COUNTING NUMERICAL SETS WITH NO SMALL ATOMS

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Abstract. A numerical set S with Frobenius number g is a set of integers with $\min(S) = 0$ and $\max(\mathbb{Z} - S) = g$, and its atom monoid is $A(S) = \{n \in \mathbb{Z} \mid n + s \in S \text{ for all } s \in S\}$. Let γ_g be the ratio of the number of numerical sets S having $A(S) = \{0\} \cup (g, \infty)$ divided by the total number of numerical sets with Frobenius number g. We show that the sequence $\{\gamma_g\}$ is decreasing and converges to a number $\gamma_\infty \approx .4844$ (with accuracy to within .0050). We also examine the singularities of the generating function for $\{\gamma_g\}$. Parallel results are obtained for the ratio γ_g^σ of the number of symmetric numerical sets S with $A(S) = \{0\} \cup (g, \infty)$ by the number of symmetric numerical sets with Frobenius number g. These results yield information regarding the asymptotic behavior of the number of finite additive 2-bases.

Introduction

Let $\mathbb{Z}$ denote the additive group of integers and let $\mathbb{N}$ denote the monoid of nonnegative integers. Both of these sets are linearly ordered by the Archimedean ordering and we will use standard interval notation to describe their convex subsets. If $n \in \mathbb{Z}$ and $S \subseteq \mathbb{Z}$ then the translate of S by n is the set $n + S = \{n + s \mid s \in S\}$.

A *numerical set* S is a cofinite subset of $\mathbb{N}$ which contains 0, and its *Frobenius number* is the maximal element in the complement $\mathbb{N} - S$.[1] Equivalently, a numerical set S with Frobenius number g is a set of integers with $\min(S) = 0$ and $\max(\mathbb{Z} - S) = g$. A numerical set which is closed under addition is called a *numerical monoid*. Every numerical set S has an associated *atom monoid* $A(S)$ defined by

$$A(S) = \{n \in \mathbb{Z} \mid n + S \subseteq S\},$$

and this is easily seen to be a numerical monoid with the same Frobenius number as S. Note that $A(S) \subseteq S$ and that S is a numerical monoid if and only if $A(S) = S$. The nonzero elements of $A(S)$ are referred to as the *atoms of* S. A small atom is an atom of S which is less than the Frobenius number. Among other uses, atoms provide basic building blocks for efficiently generating numerical sets. For example, every numerical set S can be uniquely described as $\Sigma(S) + A(S)$ where $\Sigma(S) = \{s \in S \mid \textit{if } a \in A(S) \textit{ and } a \neq 0 \textit{ then } s - a \notin S\}$ [AM].

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[1] This definition differs from that employed in [AM] where a 'numerical set' would be a translate $n + S$ of a numerical set S (in the sense given here) by an arbitrary integer n. Since the atom monoid of $n + S$ equals the atom monoid of S, this variation of the definition should not lead to any confusion.

For each $g \geq 0$ let $\mathbb{N}_g$ be the numerical monoid

$$\mathbb{N}_g = \mathbb{N} - [1, g] = \{0\} \cup (g, \infty) \ ,$$

which has Frobenius number g when $g > 0$.[2] The atom monoid of every numerical set S with Frobenius number g contains $\mathbb{N}_g$ and the complement $S - \mathbb{N}_g$ is a subset of $(0, g)$. Conversely, the union of $\mathbb{N}_g$ with any subset of $(0, g)$ is a numerical set with Frobenius number g. Therefore the set

$$\mathcal{S}(g) = \{S \subseteq \mathbb{N} \mid S \text{ is a numerical set with Frobenius number } g\}$$

is in one-to-one correspondence with the power set $\mathcal{P}(0, g)$ (which consists of all subsets of $(0, g)$), and $\mathcal{S}(g)$ has cardinality 2^{g-1}. The collection of numerical monoids in $\mathcal{S}(g)$ is a much more difficult set to enumerate. This is examined in Backelin's paper [B] where it is shown that for large values of g roughly $3 \times 2^{\lfloor (g-1)/2 \rfloor}$ of the 2^{g-1} elements of $\mathcal{S}(g)$ are numerical monoids.

