continuity, $-\mu_0(L, \rho) \sim -\mu_0(L, 0)$, with the latter being the smallest eigenvalue of $L_+(\psi_{\omega_*(L),L})$.

Lemma 2.4. For $L > L_*$ with $L_* > 0$ sufficiently large, and for $0 < \rho < \varepsilon_L$ with ε_L sufficiently small, we have

$$\mu_0(L, \rho) = (\omega_0(L) - \omega_1(L))(1 + o(1)).$$

Proof. By perturbation theory, the ground state of $L_+(\psi_{\omega_*(L),L})$ is close to the ground state of H_L . This means that we can write it in the form $g = \psi_0 + \xi$, where $\xi = P_c g$. We have

$$(2.7) \quad \begin{aligned}\langle \psi_0, L_+(\psi_{\omega_*(L),L})(\Psi_{0L} + \xi) \rangle &= -\mu_0(L, 0), \\ (H_L + \omega_*(L) + \mu_0(L, 0))\xi &= 5P_c\psi_{\omega_*(L),L_*}^4(\Psi_{0L} + \xi).\end{aligned}$$

From the second equation, we derive $\|\xi\|_{L^2} \leq C\|\psi_{\omega_*(L),L_*}^4\psi_0\|_{L^2}$ for a fixed C . By (2.4), this yields $\|\xi\|_{L^2} \leq C(\omega_*(L) - \Omega)$ for another fixed C . Substituting in the first equation in (2.7), we get

$$\begin{aligned}
 -\mu_0(L, 0) &= \omega_*(L) - \omega_0(L) - 5\langle \psi_{\omega_*(L), L}^4, \Psi_{0L}^2 \rangle + o(\omega_0(L) - \omega_1(L)) \\
 &= -4(\omega_*(L) - \omega_0(L)) + o(\omega_0(L) - \omega_1(L)) \\
 &\sim -2^{-2} \|\varphi\|_{L^6}^6 (\omega_0(L) - \omega_1(L))(1 + o(1)). \quad \square
 \end{aligned}$$

3. THE DISCRETE SPECTRUM OF THE LINEARIZATION

We fix L sufficiently large and consider the family of stable standing waves $\phi_\omega = \phi_{L, \rho}$, where $\omega = \omega_L(\rho)$ and, in particular, $\omega > \omega_*$. We set $\phi_{\omega_*} = \psi_{\omega_* L}$, and set $L_\pm(\omega) = L_\pm(\phi_\omega)$. Let

$$(3.1) \quad \mathcal{L}_\omega = \begin{pmatrix} 0 & L_-(\omega) \\ -L_+(\omega) & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Lemma 3.1. *We have $\sigma_d(\mathcal{L}_{\omega_*}) = \{0\}$. For $\{\}$ meaning “span”, we have*

$$(3.2) \quad \ker \mathcal{L}_{\omega_*} = \{e_1(\omega_*), e_2(\omega_*)\} \quad \text{with} \quad e_1(\omega) = \begin{pmatrix} 0 \\ \phi_\omega \end{pmatrix}, \quad e_2(\omega_*) = \begin{pmatrix} \phi_* \\ 0 \end{pmatrix}.$$

The generalized kernel is

$$(3.3) \quad N_g(\mathcal{L}_{\omega_*}) = \{e_j(\omega_*) \mid j = 1, \dots, 4\} \quad \text{with} \quad \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \quad e_4(\omega_*) = \begin{pmatrix} 0 \\ \gamma \end{pmatrix},$$

where $\phi_{\omega_} = L_+(\omega_*)\alpha$ and $\phi_* = L_-(\omega_*)\gamma$. We have that $\alpha(x)$ is an even function and $\gamma(x)$ is an odd function.*

Proof. The relation (3.2) follows by the fact that 0 is an eigenvalue of $L_-(\omega)$ and $L_+(\omega_*)$. The equations $\phi_{\omega_*} = L_+(\omega_*)\alpha$ and $\phi_* = L_-(\omega_*)\gamma$ admit solutions because $\langle \phi_{\omega_*}, \phi_* \rangle = 0$, since ϕ_{ω_*} is even and ϕ_* odd. We conclude that $N_g(\mathcal{L}_{\omega_*})$ contains the right hand side of (3.3), and so $\dim N_g(\mathcal{L}_{\omega_*}) \geq 4$. To see that these are equal, it is enough to check that $\dim N_g(\mathcal{L}_{\omega_*}) \leq 4$. \mathcal{L}_{ω_*} is obtained by adding to $J(-\partial_x^2 + V + \omega_*)$ a small perturbation. Since $J(-\partial_x^2 + V + \omega_*)$ has four eigenvalues, all close to 0, we get $\dim N_g(\mathcal{L}_{\omega_*}) \leq 4$. \square

Lemma 3.2. *Let $\omega > \omega_*$. Then $\sigma_d(\mathcal{L}_\omega) = \{0, i\lambda(\omega), -i\lambda(\omega)\}$ with $\lambda(\omega) > 0$ an eigenvalue of algebraic multiplicity equal to 1.*

Proof. We know that $\dim N_g(\mathcal{L}_\omega) = 2$, with $N_g(\mathcal{L}_\omega) = \{e_1(\omega), e_2(\omega)\}$ with $e_1(\omega)$ as in Lemma 3.1, and with $(e_2(\omega))^T = (\rho_1 \partial_\omega \phi_\omega, 0)$. Let D be a small disk containing the origin in its interior. We know that $\mathcal{L}_\omega - \mathcal{L}_{\omega_*}$ takes values in the space of bounded operators from $L^2(\mathbb{R})$ into itself, and is continuous in ω with respect to the uniform topology. In addition, we have that \mathcal{L}_{ω_*} has exactly four eigenvalues inside D (in fact, just one with algebraic multiplicity 4), and that ∂D is in the resolvent set of \mathcal{L}_{ω_*} . Then,

$$\sigma_d(\mathcal{L}_\omega) \cap D = \{0, i\lambda(\omega), -i\lambda(\omega)\}$$

follows by the fact that $\sigma_d(\mathcal{L}_\omega)$ is symmetric with respect to the coordinate axes, 0 has algebraic multiplicity 2 for \mathcal{L}_ω (as recalled above), and the two nonzero eigenvalues cannot lie in \mathbb{R} (since we know that the ϕ_ω for $\omega > \omega_*$ are orbitally stable).

By standard arguments in perturbation theory, exploiting only $|\phi_\mu(x)| \leq Ce^{-a|x|}$ for $a > 0$ and $C > 0$ fixed and for both $\mu = \omega_*$, ω , it is possible to prove that

$$\sigma_d(\mathcal{L}_\omega) \cap (\mathbb{C} \setminus D)$$

is empty for ω close enough to ω_* . Specifically, and quite informally, elements of $\sigma_d(\mathcal{L}_\omega) \cap (\mathbb{C} \setminus D)$ could originate by singularities of $(\mathcal{L}_{\omega_*} - z)^{-1}$ on a second sheet of the Riemann surface where it is defined, or by the points $\pm i\omega_*$ if 0 was a resonance of H_L . But the latter is excluded by hypothesis, and the former can be ruled out for ω close enough to ω_* . We skip the details: the analysis at the endpoints is similar to material in [Cu5], while the analysis of the eigenvalues coming from the second sheet can be derived from [CPV]. \square

Lemma 3.3. *Consider for $\omega > \omega_*$ the eigenvalue from Lemma 3.2 with $\lambda(\omega) > 0$. For $\omega = \omega_L(\rho)$ with ρ small enough, there exists a fixed $C > 0$ such that $\lambda(\omega) > C(\omega_0 - \omega_1)^{1/2}\rho^2$.*

Proof. We recall that if $\mathcal{L}_\omega U = i\lambda U$ and $U^T = (u, v)$, then

$$L_-(\omega)L_+(\omega)u = \lambda^2 u.$$

Then, $\langle u, \phi_\omega \rangle = 0$, and one can define $f = (L_-(\omega))^{-1/2}u$ such that

$$(L_-(\omega))^{1/2}L_+(\omega)(L_-(\omega))^{1/2}f = \lambda^2 f.$$

Since one can proceed backwards, we have

$$\begin{aligned} \lambda^2 &= \min_{\langle g, \phi_\omega \rangle = 0} \frac{\langle (L_-(\omega))^{1/2}L_+(\omega)(L_-(\omega))^{1/2}g, g \rangle}{\|g\|_{L^2}^2} \\ &\geq \min_{\langle f, \phi_\omega \rangle = 0} \frac{\langle L_+(\omega)f, f \rangle}{\|f\|_{L^2}^2} \min_{\langle g, \phi_\omega \rangle = 0} \frac{\langle L_-(\omega)g, g \rangle}{\|g\|_{L^2}^2}. \end{aligned}$$

Lemma 3.3 is thus reduced to the following inequality:

$$(3.4) \quad \min_{\langle f, \phi_\omega \rangle = 0} \frac{\langle L_+(\omega)f, f \rangle}{\|f\|_{L^2}^2} > C_1(\omega_0 - \omega_1)^{1/2}\rho^2.$$

Let $\|f\|_{L^2}^2 = 1$ and $\phi = \phi_\omega$. In addition, let $L_+(\omega)\chi_j = (-1)^{j+1}\mu_j\chi_j$ for $j = 0, 1$ with $\|\chi_j\|_{L^2} = 1$ and $\sigma_d(L_+(\omega)) = \{-\mu_0, \mu_1\}$. For a function g , let

$g_j = \langle g, \chi_j \rangle$ and $g_c = \|P_c(L_+(\omega))g\|_{L^2}$, where $P_c(L_+(\omega))g := g - g_0\chi_0 - g_1\chi_1$. Then,

$$\langle L_+(\omega)f, f \rangle \geq -\mu_0 f_0^2 + \mu_1 f_1^2 + \omega f_c^2 =: F(f_0, f_1, f_c).$$

We will prove that, subject to the constraints in the last two lines of (3.5) below, we have $F > C_2(\omega_0 - \omega_1)^{1/2}\rho^2$. Since F is continuous for the strong and weak topology in L^2 , there exists a constrained minimizer. This implies that, for a and b Lagrange multipliers, we have:

$$(3.5) \quad \begin{aligned} 2(\omega - a)P_c(L_+(\omega))f &= bP_c(L_+(\omega))\phi, \\ -2\mu_0 f_0 &= 2af_0 + b\phi_0, \\ 2\mu_1 f_1 &= 2af_1 + b\phi_1, \\ f_0^2 + f_1^2 + f_c^2 &= 1, \\ f_0\phi_0 + f_1\phi_1 + \langle P_c(L_+(\omega))f, P_c(L_+(\omega))\phi \rangle &= 0. \end{aligned}$$