If $M \in \mathcal{S}(g)$ is a numerical monoid then the *anti-atom set of* M is the set

$$\mathcal{G}(M) \ = \ \{S \in \mathcal{S}(g) \mid A(S) = M\} \, .$$

This is contained in the larger set $\mathcal{S}(M) = \{S \in \mathcal{S}(g) \mid M \subseteq A(S)\}$ whose elements might be considered to be 'M-modules'.[3] Notice that $\mathcal{S}(g) = \mathcal{S}(\mathbb{N}_g)$ and we will also write $\mathcal{G}(g) = \mathcal{G}(\mathbb{N}_g)$. This paper is motivated by the following question which we shall refer to as the *Anti-Atom Problem*.

For a given numerical monoid M with Frobenius number g
how many numerical sets in $\mathcal{S}(g)$ have atom monoid M?

Thus, for a given monoid M, the Anti-Atom Problem asks to determine the cardinality of $\mathcal{G}(M)$. This problem is certainly unwieldy given that it fundamentally presupposes an enumeration of the set of numerical monoids in $\mathcal{S}(g)$—an enumeration which Backelin has shown to be intractable at best. Nevertheless we will be able to frame aspects of the problem in a clearer light. Our main result will show that there is one monoid M in $\mathcal{S}(g)$ (that monoid being $M = \mathbb{N}_g$) which itself is the atom monoid for approximately 48.4% of all numerical sets in $\mathcal{S}(g)$ for large values of g. In order to describe this in more depth we first need to discuss symmetry and pseudosymmetry in numerical sets. These concepts are important throughout much of the theory of numerical monoids and numerical sets (see [FGH], [AM] and [A] for example), and will play a role in many of our discussions.

A numerical set $S \in \mathcal{S}(g)$ is *symmetric* if an integer x is an element of S if and only if $g - x$ is not an element of S. In other words, S is symmetric when the reflection on $\mathbb{Z}$ given by $x \mapsto g - x$ carries S onto its complement $\mathbb{Z} - S$. Notice that only numerical sets with odd Frobenius number can be symmetric. A numerical set with even Frobenius number g is said to be *pseudosymmetric* if $g/2 \notin S$ and for each integer $x \neq g/2$, x is an element of S if and only if $g - x$ is not an element of S. Symmetry and pseudosymmetry can also be described using the notion of duality of numerical sets. If $S \in \mathcal{S}(g)$ then the *dual of* S

[2]The Frobenius number of $\mathbb{N}_0 = \mathbb{N}$ is -1, and this is the only numerical set with nonpositive Frobenius number.

[3]In [BF] the elements of $\mathcal{S}(M)$ are called 'relative ideals over M'.

is the numerical set $S^* = \{n \in \mathbb{Z} \mid g - n \notin S\}$, and it is not hard to show that $S^* \in \mathcal{S}(g)$ and that $A(S^*) = A(S)$ (more background can be found in section 1 of [AM]). The numerical set S is symmetric if and only if $S^* = S$, and it is pseudosymmetric if and only if g is even and $S^* = S \cup \{g/2\}$.[4] For each numerical set $S \in \mathcal{S}(g)$ there is a rational number called the 'type of S' and denoted by $type(S)$ which is no smaller than one and satisfies the property that S is symmetric if and only if $type(S) = 1$. This concept was described for numerical monoids in [FGH] and extended to numerical sets in [AM]. The type of a numerical monoid $M \in \mathcal{S}(g)$ is always an integer, and it can be shown to equal the cardinality of the set $\mathcal{O}(M) = \big\{n \in \mathbb{Z} - M \mid n + \big(M - \{0\}\big) \subseteq M\big\}$. Since $g \in \mathcal{O}(M) \subset \mathbb{N}$, the type of a numerical monoid $M \in \mathcal{S}(g)$ is an element of the $[1, g]$, and the largest possible value $type(M) = g$ is only achieved when $M = \mathbb{N}_g$. The following elementary results allow us to solve the Anti-Atom Problem for symmetric and pseudosymmetric numerical monoids.