For $P_c(L_+(\omega))\phi$ and $P_c(L_+(\omega))f$ proportional to each other, the last equation in (3.5) is the same as $f_0\phi_0 + f_1\phi_1 + f_c\phi_c = 0$. If $P_c(L_+(\omega))\phi$ and $P_c(L_+(\omega))f$ are not proportional, then $b = 0$ and $\omega = a$. Then, the equation $(\omega + \mu_0)f_0 = 0$ implies $f_0 = 0$ since $\omega + \mu_0 > 0$. Given $(\omega - \mu_1)f_0 = 0$, we have $f_1 = 0$ since $\mu_1 = O(\rho^2)$ while $\omega > \omega_* > 0$. Then, $f_c = 1$ with $F = \omega$, which is clearly the maximum value, and not the minimum. Hence, we can assume that $P_c(L_+(\omega))\phi$ and $P_c(L_+(\omega))f$ are proportional. Then we minimize F under the constraint

$$(3.6) \quad \begin{aligned} f_0^2 + f_1^2 + f_c^2 &= 1, \\ f_0\phi_0 + f_1\phi_1 + f_c\phi_c &= 0. \end{aligned}$$

The plane can be parameterized by $f_0 = \phi_1 u + \phi_c v$, $f_1 = -\phi_0 u$, and $f_c = -\phi_0 v$. Then,

$$(3.7) \quad \begin{aligned} F(\phi_1 u + \phi_c v, -\phi_0 u, -\phi_0 v) \\ &= -\mu_0(\phi_1 u + \phi_c v)^2 + \mu_1 \phi_0^2 u^2 + \omega \phi_0^2 v^2 \\ &= (-\mu_0 \phi_1^2 + \mu_1 \phi_0^2)u^2 + (-\mu_0 \phi_c^2 + \omega \phi_0^2)v^2 - 2\mu_0 \phi_1 \phi_c uv. \end{aligned}$$

This is a quadratic form in (u, v) with eigenvalues, x , as the roots of

$$(3.8) \quad \begin{aligned} (x - (-\mu_0 \phi_1^2 + \mu_1 \phi_0^2))(x - (-\mu_0 \phi_c^2 + \omega \phi_0^2)) - \mu_0^2 \phi_1^2 \phi_c^2 \\ &= x^2 - (-\mu_0 \phi_1^2 + \mu_1 \phi_0^2 - \mu_0 \phi_c^2 + \omega \phi_0^2)x \\ &\quad + (-\mu_0 \phi_1^2 + \mu_1 \phi_0^2)(-\mu_0 \phi_c^2 + \omega \phi_0^2) - \mu_0^2 \phi_1^2 \phi_c^2 = 0. \end{aligned}$$

Notice that, by claim (2) in Lemma 2.3 and parity of functions, we have $\phi_1 = \langle \phi, \chi_1 \rangle = O(\rho)$, and $\phi_0 = \langle \phi, \chi_0 \rangle = \langle \phi_{\omega_*}, \chi_0 \rangle + O(\rho^2)$. Then,

$$(3.9) \quad \phi_0 \sim (\omega_* - \omega_0)^{1/4} \|\Psi_{0L}\|_{L^6}^{-3/2} \langle \Psi_{0L}, \chi_0 \rangle \sim (\omega_0 - \omega_1)^{1/4} \langle \Psi_{0L}, \chi_0 \rangle.$$

We have $\chi_0 = \langle \Psi_{0L}, \chi_0 \rangle \psi_0 + P_c \chi_0$, where P_c is the projection on the continuous spectrum of H_L , and we have $(H_L + \omega_0 + [\omega - \omega_0 + \mu_0])P_c \chi_0 = 5P_c \phi^4 \chi$. Then $\|P_c \chi_0\|_{L^2} \leq C(\omega_0 - \omega_1)$ for a fixed C , which implies $\langle \Psi_{0L}, \chi_0 \rangle = 1 + O(\omega_0 - \omega_1)$. So, from (3.9) we conclude $\phi_0 \sim (\omega_0 - \omega_1)^{1/4}$.

By $0 = (H_L + \omega - \phi^4)\phi = L_+(\omega)\phi + 4\phi^5$, we get

$$(3.10) \quad \phi_c \leq C' \|\phi^5\|_{L^2} \leq C(\omega_0 - \omega_1)^{5/4}.$$

Putting together the information, for the degree-one coefficient in (3.8), we get

$$(3.11) \quad -\mu_0 \phi_1^2 + \mu_1 \phi_0^2 - \mu_0 \phi_c^2 + \omega \phi_0^2 = \omega_0 \sqrt{\omega_0 - \omega_1} + o(\sqrt{\omega_0 - \omega_1}).$$

For the degree-zero coefficient in (3.8), we have

$$\begin{aligned} & -\mu_0 \omega \phi_1^2 \phi_0^2 - \mu_0 \mu_1 \phi_c^2 \phi_0^2 - \mu_0^2 \phi_1^2 \phi_c^2 + \mu_0^2 \phi_1^2 \phi_c^2 + \mu_1 \omega \phi_0^4 \\ & = (\mu_1 \omega \phi_0^2 - \mu_0 \omega \phi_1^2 - \mu_0 \mu_1 \phi_c^2) \phi_0^2. \end{aligned}$$

We claim that

$$(3.12) \quad (\mu_1 \omega \phi_0^2 - \mu_0 \omega \phi_1^2 - \mu_0 \mu_1 \phi_c^2) \phi_0^2 \gtrsim \omega_0 (\omega_0 - \omega_1)^{3/2} \rho^2.$$

This and (3.11) imply that the polynomial in (3.8) has both roots positive: one $\sim \sqrt{\omega_0 - \omega_1}$, and the other $\gtrsim (\omega_0 - \omega_1)^{3/2} \rho^2$. Then, the minimum of (3.7) for $u^2 + v^2 = 1$ is $\gtrsim (\omega_0 - \omega_1) \rho^2$. Since $f_0^2 + f_1^2 + f_c^2 = 1$ implies $u^2 + v^2 \approx (\omega_0 - \omega_1)^{-1/2}$, it follows that the minimum of F is $> C(\omega_0 - \omega_1)^{1/2} \rho^2$ for a fixed C .

Inequality (3.12) follows from (3.10), $\phi_0 \sim (\omega_0 - \omega_1)^{1/4}$, and the following:

$$\begin{aligned} & (\mu_1 \omega \phi_0^2 - \mu_0 \omega \phi_1^2) \phi_0^2 \\ & \sim \phi_0^2 \left(\frac{5}{2} \|\varphi\|_{L^6}^6 (\omega_0 - \omega_1) \rho^2 - \frac{1}{4} \|\varphi\|_{L^6}^6 (\omega_0 - \omega_1) \rho^2 \right). \quad \square \end{aligned}$$

4. SET-UP FOR THEOREM 1.1 AND DISPERSION FOR THE LINEARIZATION

Theorem 1.1 is a consequence of [Cu3, Cu4]. Notice that since the linearization has just one pair of nonzero eigenvalues, the Hamiltonian set up in [Cu1] is unnecessary, and the theory in [Cu3, Cu4, CM] is adequate. We recall that due to the absence of the endpoint Strichartz estimate in $1D$, the theory requires some adequate surrogate. For $1D$, this was provided by Mizumachi [M]. The theory in [M], however, is more complicated than necessary for the present application. The simplifications were provided in [Cu4, CT]. Subsequent papers like [KPS, PS] return to a more complicated approach, as in [M]; and so even though Theorem

1.1 is a direct consequence of [Cu4], we will take the opportunity to state the various steps of the proof in order also to point out the points in [M] and in [PS] that can be simplified.

We denote by \mathcal{O} a small interval of the form $(\omega_*, \omega_* + \delta_*)$ so that ϕ_ω is stable for $\omega \in \mathcal{O}$.

Using an Implicit Function Theorem argument relying on the regularity and decay of ϕ_ω , $\partial_\omega \phi_\omega$, given a fixed $\omega_2 \in \mathcal{O}$, there exists an $\alpha > 0$ such that for any $u_0 \in X$ with $\|u_0 - \phi_{\omega_2}\|_X < \alpha$ for $X = H^{-1}, H^1$, there exist a fixed $C(\omega_2)$ and a unique pair $(\gamma(0), \omega(0)) \in \mathbb{R}^2$ such that

$$\begin{aligned} u_0(x) &= e^{i\gamma(0)} (\phi_{\omega(0)}(x) + r(0, x)), \\ \operatorname{Re}\langle r, \partial_\omega \phi_{\omega(0)} \rangle &= \operatorname{Im}\langle r, \phi_{\omega(0)} \rangle = 0, \\ |\omega_2 - \omega(0)| + |\gamma(0)| + \|r(0, \cdot)\|_X &< C(\omega_2) \|u_0 - \phi_{\omega_2}\|_X. \end{aligned}$$

The map $u_0 \in X \rightarrow (\gamma(0), \omega(0)) \in \mathbb{R}^2$ and the map $u_0 \in X \rightarrow r(0) \in X$ are smooth. Since our ground states are orbitally stable in H^1 and we are focusing on solutions $u(t)$ of (1.1) close to stable ground states, we can assume that, for all times, the corresponding ansatz holds in H^1 ; that is,

$$u(t, x) = e^{i\Theta(t)} (\phi_{\omega(t)}(x) + r(t, x)), \quad \Theta(t) = \int_0^t \omega(s) ds + \gamma(t),$$

where $\operatorname{Re}\langle r(t), \partial_\omega \phi_{\omega(t)} \rangle = \operatorname{Im}\langle r(t), \phi_{\omega(t)} \rangle = 0$. We have used the structure of the Schrödinger equation that $u \in C^1(\mathbb{R}, H^{-1}) \cap C^0(\mathbb{R}, H^1)$. Then the chain rule implies $(\gamma, \omega) \in C^1(\mathbb{R}, \mathbb{R} \times \mathcal{O})$.

Notice that the following result, which is slightly more precise the one in Section 2 (see [Cu3, Cu4]), holds.