Proposition 1. *Suppose that M is a numerical monoid and that S is a numerical set with $A(S) = M$. Then $M \subseteq S \subseteq M^*$.*

Proof. Let S be a numerical set in $\mathcal{S}(g)$ with $A(S) = M$ and $s \in S$. If $g - s$ were an element of M then $g = s + (g - s)$ would be an element of S, which contradicts g being the Frobenius number of S. Thus $g - s \notin M$ which implies that $s \in M^*$, and $M = A(S) \subseteq S \subseteq M^*$. □

Corollary 2. *A numerical monoid $M \in \mathcal{S}(g)$ is symmetric if and only if there is just one numerical set (which must be M itself) whose atom monoid is M. If M is a pseudosymmetric numerical monoid then there are precisely two numerical sets (which must be M and M^*) whose atom monoid is M.*

Proof. Let $M \in \mathcal{S}(g)$ be a monoid. If M is not symmetric then $M \neq M^*$ but $A(M^*) = A(M) = M$, and so there are at least two distinct numerical sets in $\mathcal{G}(M)$. On the other hand, if M is symmetric and $S \in \mathcal{G}(M)$ then $M \subseteq S \subseteq M^* = M$ and $S = M$. If M is pseudosymmetric and $S \in \mathcal{G}(M)$ then $M \subseteq S \subseteq M^* = M \cup \{g/2\}$, so that S equals M or M^*. □

This corollary then provides the first positive answers to the Anti-Atom Problem: namely, that $|\mathcal{G}(M)| = 1$ when M is symmetric and that $|\mathcal{G}(M)| = 2$ when M is pseudosymmetric.[5] At the other end of the spectrum, we shall show that the anti-atom set of $\mathbb{N}_g$ (which is the numerical monoid in $\mathcal{S}(g)$ farthest removed from being symmetric, since $type(\mathbb{N}_g) = g$ is the largest possible type among all monoids in $\mathcal{S}(g)$) is an order of magnitude larger in size than that of any other numerical monoid with Frobenius number g. To establish this we will examine the sequence

[4]More generally, if the symmetric difference of S and S^* contains no more than one element then S is symmetric, pseudosymmetric or "dually pseudosymmetric" (meaning that S^* is pseudosymmetric).

[5]We showed that $|\mathcal{G}(M)| = 1$ if and only if M is symmetric, but it is not hard to construct numerical monoids M with $|\mathcal{G}(M)| = 2$ that are not pseudosymmetric.

$\gamma_g = |\mathcal{G}(g)|/|\mathcal{S}(g)|$. We introduce a combinatorially defined sequence of positive integers $\{A_k\}$ with the property that $1-\gamma_g$ is a partial sum of the convergent infinite series $\sum_{k=1}^{\infty} A_k 4^{-k}$. This allows us to show that $\{\gamma_g\}$ is a decreasing convergent sequence and that its limit γ_∞ is approximately equal to .484451, give or take .0050. An examination of the singularities of the generating function $\sum_{k=1}^{\infty} A_k z^k$ provides more detailed asymptotic information about $\{\gamma_g\}$.