Lemma 4.1. *Fix $\omega_2 \in \mathcal{O}$. Then there exist $\varepsilon_0 > 0$ and $A_0 > 0$ such that $\varepsilon \in (0, \varepsilon_0)$ and $\|u(0, x) - \phi_{\omega_2}\|_{H^1} < \varepsilon$ imply that, for the corresponding solution,*

$$\inf\{\|u(t) - e^{i\gamma} \phi_{\omega_2}\|_{H^1(\mathbb{R})} : \gamma \in \mathbb{R}\} < A_0 \varepsilon.$$

Inserting the ansatz into the NLSE (1.1), we get

$$\begin{aligned} ir_t &= H_L r + \omega(t)r - 3\phi_{\omega(t)}^4 r - 2\phi_{\omega(t)}^4 \bar{r} \\ &\quad + \dot{\gamma}(t)\phi_{\omega(t)} - i\dot{\omega}(t)\partial_\omega \phi_{\omega(t)} + \dot{\gamma}(t)r + O(r^2), \end{aligned}$$

where, for appropriate Schwartz functions $\chi_{ij}(\omega) \sim \phi_\omega^{5-i-j}$, depending smoothly on ω , we have

$$O(r^2) = \sum_{i+j=2}^4 \chi_{ij}(\omega(t)) r^i \bar{r}^j - |r|^4 r.$$

We set ${}^tR = (r, \bar{r})$, ${}^t\Phi = (\phi_\omega, \bar{\phi}_\omega)$ (using a different frame from the one in Section 3), and rewrite the above equation as

$$(4.1) \quad iR_t = \mathcal{H}_\omega R + \sigma_3 \dot{y}R + \sigma_3 \dot{y}\Phi - i\omega \partial_\omega \Phi + O(R^2)$$

where, setting $(O(r^2))^*$ as the complex conjugate of $O(r^2)$, we have the following implicitly defined:

$$\begin{aligned} O(R^2) &= \sigma_3 \begin{pmatrix} O(r^2) \\ (O(r^2))^* \end{pmatrix}; \\ \mathcal{H}_{\omega,0} &= \sigma_3(H_L + \omega), \quad \mathcal{V}_\omega = -3\phi_\omega^4 \sigma_3 + 2i\phi_\omega^4 \sigma_2; \\ \mathcal{H}_\omega &= \mathcal{H}_{\omega,0} + V_\omega, \end{aligned}$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as the standard Pauli matrices.

We know that 0 is an isolated eigenvalue of \mathcal{H}_ω and that $\dim N_g(\mathcal{H}_\omega) = 2$. We have

$$\mathcal{H}_\omega \sigma_3 \Phi_\omega = 0, \quad \mathcal{H}_\omega \partial_\omega \Phi_\omega = -\Phi_\omega.$$

Since $\mathcal{H}_\omega^* = \sigma_3 \mathcal{H}_\omega \sigma_3$, we have $N_g(\mathcal{H}_\omega^*) = \text{span}\{\Phi_\omega, \sigma_3 \partial_\omega \Phi_\omega\}$. Let $\xi(\omega)$ be a real eigenfunction of \mathcal{H}_ω^* with eigenvalue $\lambda(\omega)$. Note that the components of $\xi(\omega, x)$ are Schwartz functions. We have

$$\mathcal{H}_\omega \xi(\omega) = \lambda(\omega) \xi(\omega), \quad \mathcal{H}_\omega \sigma_1 \xi(\omega) = -\lambda(\omega) \sigma_1 \xi(\omega).$$

Notice that $\langle \xi, \sigma_3 \xi \rangle > 0$ since $\langle \sigma_3 \mathcal{H}_\omega \cdot, \cdot \rangle$ can be proved by standard arguments to be positive definite on $N_g^\perp(\mathcal{H}_\omega^*)$. In particular, we can pick $\xi(\omega)$ smoothly dependent on ω and normalized such that $\langle \xi, \sigma_3 \xi \rangle = 1$.

For $\omega \in \mathcal{O}$, we have the \mathcal{H}_ω -invariant Jordan block decomposition

$$(4.2) \quad L^2(\mathbb{R}, \mathbb{C}^2) = N_g(\mathcal{H}_\omega) \oplus (\oplus_\pm \ker(\mathcal{H}_\omega \mp \lambda(\omega))) \oplus L_c^2(\mathcal{H}_\omega),$$

where $L_c^2(\mathcal{H}_\omega) := \{N_g(\mathcal{H}_\omega^*) \oplus (\oplus_\pm \ker(\mathcal{H}_\omega^* \mp \lambda(\omega)))\}^\perp$. Here, by \oplus_\pm we denote the direct product in our decomposition of L^2 stemming from the \pm symmetry of the spectrum of \mathcal{H}_ω . This symmetry is generated by the fact that $\sigma_1 \mathcal{H}_\omega \sigma_1 = -\mathcal{H}_\omega$ for ϕ any real eigenfunction of \mathcal{H}_ω . Thus all eigenvalues of \mathcal{H}_ω will have symmetric counterparts on the real and imaginary axes. Correspondingly, we set

$$(4.3) \quad R(t) = z(t)\xi(\omega(t)) + \bar{z}(t)\sigma_1\xi(\omega(t)) + f(t),$$

$$(4.4) \quad R(t) \in N_g^\perp(\mathcal{H}_{\omega(t)}^*) \quad \text{and} \quad f(t) \in L_c^2(\mathcal{H}_{\omega(t)}).$$

Notice that by orbital stability the decomposition (4.3)–(4.4) holds for all times, and that $u \in C^1(\mathbb{R}, L^2)$ implies $z \in C^1(\mathbb{R}, \mathbb{C})$ and $f \in C^1(\mathbb{R}, L^2)$.

Substituting (4.3) into $O(R^2)$, we get an expansion

$$\begin{aligned} O(R^2) = & \sum_{2 \leq m+n \leq 5} R_{m,n}(\omega) z^m \bar{z}^n + \sum_{1 \leq m+n \leq 4} z^m \bar{z}^n A_{m,n}(\omega) f \\ & + \sum_{i=2}^4 \sum_{m+n+i \leq 5} z^m \bar{z}^n G_{i,m,n}(\omega) f^i - \frac{1}{4} \sigma_3 |f|^4 f, \end{aligned}$$

with the following:

- (1) $R_{m,n}(\omega, x)$ are real vectors with Schwartz functions as entries;
- (2) $A_{m,n}(\omega, x)$ are real 2×2 matrices with Schwartz functions as entries; and
- (3) $G_{i,m,n}(\omega, x)$ are i -linear forms mapping \mathbb{C}^{2i} into \mathbb{C}^2 , also with Schwartz coefficients.

All these Schwartz functions are fixed linear combinations of products of components of $\xi(\omega)$ with appropriate powers of $\phi(\omega)$. Thus,

(4.5)

$$\begin{aligned} i\dot{f} = & (\mathcal{H}_{\omega(t)} + \sigma_3 \dot{y})f + \sigma_3 \dot{y} \Phi_\omega - i\dot{\omega} \partial_\omega \Phi_\omega + (z\lambda(\omega) - i\dot{z})\xi(\omega) \\ & - (\bar{z}\lambda(\omega) + i\dot{\bar{z}})\sigma_1 \xi(\omega) + \sigma_3 \dot{y} (z\xi + \bar{z}\sigma_1 \xi) - i\dot{\omega} (z \partial_\omega \xi + \bar{z} \sigma_1 \partial_\omega \xi) \\ & + \sum_{2 \leq m+n \leq 5} R_{m,n}(\omega) z^m \bar{z}^n + \sum_{1 \leq m+n \leq 4} z^m \bar{z}^n A_{m,n}(\omega) f \\ & + \sum_{2 \leq i \leq 4} \sum_{m+n+i \leq 5} z^m \bar{z}^n G_{i,m,n}(\omega) f^i - \frac{1}{4} \sigma_3 |f|^4 f. \end{aligned}$$

Taking inner products of the equation with the generators of $N_g(H_\omega^*)$ and of $\ker(H_\omega^* - \lambda)$, we obtain modulation and discrete modes equations:

$$(4.6) \quad \begin{aligned} i\dot{\omega} q' = & \langle \mathcal{X}, \Phi \rangle, \quad \dot{y} q' = \langle \mathcal{X}, \sigma_3 \partial_\omega \Phi \rangle, \quad i\dot{z} - \lambda(\omega)z = \langle \mathcal{X}, \sigma_3 \xi \rangle, \\ \mathcal{X} := & \sigma_3 \dot{y} (z\xi + \bar{z}\sigma_1 \xi) - i\dot{\omega} (z \partial_\omega \xi + \bar{z} \sigma_1 \partial_\omega \xi) + O(R^2), \end{aligned}$$

where $q' := d\|\phi_\omega\|_2^2/d\omega$, and $O(R^2)$ can be expanded as done above.

We now go through the dispersive estimates (proofs are in [Cu4]). We call admissible a pair (p, q) such that

$$(4.7) \quad \frac{2}{p} + \frac{1}{q} = \frac{1}{2}, \quad p \geq 4, \quad q \geq 2.$$

Theorem 4.2 (Strichartz estimates). *For $k = [0, 2]$, there exist positive numbers $C(\omega, k, p)$ and $C(\omega, k, p_1, p_2)$ upper semicontinuous in their arguments such that:*

(1) For any $f \in L_c^2(\omega)$ and any admissible pair (p, q) with $p > 4$, we have

$$(4.8) \quad \|e^{-it\mathcal{H}_\omega} f\|_{L_t^p W_x^{k,q}} \leq C \|f\|_{H^k};$$

(2) For any $g(t, x)$ and any two admissible pairs (p_j, q_j) for $j = 1, 2$ with $p_j > 4$, we have

$$(4.9) \quad \left\| \int_0^t e^{-i(t-s)\mathcal{H}_\omega} P_c(\omega) g(s, \cdot) ds \right\|_{L_t^{p_1} W_x^{k,q_1}} \leq C \|g\|_{L_t^{p'_2} W_x^{k,q'_2}}.$$

In the case $k = 0$, we can include also cases $p = 4$ in (4.8) and $p_j = 4$ for any of $j = 1, 2$ in (4.9).

For the proof, see [Cu3, Cu4]. The case $k > 0$ requires interpolation. The case $k = 0$ is like the one for $e^{-it\partial_x^2}$. Specifically, we can use dispersive estimates (see [KS]) and an appropriate version of the so-called TT^* argument. In particular, this yields the $L_t^4 L_x^\infty$ bound, which is not reached in [KS]. See [DMW] for how to extend such results to Schrödinger operators, H , formed by singular perturbations of the Laplacian with $k \leq 1$.

Lemma 4.3. Fix $\tau > \frac{3}{2}$.

(1) There exists $C = C(\tau, \omega)$, upper semicontinuous in ω such that for any $\varepsilon \neq 0$,

$$\|R_{\mathcal{H}_\omega}(\lambda + i\varepsilon) P_c(\mathcal{H}_\omega) u\|_{L_\lambda^3 L_x^{2-\tau}} \leq C \|u\|_{L^2}.$$

(2) For any $u \in L_x^{2,\tau}$, the following limit exists:

$$\lim_{\varepsilon \searrow 0} R_{\mathcal{H}_\omega}(\lambda \pm i\varepsilon) u = R_{\mathcal{H}_\omega}^\pm(\lambda) u \quad \text{in } C^0(\sigma_e(H_\omega), L_x^{2,-\tau}).$$

(3) There exists $C = C(\tau, \omega)$, upper semicontinuous in ω such that

$$\|R_{\mathcal{H}_\omega}^\pm(\lambda) P_c(\mathcal{H}_\omega)\|_{B(L_x^{2,\tau}, L_x^{2,-\tau})} < C \langle \lambda \rangle^{-1/2}.$$

(4) Given any $u \in L_x^{2,\tau}$, we have

$$P_c(\mathcal{H}_\omega) u = \frac{1}{2\pi i} \int_{\sigma_e(\mathcal{H}_\omega)} (R_{\mathcal{H}_\omega}^+(\lambda) - R_{\mathcal{H}_\omega}^-(\lambda)) u d\lambda.$$

The above resolvent estimates are consequences of the fact that the set $\sigma_e(H_\omega)$ does not contain eigenvalues and the values $\pm\omega$ are not resonances of \mathcal{H}_ω , as well as the construction of spectral measures in [KS] using plane wave theory and resolvent expansions. In fact, resolvent estimates and construction of a spectral measure follow from simpler techniques than those in [KS], since \mathcal{H}_ω is a small perturbation of $\sigma_3(H_L + \omega)$.

Claim (1) of the following smoothing lemma is a consequence of Lemma 4.3 by [K], while (2) follows from (1) by duality.

Lemma 4.4. For any $k, \tau > \frac{3}{2}$, $\exists C = C(\tau, k, \omega)$ upper semicontinuous in ω such that:

- (1) For any f , $\|e^{-it\mathcal{H}_\omega} P_c(\mathcal{H}_\omega) f\|_{L_t^2 H_x^{k-\tau}} \leq C \|f\|_{H^k}$;
- (2) For any $g(t, x)$,

$$\left\| \int_{\mathbb{R}} e^{it\mathcal{H}_\omega} P_c(\mathcal{H}_\omega) g(t, \cdot) dt \right\|_{H_x^k} \leq C \|g\|_{L_t^2 H_x^{k,\tau}}.$$

Lemma 4.5. For any $k, \tau > \frac{3}{2}$, $\exists C = C(\tau, k, \omega)$ as above such that $\forall g(t, x)$,

$$\left\| \int_0^t e^{-i(t-s)\mathcal{H}_\omega} P_c(\mathcal{H}_\omega) g(s, \cdot) ds \right\|_{L_t^2 H_x^{k-\tau}} \leq C \|g\|_{L_t^2 H_x^{k,\tau}}.$$

Proof. To get this proof, there is no need of Lemma 11 in [M] or of the analogous result (Lemma 2 in [PS, Section 7]). Rather, we simply use Plancherel and Hölder inequalities and Lemma 4.3 (3):

$$\begin{aligned} & \left\| \int_0^t e^{-i(t-s)\mathcal{H}_\omega} P_c(\mathcal{H}_\omega) g(s, \cdot) ds \right\|_{L_t^2 L_x^{2-\tau}} \\ & \leq \|R_{\mathcal{H}_\omega}^+(\lambda) P_c(\mathcal{H}_\omega) \hat{\chi}_{[0,+\infty)} * \lambda \hat{g}(\lambda, x)\|_{L_\lambda^2 L_x^{2-\tau}} \\ & \leq \| \|R_{\mathcal{H}_\omega}^+(\lambda) P_c(\mathcal{H}_\omega)\|_{B(L_x^{2,\tau}, L_x^{2-\tau})} \| \hat{\chi}_{[0,+\infty)} * \lambda \hat{g}(\lambda, x)\|_{L_\lambda^{2,\tau}} \|_{L_\lambda^2} \\ & \leq \|R_{\mathcal{H}_\omega}^+(\lambda) P_c(\mathcal{H}_\omega)\|_{L_\lambda^\infty(\mathbb{R}, B(L_x^{2,\tau}, L_x^{2-\tau}))} \|g\|_{L_t^2 L_x^{2,\tau}} \\ & \leq C \|g\|_{L_t^2 L_x^{2,\tau}}. \end{aligned} \quad \square$$

Lemma 4.6. For any $k, \tau > \frac{3}{2}$ $\exists C = C(\tau, k, \omega)$ as above such that $\forall g(t, x)$,

$$\left\| \int_0^t e^{-i(t-s)\mathcal{H}_\omega} P_c(\mathcal{H}_\omega) g(s, \cdot) ds \right\|_{L_t^\infty H_x^k \cap L_t^4(\mathbb{R}, W_x^{k,\infty})} \leq C \|g\|_{L_t^2 H_x^{k,\tau}}.$$

Proof. By density, it is enough to focus on $g(t, x)$ being a Schwartz function in \mathbb{R}^2 . Set $Tg(t) = \int_0^{+\infty} e^{-i(t-s)\mathcal{H}_\omega} P_c(\mathcal{H}_\omega) g(s) ds$. Lemma 4.4 (2) implies that $f := \int_0^{+\infty} e^{is\mathcal{H}_\omega} P_c(\omega) g(s) ds \in L^2(\mathbb{R})$; thus Lemma 4.6 is a direct consequence of [CK]. \square

Lemma 4.7. The following operators $P_\pm(\omega)$ are well defined:

$$\begin{aligned} P_+(\omega)u &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \lim_{M \rightarrow +\infty} \int_\omega^M [R_{\mathcal{H}_\omega}(\lambda + i\varepsilon) - R_{\mathcal{H}_\omega}(\lambda - i\varepsilon)] u d\lambda, \\ P_-(\omega)u &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \lim_{M \rightarrow +\infty} \int_{-M}^{-\omega} [R_{\mathcal{H}_\omega}(\lambda + i\varepsilon) - R_{\mathcal{H}_\omega}(\lambda - i\varepsilon)] u d\lambda. \end{aligned}$$

For any $k \in \mathbb{N}$, $M > 0$ and $N > 0$, and for $C = C(N, M, \omega)$ upper semicontinuous in $\omega > \omega^*$, we have $\|(P_+(\omega) - P_-(\omega) - P_c(\omega)\sigma_3)f\|_{H^{k,M}} \leq C \|f\|_{H^{k,-N}}$.

Proof. In [Cu3, Cu4], this is proved for $k = 0$. The proof extends to the case of all k positive integers. \square

5. NORMAL FORM EXPANSION

Here we repeat the theory in [CM, Cu4], which is somewhat more elementary than [Cu1], but still adequate in our setting.

We consider $N \in \mathbb{N}$ such that for any $t \geq 0$, for $\rho(t)$ and for the corresponding $\omega(t) = \omega(\rho(t))$ and $\lambda(t) = \lambda(\omega(\rho(t)))$, we have

$$(5.1) \quad N\lambda(t) < \omega(t) < (N+1)\lambda(t).$$

Notice that, for $\rho \geq 0$, the function $\lambda(\omega(\rho))/\omega(\rho)$ is continuous and strictly increasing, equal to 0 at $\rho = 0$, and so for small values of ρ it has no values in \mathbb{N} . Then, by continuity and orbital stability, we can then assume (5.1) for all t and for a fixed $N \in \mathbb{N}$.

5.1. Changes of variables on f . For the N of (5.1), we consider $k = 1, 2, \dots, N$, and set $f = f_k$ for $k = 1$. The other f_k are defined below. In the ODEs, there will be error terms schematically of the form

$$\begin{aligned} E_{\text{ODE}}(k) = & \sum_{m+n \geq 2N+2} z^m \bar{z}^n \Lambda_{m,n}^{(k)}(\omega) + \sum_{m+n \geq N+1} z^m \bar{z}^n A_{m,n}^{(k)}(\omega) f_k \\ & + \sum_{\substack{m+n \geq 1 \\ 2 \leq i+|j| \leq 4}} z^m \bar{z}^n G_{i,m,n}^{(k)}(\omega) f_k^i \langle B_{j,m,n}^{(\ell)}(\omega), f_k^{j\ell} \rangle^\ell - \frac{1}{4} \sigma_3 |f_k|^4 f_k. \end{aligned}$$

All coefficients $\Lambda_{m,n}^{(k)}(\omega)$, $A_{m,n}^{(k)}(\omega)$, $G_{i,m,n}^{(k)}(\omega)$, $B_{j,m,n}^{(\ell)}(\omega)$ are vectors, matrices, i -forms and j_ℓ -forms with Schwartz coefficients dependent smoothly on ω .

In the PDEs, there will be error terms of the form

$$\begin{aligned} E_{\text{PDE}}(k) = & \sum_{m+n \geq N+2} z^m \bar{z}^n \Lambda_{m,n}^{(k)}(\omega) \\ & + \sum_{\substack{m+n \geq 1 \\ 2 \leq i+|j| \leq 4}} z^m \bar{z}^n \mathbf{G}_{i,m,n}^{(k)}(\omega) f_k^i \langle \mathbf{B}_{j,m,n}^{(\ell)}(\omega), f_k^{j\ell} \rangle^\ell - \frac{1}{4} \sigma_3 |f_k|^4 f_k. \end{aligned}$$

The coefficients are similar to those of $E_{\text{ODE}}(k)$. In what follows, we will use that $\|z\|_{L^\infty} < c\varepsilon$, by Lemma 4.1. In the right-hand side of (4.5) and of the equation of z in (4.6), we substitute \dot{y} and $\dot{\omega}$ using the modulation equations (i.e., the equations for \dot{y} and $\dot{\omega}$ in (4.6)). We repeat the procedure a sufficient number of

times until, for $k = 1$, $f_1 = f$, and $q'(\omega) = d\|\phi_\omega\|_2^2/d\omega$, we can write

$$\begin{aligned}
 (5.2) \quad & i\dot{\omega}q'(\omega) \\
 &= \left\langle \sum_{m+n=2}^{2N+1} z^m \bar{z}^n \Lambda_{m,n}^{(k)}(\omega) + \sum_{m+n=1}^N z^m \bar{z}^n A_{m,n}^{(k)}(\omega) f_k + E_{\text{ODE}}(k), \Phi(\omega) \right\rangle, \\
 & i\dot{z} - \lambda z \\
 &= \left\langle \sum_{m+n=2}^{2N+1} z^m \bar{z}^n \Lambda_{m,n}^{(k)}(\omega) + \sum_{m+n=1}^N z^m \bar{z}^n A_{m,n}^{(k)}(\omega) f_k + E_{\text{ODE}}(k), \sigma_3 \xi(\omega) \right\rangle, \\
 & i\dot{f}_k = (\mathcal{H}_\omega + \sigma_3 \dot{y}) f_k + E_{\text{PDE}}(k) + \sum_{k+1 \leq m+n \leq N+1} z^m \bar{z}^n R_{m,n}^{(k)}(\omega),
 \end{aligned}$$

where $A_{m,n}^{(k)}$, $R_{m,n}^{(k)}$ and $\Lambda_{m,n}^{(k)}(\omega, \mathbf{x})$ have Schwartz coefficients which are smooth in ω . Exploiting $|(m-n)\lambda(\omega)| < \omega$ for $m+n \leq N$, $m \geq 0$, $n \geq 0$, we define inductively f_k with $k \leq N$ by

$$\begin{aligned}
 (5.3) \quad f_k &= \sum_{m+n=k} z^m \bar{z}^n \Psi_{m,n}(\omega) + f_{k-1} \\
 &\quad \text{for } \Psi_{m,n}(\omega) := R_{\mathcal{H}_\omega}((m-n)\lambda(\omega)) R_{m,n}^{(k)}(\omega).
 \end{aligned}$$

Notice that if $R_{m,n}^{(k-1)}(\omega, \mathbf{x})$ is a vector with Schwartz functions as entries, and if it depends smoothly on ω , the same is true for $\Psi_{m,n}(\omega)$ by $|(m-n)\lambda(\omega)| < \omega$. When, using (5.3), we replace f_{k-1} by f_k in the equations for ω and z of step $k-1$ in (5.2), we then obtain new equations as of step k . When we differentiate (5.3), we obtain

$$(5.4) \quad i\dot{f}_k = i\dot{f}_{k-1} + \sum_{m+n=k} \lambda \cdot (m-n) z^m \bar{z}^n \Psi_{m,n}(\omega) + (*),$$

where $(*)$ is formed by terms that can be absorbed in the last two terms of the equation of f_k in (5.2). Terms of the same type are obtained by substituting f_{k-1} by f_k inside $E_{\text{PDE}}(k-1)$. Finally, the second term in the right-hand side of (5.4) cancels with the terms

$$\sum_{m+n=k} z^m \bar{z}^n R_{m,n}^{(k-1)}(\omega)$$

in (5.2) at step $k-1$. We conclude that, by induction and elementary algebra, (5.2) is true.