In addition to forming a large subset of $\mathcal{S}(g)$, the numerical sets in $\mathcal{G}(g)$ have nice properties in terms of the direct sum decompositions discussed in [AM]. Given numerical sets S and T and relatively prime atoms $a \in A(S)$ and $b \in A(T)$ the *direct sum of* S *and* T is the numerical set $bS \oplus aT = \{bs + at \mid s \in S \text{ and } t \in T\}$. A numerical set S can always be trivially decomposed as $S = 1S \oplus a\mathbb{N}$ for any nonzero $a \in A(S)$, but if this is the only kind of direct sum decomposition of S then we say that S is *irreducible*. Every numerical set can be expressed as a finite direct sum of irreducibles. By [AM, Proposition 4.4], the only numerical set in $\bigcup\{\mathcal{G}(g) \mid g \geq 1\}$ which is not irreducible is $\mathbb{N}_1 = 2\mathbb{N} \oplus 3\mathbb{N}$. Thus our results show that at least 47.94% of all numerical sets in $\mathcal{S}(g)$ are irreducible. Another nice property is that the type function is multiplicative when restricted to $\bigcup\{\mathcal{G}(g) \mid g \geq 1\}$ by [AM, Proposition 5.3] (that is, the type of a direct sum is the product of the types of its factors, if the factors have no small atoms). Multiplicativity of type was a central theme in [AM]. We also mention that when a numerical set S is in $\mathcal{G}(g)$ its type can be computed via the formula

$$type(S) = \frac{|S \cap [0,g)|\,|S^* \cap [0,g)|}{|S \cap S^* \cap [0,g)|^2},$$

which is readily derived from the general formula for the type of an arbitrary numerical set given in [AM].

Let M be a numerical monoid and let k be a field. The 'semigroup ring' $R(M) = k[[t^M]]$ which consists of all formal power series $\sum_{m\in M} a_m t^m$ over k is a one dimensional complete Noetherian local domain with integral closure $k[[t]]$, and it is a Cohen-Macaulay ring. If S is a numerical set whose atom monoid is M then $k[[t^S]] = \left\{\sum_{s\in S} a_s t^s \mid a_s \in k\right\}$ is a finite-dimensional $R(M)$-module. This provides a connection between the anti-atom set $\mathcal{G}(M)$ of M and the family of Cohen-Macaulay modules over the domain $R(M)$. For example, the canonical module of the Cohen-Macaulay ring $R(M)$ is the module $k[[t^{M^*}]]$ associated with the numerical set M^* dual to M; in particular, $R(M)$ is a Gorenstein ring precisely when $R(M) = k[[t^{M^*}]]$, that is, when M is a symmetric monoid (see [BH] and discussions in [AM]). More detailed related information about the correspondence between numerical monoids and one dimensional analytically irreducible Noetherian local domains can be found in [K], [BF], or [BDF]. The book [A] describes a variety of other settings in which numerical monoids arise.

Outline of results

To give a basic overview of the paper, the main focus is to enumerate the two sets

$$\mathcal{G}(g) \;=\; \{S \in \mathcal{S}(g) \mid A(S) = \mathbb{N}_g\}$$

and

$$\mathcal{G}^\sigma(g) \;=\; \{S \in \mathcal{G}(g) \mid S \text{ is symmetric}\} \;=\; \mathcal{G}(g) \cap \mathcal{S}^\sigma(g),$$

where $\mathcal{S}(g)$ is the collection of all numerical sets with $g(S) = g$ and $\mathcal{S}^\sigma(g)$ is the subset of $\mathcal{S}(g)$ consisting of symmetric numerical sets. Our study of $\mathcal{G}^\sigma(g)$ is suggested and motivated by Backelin's examination of the number of symmetric numerical monoids in $\mathcal{S}(g)$ [B]. We shall obtain information about both sets $\mathcal{G}(g)$ and $\mathcal{G}^\sigma(g)$ by employing and comparing two essentially different approaches.

In the first approach, we describe a natural partition of the complementary sets $\mathcal{B}(g) = \{S \in \mathcal{S}(g) \mid A(S) \neq \mathbb{N}_g\}$ and $\mathcal{B}^\sigma(g) = \{S \in \mathcal{S}^\sigma(g) \mid A(S) \neq \mathbb{N}_g\}$, and use this to show that the sequences $\gamma_g = |\mathcal{G}(g)|/|\mathcal{S}(g)|$ and $\gamma_g^\sigma = |\mathcal{G}^\sigma(g)|/|\mathcal{S}^\sigma(g)|$ are bounded and decreasing. We also obtain representations of their respective limits γ_∞ and γ_∞^σ as sums of positive infinite series (see the discussions of corollaries 6 and 16). In the case of $\mathcal{G}(g)$ and γ_∞, this involves analyzing the integral sequence A_k described above, and leads to the approximation $\gamma_\infty \approx .484451$ (see table 1). For $\mathcal{G}^\sigma(g)$ and γ_g^σ, a similar approach produces an integer sequence A_k^σ and the approximation $\gamma_\infty^\sigma \approx .230653$ (see table 2). The computer routines that we have used to generate these estimates are quite tedious and it does not appear likely that there is a polynomial time algorithm for them. We have posted Fortran codes for the routines in the descriptions of sequences A164047 and A164048 at the web site [S].

The second approach that we employ to study $\mathcal{G}(g)$ and $\mathcal{G}^\sigma(g)$ involves a more direct examination via one-to-one correspondences between them and the sets

$$\mathcal{A}(g)' \;=\; \{L \subseteq (0,g) \mid \forall x \in L,\ \exists y \in L \text{ s.t. } x+y \in (0,g] - L\}$$

and

$$\mathcal{A}^\sigma(g)' \;=\; \{L \in \mathcal{A}(g)' \mid \text{if } x \in (0,g) \text{ then } |L \cap \{x, g-x\}| = 1\}$$

respectively. The cardinalities $A_g' = |\mathcal{A}(g)'|$ and $A_g^{\sigma\prime} = |\mathcal{A}^\sigma(g)'|$ of these two sets are recursively related to the integer sequences A_k and A_k^σ (theorems 11 and 19). From the descriptions of $\mathcal{A}(g)'$ and $\mathcal{A}^\sigma(g)'$ we obtain information about the generating functions $f(z) = \sum_{g=1}^\infty A_g' z^g$ and $f^\sigma(z) = \sum_{g=1}^\infty A_g^{\sigma\prime} z^g$. The descriptions also lead into the construction of two rooted trees which encode the relationships between $\mathcal{G}(2k-1)$ and $\mathcal{G}(2k+1)$, and between $\mathcal{G}^\sigma(2k-1)$ and $\mathcal{G}^\sigma(2k+1)$ (see figures 4 and 5). In the rooted tree for $\cup\mathcal{G}(2k+1)$, the vertices at height k correspond with the elements of $\mathcal{G}(2k+1)$, and each vertex spawns either three or four adjacent vertices below it on the tree. In the rooted tree for $\cup\mathcal{G}^\sigma(2g+1)$ the vertices at height k correspond with the elements of $\mathcal{G}^\sigma(2k+1)$ and each vertex spawns either one or two adjacent vertices below it on the graph (but a pattern of which spawn one vertex and which spawn two is not easy to decipher). We will also show that when g is odd the set $\mathcal{A}^\sigma(g)'$ is in one-to-one correspondence with the collection of subsets of $[0,k)$ which form additive two-bases of $k = (g-1)/2$.[6] Our results on the generating function $f^\sigma(z)$ provide asymptotic information about the number of such additive two bases for increasing values of k.

[6] An additive two-basis of k is a set of integers A such that every element of $[0,k)$ is the sum of two (not necessarily distinct) elements of A.

Outside of the introduction and the present outline of results, the paper is structured into four sections. The initial two sections are entitled '*Numerical sets with no small atoms*' and '*The generating function for* $\{\gamma_k\}$'. They examine $\mathcal{G}(g)$ and are roughly broken in two parts according to the two different approaches discussed above. Some specific additional information addressing the Anti-Atom Problem is obtained at the end of the first section. The final two sections are entitled '*Symmetric numerical sets with no small atoms*' and '*The generating function for* $\{\gamma_k^\sigma\}$'. They examine $\mathcal{G}^\sigma(g)$ and again are broken apart according to the two approaches. The last section includes a discussion of the connection with additive 2-bases.