We are now ready to state the result which directly implies Theorem 1.1. Note that this theorem is quite similar to the result posed in [Cu3] in higher dimensions. In our case, the choice of quintic nonlinearity as well as the spectral analysis in Section 3 give that results similar to main hypotheses (H1)–(H10) of [Cu3] hold in our case without assumption.

Theorem 5.1. *Assume that the Fermi Golden Rule hypothesis is nondegenerate (see (HF), below). Let ε_0 and ε be the constants of Theorem 1.1, Let u be a solution*

of (1.1), let $U = {}^t(u, \bar{u})$, and let $\Psi_{m,n}(\omega)$ be as above. Then if ε_0 is sufficiently small, there exist C^1 -functions $\omega(t)$ and $\theta(t)$, and a constant $\omega_+ > \omega_*$ such that we have $\sup_{t \geq 0} |\omega(t) - \omega_0| = O(\varepsilon)$. Further, $\lim_{t \rightarrow +\infty} \omega(t) = \omega_+$, and we can write

$$\begin{aligned} U(t, x) &= e^{i\theta(t)\sigma_3} \left(\Phi_{\omega(t)}(x) + z(t)\xi(\omega(t)) + \overline{z(t)}\sigma_1\xi(\omega(t)) \right) \\ &\quad + e^{i\theta(t)\sigma_3} \sum_{2 \leq m+n \leq N} \Psi_{m,n}(\omega(t)) z(t)^m \overline{z(t)}^n + e^{i\theta(t)\sigma_3} f_N(t, x), \\ &\quad \text{with } \|z(t)\|_{L_t^{2N+2}}^{N+1} + \|f_N(t, x)\|_{L_t^\infty H_x^1 \cap L_t^5 W_x^{1,10} \cap L_t^4 L_x^\infty} \leq C\varepsilon. \end{aligned}$$

Furthermore, there exists $f_+ \in H^1(\mathbb{R}, \mathbb{C}^2)$ such that

$$(5.5) \quad \lim_{t \rightarrow +\infty} \|e^{i\theta(t)\sigma_3} f_N(t) - e^{it\partial_x^2 \sigma_3} f_+\|_{H^1} = 0.$$

Obviously, the scattering result (5.5) holds also for $t \rightarrow -\infty$. We do not prove this theorem explicitly, but we recall Lemma 4.3 in [Cu4], which states:

Lemma 5.2. *Let ε and ε_0 be the constants of Theorem 1.1. For a fixed constant c , there are fixed constants positive C_d , C_c , and ε_0 such that, for any $0 < \varepsilon \leq \varepsilon_0$, if $|z(0)| \leq c\varepsilon$ and $\|f_N(0)\|_{H^1} \leq c\varepsilon$, and if in an interval $[0, T]$ the estimates*

$$(5.6) \quad \begin{aligned} \|z\|_{L_t^{2N+2}}^{N+1} &\leq 2C_d\varepsilon, \\ \|f_N\|_{L_t^\infty H_x^1 \cap L_t^5 W_x^{1,10} \cap L_t^4 L_x^\infty \cap L_t^2 H_x^{1-2}} &\leq 2C_c\varepsilon, \end{aligned}$$

hold, then in $[0, T]$ we obtain the improved inequalities

$$(5.7) \quad \|f_N\|_{L_t^\infty H_x^1 \cap L_t^5 W_x^{1,10} \cap L_t^4 L_x^\infty \cap L_t^2 H_x^{1-2}} \leq C_c\varepsilon,$$

$$(5.8) \quad \|z\|_{L_t^{2N+2}}^{N+1} \leq C_d\varepsilon.$$

Lemma 5.2 implies that (5.7)–(5.8) hold on $[0, \infty)$. We sketch only the main steps of the proof of Lemma 5.2. That such bounds imply the necessary bootstrap and scattering arguments can be seen generally in, for instance, [Tao, Sections 1.3 and 3.6].

First, we rewrite the equation for f_N . Set $\omega(0) := \omega(\rho(0))$, and write

$$\begin{aligned} i\partial_t f_N - (\mathcal{H}_{\omega(0)} + \sigma_3(\dot{\gamma} + \omega - \omega(0)))f_N \\ = \sum_{m+n=N+1} z^m \bar{z}^n R_{m,n}^{(N)}(\omega) + \tilde{E}_{\text{PDE}}(N), \end{aligned}$$

where $\tilde{E}_{\text{PDE}}(N) := E_{\text{PDE}}(N) + (\mathcal{V}_\omega - \mathcal{V}_{\omega(0)})f_N$. It is easy to see that the estimate in equation (5.7) with $f_N (= P_c(\omega)f_N)$ is equivalent to the same estimate holding

for $P_c(\omega(0))f_N$, because $P_c(\omega) - P_c(\omega(0)) = P_d(\omega(0)) - P_d(\omega)$ is a small, smoothing operator. For $\varphi(t) = \varphi = \dot{y} + \omega - \omega(0)$, we then write

$$(5.9) \quad i\partial_t P_c(\omega(0))f_N - \{\mathcal{H}_{\omega(0)} + \varphi[P_+(\omega(0)) - P_-(\omega(0))]\}P_c(\omega(0))f_N \\ = \sum_{m+n=N+1} z^m \bar{z}^n P_c(\omega(0))R_{m,n}^{(N)}(\omega) + \mathcal{R},$$

(5.10)

$$\mathcal{R} := P_c(\omega(0))\tilde{E}_{\text{PDE}}(N) + \varphi[P_+(\omega(0)) - P_-(\omega(0)) - P_c(\omega(0))\sigma_3]f_N.$$

Lemma 5.3. *Assume that in an interval $[0, T]$, inequalities (5.6) are true. Then there is a fixed constant κ_0 such that if ε_0 is sufficiently small, we have*

$$(5.11) \quad \|f_N\|_{L_t^\infty H_x^1 \cap L_t^5 W_x^{1,10} \cap L_t^4 L_x^\infty \cap L_t^2 H_x^{1,-2}} \leq \kappa_0(1 + C_d)\varepsilon.$$

Proof. As a preliminary, we claim that $P_\pm(\omega)$ is a bounded operator in any $H^{k,s}$ with $k \in \mathbb{Z}$ (where $k \geq 0$) and any $s \in \mathbb{R}$. This is the case for $P_c(\omega)$. We thus have

$$P_c(\omega) = P_+(\omega) + P_-(\omega) \\ = 2P_+(\omega) - (P_+(\omega) - P_-(\omega) - P_c(\omega)\sigma_3) - P_c(\omega)\sigma_3.$$

Solving with respect to $P_+(\omega)$, by Lemma 4.7 the claim is true for $P_+(\omega)$, and so also for $P_-(\omega)$. We use

$$P_\pm(\omega(0))f_N(t) \\ = e^{-it\mathcal{H}_{\omega(0)}} e^{-i\int_0^t \varphi(\tau) d\tau} P_\pm(\omega(0))f_N(0) \\ \times \sum_{m+n=N+1} -i \int_0^t e^{-i(t-s)\mathcal{H}_{\omega(0)}} e^{\pm i\int_s^t \varphi(\tau) d\tau} P_\pm(\omega(0))z^m \bar{z}^n P_c(\omega(0))R_{m,n}^{(N)}(\omega) ds \\ - i \int_0^t e^{-i(t-s)\mathcal{H}_{\omega(0)}} e^{\pm i\int_s^t \varphi(\tau) d\tau} P_\pm(\omega(0))\mathcal{R}(s) ds.$$

Set $\mathbf{X} := L_t^\infty H_x^1 \cap L_t^5 W_x^{1,10} \cap L_t^4 L_x^\infty$. Then, for fixed constants, we have

$$\|e^{-it\mathcal{H}_{\omega(0)}} e^{-i\int_0^t \varphi(\tau) d\tau} P_\pm(\omega(0))f_N(0)\|_{\mathbf{X} \cap L_t^2 H_x^{1,-2}} \\ \leq c_0 \|P_\pm(\omega(0))f_N(0)\|_{H_x^1} \leq c_1 \|f_N(0)\|_{H_x^1} \leq c c_1 \varepsilon$$

by Theorem 4.2, Lemma 4.4, and the above claim on $P_\pm(\omega(0))$ if we take ε_0 sufficiently small. Similarly, for fixed c_2 and c_3 ,

$$\left\| \int_0^t e^{-i(t-s)\mathcal{H}_{\omega(0)}} e^{\pm i\int_s^t \varphi(\tau) d\tau} P_\pm(\omega(0))z^m \bar{z}^n R_{m,n}^{(N)}(\omega) ds \right\|_{\mathbf{X} \cap L_t^2 H_x^{1,-2}} \\ \leq c_2 \|z^{N+1}\|_{L^2} \leq c_3 C_d \varepsilon,$$

and for fixed c_4, c_5 ,

$$\begin{aligned} & \left\| \int_0^t e^{-i(t-s)\mathcal{H}_{\omega(0)}} e^{\pm i \int_s^t \varphi(\tau) d\tau} P_{\pm}(\omega(0)) \mathcal{R}(s) ds \right\|_{\mathbf{X}} \\ & \leq c_4 \|P_{\pm}(\omega(0)) \mathcal{R}\|_{L_t^1 H_x^1 + L_t^2 H_x^{1,2}} \leq c_5 \|\mathcal{R}\|_{L_t^1 H_x^1 + L_t^2 H_x^{1,2}}. \end{aligned}$$

We now split $\mathcal{R} = \mathcal{R}_1 - \frac{1}{4}|f|^4 \sigma_3 f$. The inequalities of (5.6) yield

$$\begin{aligned} \|\mathcal{R}_1\|_{L_t^2 H_x^{1,2}} & \leq C \|z\|_{L_t^{\infty}} \|z^{N+1}\|_{L_t^2} + \|O_{\text{loc}}(z f_N) + O_{\text{loc}}(f_N^2)\|_{H_x^{1,2}} \\ & \quad + \|\varphi\|_{L_t^{\infty}} \|[P_+(\omega(0)) - P_-(\omega(0)) - \sigma_3 P_c(\omega(0))] f_N\|_{L_t^2 H_x^{1,2}} \\ & \quad + \|(\mathcal{V}_{\omega} - \mathcal{V}_{\omega(0)}) f_N\|_{L_t^2 H_x^{1,2}} \leq C(C_d, C_c) \varepsilon^2 \end{aligned}$$

and

$$\| |f|^4 f \|_{L_t^1 H_x^1} \lesssim \|f_N\|_{W_x^{1,10}} \|f_N\|_{L_x^4}^4 \|f_N\|_{L_t^1} \lesssim \|f_N\|_{L_t^2 W_x^{1,10}}^5 \leq C(C_d, C_c) \varepsilon^5.$$

Then, by Lemma 4.4 (1), we have

$$\begin{aligned} & \left\| \int_0^t e^{-i(t-s)\mathcal{H}_{\omega(0)}} e^{\pm i \int_s^t \varphi(\tau) d\tau} P_{\pm}(\omega(0)) \frac{1}{4} |f|^4 \sigma_3 f ds \right\|_{L_t^2 H_x^{1,-2}} \\ & \leq \int_0^T ds \left\| e^{-i(t-s)\mathcal{H}_{\omega(0)}} e^{\pm i \int_s^t \varphi(\tau) d\tau} P_{\pm}(\omega(0)) \frac{1}{4} |f|^4 \sigma_3 f ds \right\|_{L_t^2 H_x^{1,-2}} \\ & \lesssim \| |f|^4 f \|_{L_t^1 H_x^1} \leq C(C_d, C_c) \varepsilon^5. \end{aligned}$$

We have, by Lemma 4.5,

$$\begin{aligned} & \left\| \int_0^t e^{-i(t-s)\mathcal{H}_{\omega(0)}} e^{\pm i \int_s^t \varphi(\tau) d\tau} P_{\pm}(\omega(0)) \mathcal{R}_1(s) ds \right\|_{L_t^2 H_x^{1,-2}} \\ & \leq C \|\mathcal{R}_1\|_{L_t^2 H_x^{1,2}} \leq C(C_d, C_c) \varepsilon^2. \end{aligned}$$

Thus (5.11) is obtained by fixing κ_0 sufficiently large and ε_0 sufficiently small. \square

Notice that if we pick $C_c = 2\kappa_0(1 + C_d)$, we get that (5.6) implies (5.7). To complete the proof of Lemma 5.2, it remains to show that (5.6) implies (5.8).

5.2. A further change of variable in f . In the argument, there is need for a decomposition of f_N , namely, setting

$$(5.12) \quad f_N = - \sum_{m+n=N+1} z^m \bar{z}^n R_{\mathcal{H}_{\omega(0)}}^+((m-n)\lambda) P_c(\omega(0)) R_{m,n}^{(N)}(\omega) + g,$$

with $R_{\mathcal{H}_{\omega(0)}}^+(\Lambda) = R_{\mathcal{H}_{\omega(0)}}(\Lambda)$ if $|\Lambda| < \omega(0)$. If we eliminate f_N in (5.9) by (5.12),

$$\sum_{m+n=N+1} z^m \bar{z}^n P_c(\omega(0)) R_{m,n}^{(N)}(\omega)$$

in (5.9) cancels out, so that g satisfies an equation of the form

$$\begin{aligned} i \partial_t P_c(\omega(0))g &= (\mathcal{H}_{\omega(0)} + \varphi(P_+(\omega(0)) - P_-(\omega(0))))P_c(\omega(0))g \\ &+ \sum_{\pm} O(\varepsilon|z|^{N+1})R_{\mathcal{H}_{\omega(0)}}^{\pm}(\pm(N+1)\lambda(\omega(0)))R_{\pm} + P_c(\omega(0))\hat{E}_{\text{PDE}}(N), \end{aligned}$$

where

$$R_+ := R_{N+1,0}^{(N)}, \quad R_- := R_{0,N+1}^{(N)} \quad \text{and} \quad \hat{E}_{\text{PDE}}(N) := \tilde{E}_{\text{PDE}}(N) + O_{\text{loc}}(\varepsilon z^{N+1}).$$

This leads us to the following lemma.

Lemma 5.4. *Assume the hypotheses of Lemma 5.2. Then, there exists a fixed $\kappa_1 = \kappa(\omega(0))$ such that for a fixed s sufficiently large,*

$$(5.13) \quad \|g\|_{L_t^2 L_x^{2-s}} \leq \kappa_1 \varepsilon + O(\varepsilon^2).$$

See [Cu3, Lemma 4.6] for the proof.

5.3. Change of ω and z . Consider now the equations of ω and z in (5.2). We have the following lemma.

Lemma 5.5. *There is a change of variables*

$$(5.14) \quad \begin{aligned} \tilde{\omega} &= \omega + q(\omega, z, \bar{z}) + \sum_{1 \leq m+n \leq N} z^m \bar{z}^n \langle f_N, A_{mn}(\omega) \rangle, \\ \zeta &= z + p(\omega, z, \bar{z}) + \sum_{1 \leq m+n \leq N} z^m \bar{z}^n \langle f_N, B_{mn}(\omega) \rangle, \end{aligned}$$

with $p(\omega, z, \bar{z}) = \sum p_{m,n}(\omega) z^m \bar{z}^n$ and $q(\omega, z, \bar{z}) = \sum q_{m,n}(\omega) z^m \bar{z}^n$ polynomials in (z, \bar{z}) with real coefficients, and $O(|z|^2)$ near 0, such that, for $a_m(\omega)$ real, we get

$$(5.15) \quad \begin{aligned} i \dot{\tilde{\omega}} &= \langle E_{\text{PDE}}(N), \Phi \rangle \\ i \dot{\zeta} - \lambda(\omega)\zeta &= \sum_{1 \leq m \leq N} a_m(\omega) |\zeta|^{2m} \zeta + \langle E_{\text{ODE}}(N), \sigma_3 \xi \rangle \\ &+ \tilde{\zeta}^N \langle A_{0,N}^{(N)}(\omega) f_N, \sigma_3 \xi \rangle. \end{aligned}$$

Proof. The proof is elementary and proceeds as follows (see [CM] for details). For $\ell = 0, \dots, 2N$, taking $z_0 = z$ and defining $(x \wedge y := \min\{x, y\})$, we consider recursively the equations

$$(5.16) \quad \begin{aligned} i \dot{z}_\ell - \lambda z_\ell &= \sum_{1 \leq j \leq \ell} a_{j,\ell}(\omega) |z_\ell|^{2j} z_\ell + \sum_{\ell+2 \leq m+n \leq 2N+1} z_\ell^m \bar{z}_\ell^n \alpha_{m,n}^{(\ell)}(\omega) \\ &+ \sum_{(\ell+1) \wedge N \leq m+n \leq N} z_\ell^m \bar{z}_\ell^n \langle A_{m,n}^{(\ell)}(\omega), f_N \rangle + \langle E_{\text{ODE}}(\ell), \sigma_3 \xi \rangle, \end{aligned}$$

with $\alpha_{m,n}^{(\ell)}(\omega) \in \mathbb{R}$ and $A_{m,n}^{(\ell)}(\omega)$ with Schwartz real entries, with smooth dependence in ω . Note that, here, $E_{\text{ODE}}(\ell)$ is like in Section 5.1 in terms of z_0 .

Suppose (5.16) holds for $\ell < 2N$. Then set

$$(5.17) \quad \begin{aligned} z_{\ell+1} &= z_\ell + \sum_{m+n=\ell+2} z_\ell^m \bar{z}_\ell^n \beta_{m,n}^{(\ell)}(\omega) + \sum_{m+n=\ell+1} z_\ell^m \bar{z}_\ell^n \langle B_{m,n}^{(\ell)}(\omega), f_N \rangle, \\ \beta_{m,n}^{(\ell)}(\omega) &:= \frac{\alpha_{m,n}^{(\ell)}(\omega)}{(m-1-n)\lambda} \quad \text{for } m \neq n+1, \quad \beta_{n+1,n}^{(\ell)}(\omega) := 0, \\ B_{m,n}^{(\ell)}(\omega) &:= -R_{\mathcal{H}_\omega^*}((m-1-n)\lambda)A_{m,n}^{(\ell)}(\omega) \quad \text{for } \ell < N, \\ B_{m,n}^{(N)}(\omega) &:= -R_{\mathcal{H}_\omega^*}((m-1-n)\lambda)A_{m,n}^{(\ell)}(\omega) \quad \text{for } (m,n) \neq (0,N), \end{aligned}$$

where $B_{0,N}^{(N)}(\omega) := 0$ and $B_{m,n}^{(\ell)}(\omega) := 0$ for $\ell > N$. The “0 terms” in the above substitution correspond to the *resonant* terms in (5.16). Notice that the only resonant term in the second line of (5.16) is that with $(m,n) = (0,N)$. This is because, by (5.1),

$$m+n \leq N \text{ and } |(m-1-n)\lambda| \geq \omega \implies (m,n) = (0,N).$$

When we compute the derivative $\dot{z}_{\ell+1}$, we see that the nonresonant terms in the second (respectively third) summation with $m+n = \ell+2$ (respectively $m+n = \ell+1$) in (5.16) cancel out because of the last two summations in the definition of $z_{\ell+1}$ in (5.17). We get an equation of the form

$$\begin{aligned} i\dot{z}_{\ell+1} - \lambda z_{\ell+1} &= \sum_{1 \leq j \leq \ell+1} a_{j,\ell}(\omega) |z_\ell|^{2j} z_\ell + \sum_{\ell+2 \leq m+n \leq 2N+1} z_\ell^m \bar{z}_\ell^n \alpha_{m,n}^{(\ell)}(\omega) \\ &+ \sum_{(\ell+2) \wedge N \leq m+n \leq N} z_\ell^m \bar{z}_\ell^n \langle A_{m,n}^{(\ell)}(\omega), f_N \rangle + \langle E_{\text{ODE}}(\ell) + E'_{\text{ODE}}(\ell+1), \sigma_3 \xi \rangle, \end{aligned}$$

where $E'_{\text{ODE}}(\ell+1)$ contains terms from the differentiation of (5.17) not exploited in the cancellation. Thus, $E'_{\text{ODE}}(\ell+1)$ can be absorbed in $E_{\text{ODE}}(\ell+1)$.

When we substitute z_ℓ in the first line of the above equation using (5.17), we get terms as in the first line of (5.16) for $\ell+1$, plus other terms which can be absorbed in $E_{\text{ODE}}(\ell+1)$. Finally, there will be some terms in $E_{\text{ODE}}(\ell)$ which can be expressed as a sum of terms which go into the second and third summations in (5.16) of step $\ell+1$, plus terms which can be absorbed in $E_{\text{ODE}}(\ell+1)$. In this way, we get (5.16) for $\ell+1$. Eventually, for $\zeta = z_{2N}$, we get the desired result. In particular, for $\ell = 2N$, the only nonzero term in the third summation of (5.16) is $\bar{\zeta}^N \langle A_{0,N}^{(2N)}(\omega), f_N \rangle$.

In the final step, we look at ω . We consider the equations for ω and z in (5.2) for $k = N$. For $\ell = 0, \dots, 2N+1$, and now taking $\Omega_0 = \omega$, we have the recursively defined equations

$$(5.18) \quad \begin{aligned} i\dot{\Omega}_\ell &= \sum_{\ell+2 \leq m+n \leq 2N+1} z^m \bar{z}^n \gamma_{m,n}^{(\ell)}(\omega) \\ &+ \sum_{\ell+1 \leq m+n \leq N} z^m \bar{z}^n \langle \Gamma_{m,n}^{(\ell)}(\omega), f_N \rangle + \langle E_{\text{ODE}}(\ell), \Phi \rangle, \end{aligned}$$

with $y_{m,n}^{(\ell)}(\omega) \in \mathbb{R}$ and $\Gamma_{m,n}^{(\ell)}(\omega)$ with Schwartz entries, all with smooth dependence in ω . Suppose this holds for $\ell < 2N + 1$. Then set

$$\begin{aligned}\Omega_{\ell+1} &= \Omega_\ell + \sum_{m+n=\ell+2} z^m \bar{z}^n \delta_{m,n}^{(\ell)}(\omega) + \sum_{m+n=\ell+1} z^m \bar{z}^n \langle \Delta_{m,n}^{(\ell)}(\omega), f_N \rangle, \\ \delta_{m,n}^{(\ell)}(\omega) &:= \frac{y_{m,n}^{(\ell)}(\omega)}{(m-n)\lambda} \quad \text{for } m \neq n, \quad \delta_{n,n}^{(\ell)}(\omega) = 0, \\ \Delta_{m,n}^{(\ell)}(\omega) &= -R_{\mathcal{G}_\omega^*}((m-n)\lambda)\Gamma_{m,n}^{(\ell)}(\omega) \quad \text{for } \ell \leq N,\end{aligned}$$

with $\Delta_{m,n}^{(\ell)}(\omega) = 0$ for $\ell > N$. The procedure is similar to that used for (5.16)—in fact, somewhat simpler, because we are using z rather than z_ℓ . This time, however, there are no resonant terms in the first term of the second line of (5.18), because $m + n \leq N$ implies $|(m-n)\lambda| < \omega$. Notice that the resonant terms in the first summation of (5.18) are those with $m = n$. These, however, are irrelevant because $y_{m,m}^{(\ell)}(\omega) = 0$. This identity follows from the fact, which can be proved by induction, that $y_{m,n}^{(\ell)}(\omega) \in \mathbb{R}$, $\Gamma_{m,n}^{(\ell)}(\omega)$ are real and Ω_ℓ are real valued.

Eventually, for $\tilde{\omega} = \Omega_{2N+1}$, we get the desired result. \square

By (5.6), we have $\|\dot{\tilde{\omega}}\|_{L^1_t} = O(\varepsilon^2)$.

Remark 5.6. Setting $\tilde{\omega}(t) \equiv \tilde{\omega}(0)$, $f_N \equiv 0$, and considering the equation

$$i\dot{\zeta} - \lambda(\omega)\zeta = \sum_{1 \leq m \leq N} a_m(\omega) |\zeta|^{2m} \zeta$$

yield a finite dimensional approximation of the NLSE. We do not check here the time span when the solutions of this approximation are good approximations of solutions of the full NLSE; however, we discuss the dynamics resulting from resonance in the remaining section.

6. THE FERMI GOLDEN RULE

In the equation of ζ , we substitute f_N using (5.12) to get

$$\begin{aligned}i\dot{\zeta} - \lambda(\omega)\zeta &= \sum_{1 \leq m \leq N} a_m(\omega) |\zeta|^{2m} \zeta \\ &\quad - |\zeta|^{2N} \zeta \langle A_{0,N}^{(N)}(\omega) R_{H_{\omega(0)}}^+((N+1)\lambda(\omega(0))) P_C(\omega_0) R_{N+1,0}^{(N)}(\omega), \sigma_3 \xi \rangle \\ &\quad + \tilde{\zeta}^N \langle A_{0,N}^{(N)}(\omega) \mathcal{g}, \sigma_3 \xi \rangle + \langle E_{\text{ODE}}(N), \sigma_3 \xi \rangle,\end{aligned}$$

with $a_m(\omega)$, $A_{0,N}^{(N)}(\omega)$, and $R_{N+1,0}^{(N)}(\omega)$ being real and smoothly dependent on ω , and the latter two with entries which are Schwartz functions. Set

$$\begin{aligned}
& \Gamma(\omega, \omega(0)) \\
&= \operatorname{Im}(\langle A_{0,N}^{(N)}(\omega) R_{\mathcal{H}_{\omega(0)}}^+((N+1)\lambda(\omega(0))) P_c(\omega(0)) R_{N+1,0}^{(N)}(\omega), \sigma_3 \xi(\omega) \rangle) \\
&= \pi \langle A_{0,N}^{(N)}(\omega) \delta(\mathcal{H}_{\omega(0)} - (N+1)\lambda(\omega(0))) P_c(\omega(0)) R_{N+1,0}^{(N)}(\omega), \sigma_3 \xi(\omega) \rangle,
\end{aligned}$$

with the last formula well defined by the *Limiting Absorption Principle* in (4.3).

We assume the following:

(HF) There is a fixed constant $\Gamma > 0$ such that $|\Gamma(\omega, \omega)| > \Gamma$.

Remark 6.1. It is easy to see (e.g., using [Cu4, Corollary 4.7]) that $\Gamma(\omega, \omega) \neq 0$ and orbital stability together imply $\Gamma(\omega, \omega) > 0$. In [Cu1], with generic nonresonance hypotheses and for a broader set of potentials and nonlinearities than the present paper considers, it is proved that $\Gamma(\omega, \omega) \geq 0$, since, in essence,

$$A_{0,N}^{(N)}(\omega) = \sigma_3 R_{N+1,0}^{(N)}(\omega) \sigma_3 \xi(\omega),$$

in a more carefully chosen coordinate system (notice that anything of this sort is by no means obvious in the set-up used here). The key insight in [Cu1] is that the Hamiltonian nature of (1.1) yields the necessary algebraic identities between coefficients of the system. Notice that the square structure of $\Gamma(\omega, \omega)$ was known for $N = 1$ by [BP2]. (Case $N = 2, 3$ was explored in [G1].) Finally, note that we do not need a $\Gamma(\omega, \omega) \geq 0$ here.

By continuity, we can assume $|\Gamma(\omega, \omega(0))| > \Gamma/2$. Then, we write

$$\begin{aligned}
\frac{d}{dt} \frac{|\zeta|^2}{2} &= -\Gamma(\omega, \omega(0)) |\zeta|^{2N+2} + \operatorname{Im}(\langle A_{0,N}^{(N)}(\omega) f_{N+1}, \sigma_3 \xi(\omega) \rangle \bar{\zeta}^{N+1}) \\
&\quad + \operatorname{Im}(\langle E_{\text{ODE}}(N), \sigma_3 \xi(\omega) \rangle \bar{\zeta}).
\end{aligned}$$

For A_0 , an upper bound of the constants $A_0(\omega)$ of Lemma 4.1, we get

$$(6.1) \quad \frac{\Gamma}{2} \|\zeta\|_{L_t^{2N+2}}^{2N+2} \leq A_0 \varepsilon^2 + 2\kappa_1 \varepsilon \|\zeta\|_{L_t^{2N+2}}^{N+1} + o(\varepsilon^2),$$

with κ_1 as in Lemma 5.4. Then we can pick $C_d = 2(A_0 + 2\kappa_1 + 1)/\Gamma$ in Lemma 5.2. So (5.6) implies (6.1), and the latter by (5.14) implies (5.8). In particular, (5.8) and $\|\dot{\omega}\|_{L_t^1} = O(\varepsilon^2)$ hold in $[0, \infty)$. Furthermore, $z(t) \rightarrow 0$ since $\dot{z}(t) = O(\varepsilon)$. Indeed, by $z \in L^{2N+2}(\mathbb{R})$ and by $\dot{z} \in L^\infty(\mathbb{R})$, we get that $z^{2N+3}(t)$ has limit as $t \rightarrow \infty$, and, necessarily, this limit is 0.

By (5.14) and $\dot{\omega} \in L^1(\mathbb{R})$, we conclude that $\omega(t)$ converges to some ω_+ for $t \rightarrow \infty$. Also, more precisely, we have $\|\omega(t) - \omega_+\|_{L_t^\infty} = O(\varepsilon^2)$ in $[0, \infty)$ by $\|\dot{\omega}\|_{L_t^1} = O(\varepsilon^2)$.

We recall that in the literature (e.g., see [BP2, GS]), solutions of (5.15) are displayed such that, approximately,

$$|\zeta(t)| \approx \frac{|\zeta(0)|}{(|\zeta(0)|^{2N} N t + 1)^{1/(2N)}} \quad \text{for } t \ll |\zeta(0)|^{-2N}.$$

APPENDIX A. FINITE DIMENSIONAL DYNAMICS

In [PP], for equations of the form (1.1), the authors describe oscillatory solutions as perturbations of the nonlinear bound states described in Section 2 under certain spectral conditions. Namely, the authors show that solutions exist of the form

$$u(x, t) = e^{i\theta(t)}(\phi(x, \omega(t)) + A(t)\psi(x, \omega(t)) + iB\chi(x, \omega(t))) + \tilde{u}(x, t),$$

where

$$L_+(\omega)\psi = -\Lambda^2\chi, \quad L_-(\omega)\chi = \psi \quad \text{for } \langle \chi, \psi \rangle = 1$$

and

$$\Lambda^2 = -\lambda'(\omega_*)\|\phi_*\|_{L^2}^2(\omega - \omega_*) + \mathcal{O}(\omega - \omega_*)^2,$$

and where λ is like μ_1 in the discussion above. Specifically, the existence of dynamics similar to those in Figure A.1 is related to the following quantities:

$$N(\omega) = \|\phi_\omega\|_{L^2}^2,$$

$$\Delta N = N_0 - N(\omega_*),$$

$$L_{+, \omega}\varphi = \lambda(\omega)\varphi, \quad \text{for } \psi \text{ the anti-symmetric,}$$

λ the second eigenvalue of $L_{+, \omega}$,

$$\mathcal{Q} = 200\langle \phi_*^3 \psi_*^2, L_{+, \omega_*}^{-1} \phi_*^3 \psi_*^2 \rangle + 10\langle \psi_*^2, \phi_*^2 \psi_*^2 \rangle,$$

$$S = N'(\omega_*) + \mathcal{Q}^{-1}(\lambda'(\omega_*)\|\psi_*\|_{L^2}^2).$$

Specifically, [PP] derive the pendulum-like dynamical system

$$\dot{A} = B,$$

$$\dot{B} = \frac{\lambda'(\omega_*)\|\psi_*\|_{L^2}^2(\Delta N)A + \mathcal{Q}SA^3}{N'(\omega_*)}$$

under the assumptions that we put on the double well in the [Introduction](#). In this case, there exists a bifurcation point ω_* , and

$$\lambda'(\omega_*) < 0, \quad N'(\omega_*) > 0, \quad \mathcal{Q} > 0.$$

Each of these quantities can be computed for a double well potential of the form V_L as $L \rightarrow \infty$ using the analysis in [KKP], where indeed some numerical studies of the bifurcation curve for various p in (1.1) are carried out in detail.

We wish to partially motivate our discussion in the [Introduction](#) related to periodic solutions in [MW] by numerically observing periodic orbits in finite dimensional dynamics for $p = 5$. Such orbits are only expected to persist on finite

time scales because of the nonlinear coupling-to-radiation property we observe in these results for small perturbations of the soliton. However, further investigations of damping and possible dynamics in the quintic NLSE should include larger deviations from the nonlinear bound states, such as those studied in [MW]. The phase plane diagrams in A.1 for the finite dimensional dynamics are plotted using the MATLAB software program *pplane7* (see [AP]). We may numerically solve the PDE system (1.1) with initial data corresponding to that necessary for the three types of oscillation described in Figure A.1; see [MW] for details. Though, as mentioned in the Introduction, this differs from the asymptotic stability result presented here, in the interest of motivating future analysis for such a problem, we plug the ansatz

$$u(x, t) = c_0(t)\psi_0 + c_1(t)\psi_1 + R(x, t)$$

into (1.1), where $H\psi_j = (-\partial_x^2 + V)\psi_j = -\omega_j\psi_j$ for $j = 0, 1$. As a result, we have the equation

$$\begin{aligned} i\dot{c}_0\psi_0 + i\dot{c}_1\psi_1 + iR_t(x, t) &= -\omega_0c_0\psi_0 - \omega_1c_1\psi_1 + HR \\ &\quad - |c_0\psi_0 + c_1\psi_1 + R|^4(c_0\psi_0 + c_1\psi_1 + R). \end{aligned}$$

The nonlinear contribution is then given by

$$\begin{aligned} &(c_0\psi_0 + c_1\psi_1)^3(\bar{c}_0\psi_0 + \bar{c}_1\psi_1)^2 + \mathcal{O}(R) \\ &= (c_0^3\psi_0^3 + 3c_0^2c_1\psi_0^2\psi_1 + 3c_0c_1^2\psi_0\psi_1^2 + c_1^3\psi_1^3) \\ &\quad \times (\bar{c}_0^2\psi_0^2 + 2\bar{c}_0\bar{c}_1\psi_0\psi_1 + \bar{c}_1^2\psi_1^2) + \mathcal{O}(R) \\ &= (c_0^3\bar{c}_0^2)\psi_0^5 + (3c_0^2\bar{c}_0^2c_1 + 2c_0^3\bar{c}_0\bar{c}_1)\psi_0^4\psi_1 \\ &\quad + (3c_0\bar{c}_0^2c_1^2 + 6c_0^2\bar{c}_0c_1\bar{c}_1 + c_0^3\bar{c}_1^2)\psi_0^3\psi_1^2 \\ &\quad + (\bar{c}_0^2c_1^3 + 6c_0\bar{c}_0c_1^2\bar{c}_1 + 3c_0^2c_1\bar{c}_1^2)\psi_0^2\psi_1^3 \\ &\quad + (2\bar{c}_0c_1^3\bar{c}_1 + 3c_0c_1^2\bar{c}_1^2)\psi_0\psi_1^4 + (c_1^3\bar{c}_1^2)\psi_1^5 + \mathcal{O}(R). \end{aligned}$$

Projecting onto ψ_0 and ψ_1 , respectively, and ignoring (for now) components with dependence upon R (see [MW]), we arrive at the finite dimensional Hamiltonian system of equations given by

$$\begin{aligned} i\dot{\rho}_0\langle\psi_0, \psi_0\rangle &= -\omega_0\rho_0\langle\psi_0, \psi_0\rangle - \rho_0^3\bar{\rho}_0^2\langle\psi_0^5, \psi_0\rangle \\ &\quad - (3\rho_0\bar{\rho}_0^2\rho_1^2 + 6\rho_0^2\bar{\rho}_0\rho_1\bar{\rho}_1 - \rho_0^3\bar{\rho}_1^2)\langle\psi_0^4, \psi_1^2\rangle \\ &\quad - (2\bar{\rho}_0\rho_1^3\bar{\rho}_1 + 3\rho_0\rho_1^2\bar{\rho}_1^2)\langle\psi_0^2, \psi_1^4\rangle, \\ i\dot{\rho}_1\langle\psi_1, \psi_1\rangle &= -\omega_1\rho_1\langle\psi_1, \psi_1\rangle - \rho_1^3\bar{\rho}_1^2\langle\psi_1^5, \psi_1\rangle \\ &\quad - (3\rho_0^2\bar{\rho}_0^2\rho_1 + 2\rho_0^3\bar{\rho}_0\bar{\rho}_1)\langle\psi_0^4, \psi_1^2\rangle \\ &\quad - (6\rho_0\bar{\rho}_0\rho_1^2\bar{\rho}_1 + 3\rho_0^2\rho_1\bar{\rho}_1^2 + \bar{\rho}_0^2\rho_1^3)\langle\psi_0^2, \psi_1^4\rangle \end{aligned}$$

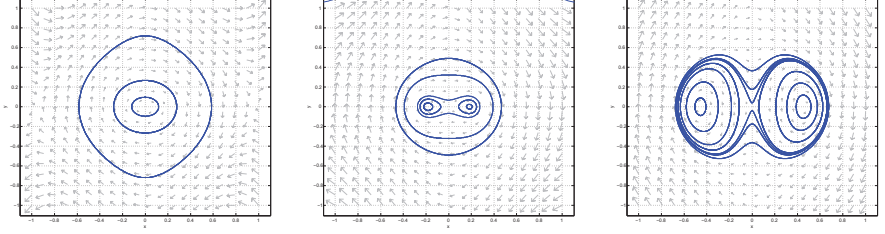


FIGURE A.1. The symmetric state for $N - N_{cr}^{FD} < 0$ corresponds to an equilibrium elliptic point $(\alpha, \beta) = (0, 0)$, and asymmetric states for $N - N_{cr}^{FD} > 0$ correlate to equilibrium points at $(\alpha, \beta) = (\pm\alpha_{cr}, 0)$. The plotted phase portrait corresponds to parameter values $\omega_0 - \omega_1 = .1$ and $N = .1, .2, .5$, respectively.

and the corresponding conjugate equations. Note that mass is conserved in this finite dimensional system; hence, we have $|\rho_0|^2 + |\rho_1|^2 = N$, for all t .

Plugging in alternative coordinates designed to give rise to a simple classification of the finite dimensional dynamics, we set $\rho_0(t) = A(t)e^{i\theta(t)}$ and $\rho_1(t) = (\alpha(t) + i\beta(t))e^{i\theta(t)}$. As a result, we have the following

$$\begin{aligned} i\dot{A} - A\dot{\theta} &= -\omega_0 A - A^5 - 3A^3(\alpha + i\beta)^2 - 6A^3(\alpha^2 + \beta^2) \\ &\quad - 2A(\alpha + i\beta)^2(\alpha^2 + \beta^2) - A^3(\alpha - i\beta)^2 - 3A(\alpha^2 + \beta^2)^2; \\ i(\dot{\alpha} + i\dot{\beta}) - (\alpha + i\beta)\dot{\theta} &= -\omega_1(\alpha + i\beta) - 3A^2(\alpha^2 + \beta^2)(\alpha - i\beta) - A^2(\alpha + i\beta)^3 - 2A^4(\alpha - i\beta) \\ &\quad - 6A^2(\alpha^2 + \beta^2)(\alpha + i\beta) - 3A^4(\alpha + i\beta) + (\alpha^2 + \beta^2)^2(\alpha + i\beta), \end{aligned}$$

where, for simplicity, we set $\langle \psi_0^{2j}, \psi_1^{6-2j} \rangle = 1$ for all $j = 0, 1, 2, 3$. This will simply rescale the dynamical system and not impact the general shape of the phase diagram.

In the end, we have

$$\begin{aligned} \dot{\alpha} &= (\omega_0 - \omega_1 + 4A^2\alpha^2 + 2(\alpha^2 + \beta^2)^2 + 2(\alpha^2 + \beta^2)(\alpha^2 - \beta^2))\beta, \\ \dot{\beta} &= -(\omega_0 - \omega_1 - 4A^4 - 4A^2\beta^2 + A^2\alpha^2 + 2(\alpha^2 + \beta^2)^2 \\ &\quad + 2(\alpha^2 + \beta^2)(\alpha^2 - \beta^2))\alpha, \\ \dot{A} &= -4A(A^2 + (\alpha^2 + \beta^2))\alpha\beta, \\ \dot{\theta} &= \omega_0 + A^4 + 10A^2\alpha^2 + 2A^2\beta^2 + 2(\alpha^2 + \beta^2)(\alpha^2 - \beta^2) + 3(\alpha^2 + \beta^2)^2, \end{aligned}$$

where $N = A^2 + \alpha^2 + \beta^2$.

Using the mass conservation, we can write a closed system for (α, β) . From the equation for β , it is clear that in this rescaled dynamical system, we have

$$N_{cr}^{FD} = \left(\frac{\omega_0 - \omega_1}{4} \right)^{1/4}.$$

We observe in Figure A.1 several phase diagrams for varying values of N , which point out the existence of periodic solutions above, near, and below the bifurcation point. It is our goal in this section purely to give further evidence that the quintic NLSE with double well potential presents similar dynamics to that of the cubic NLSE with double well potential. For a dynamics approach to classifying these solutions and studying their stability properties, we refer to the finite dimensional results in [MW] for techniques which directly apply to reducible Hamiltonian systems of this type, particularly for the proof of existence of periodic orbits and the resulting Floquet stability analysis. However, as the intent of this note is to prove asymptotic stability, we do not explore this topic further here.

Acknowledgments. J. L. M. is partially supported by an IBM Junior Faculty Development Grant at the University of North Carolina. The authors wish to thank Michael Weinstein for helpful conversations, as well as the anonymous reviewer for a careful reading of the draft and helpful suggestions. In addition, the authors would like to thank the Mathematisches Forschungsinstitut Oberwolfach for graciously hosting them during the start of this research.

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Received: September 4, 2011.