Numerical sets with no small atoms

Let S be a numerical set with Frobenius number g. A *small atom* for S is a (nonzero) atom for S which is less than g.

Lemma 3. *Let S be a numerical set in $\mathcal{S}(g)$. If S has a small atom then S has a small atom larger than $g/2$.*

Proof. If g is even then $g/2$ is not an atom of S since $g/2 + g/2 \notin S$. Suppose S has an atom less than $g/2$ and let k be the largest such atom. Then $2k$ is a small atom of S, and $2k$ is greater than $g/2$ by the choice of k. □

The set $\mathcal{S}(g)$ is partitioned into two subsets

$$\mathcal{G}(g) \;=\; \mathcal{G}(\mathbb{N}_g) \;=\; \{S \in \mathcal{S}(g) \mid S \text{ has no small atoms}\}$$

and

$$\mathcal{B}(g) \;=\; \{S \in \mathcal{S}(g) \mid S \text{ has at least one small atom}\}\ .$$

For each $g > 0$, $\mathbb{N}_g \in \mathcal{G}(g)$ and $\mathcal{G}(g)$ is nonempty. On the other hand, $\mathcal{B}(g)$ contains all of the numerical monoids in $\mathcal{S}(g)$ other than $\mathbb{N}_g$, and $\mathcal{B}(g)$ is nonempty when $g > 2$. We are interested in the two ratios

$$\beta_g = \frac{|\mathcal{B}(g)|}{|\mathcal{S}(g)|} = \frac{|\mathcal{B}(g)|}{2^{g-1}}$$

and

$$\gamma_g = \frac{|\mathcal{G}(g)|}{|\mathcal{S}(g)|} = \frac{|\mathcal{G}(g)|}{2^{g-1}}.$$

Observe that $0 \le \beta_g, \gamma_g \le 1$ and that $\beta_g + \gamma_g = 1$.

For each $S \in \mathcal{S}(2n-1)$ and $\epsilon \in \mathbb{Z}_2 = \{0, 1\}$ we define

$$S'_\epsilon = \big(S \cap [0, n-1]\big) \cup \{\epsilon\, n\} \cup \big(1 + S \cap [n, \infty)\big).$$

Lemma 4. *The correspondence $(S, \epsilon) \mapsto S'_\epsilon$ is a bijection from $\mathcal{S}(2n-1) \times \mathbb{Z}_2$ to $\mathcal{S}(2n)$ which carries $\mathcal{G}(2n-1) \times \mathbb{Z}_2$ onto $\mathcal{G}(2n)$. Furthermore, $\gamma_{2n} = \gamma_{2n-1}$ and $\beta_{2n} = \beta_{2n-1}$.*

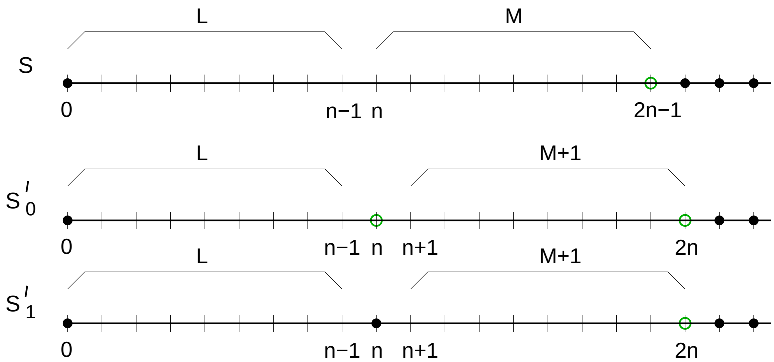

FIGURE 1. *The numerical sets S'_0 and S'_1*

Proof. The correspondence $(S, \epsilon) \mapsto S'_\epsilon$ is injective by definition, and it is also surjective: if $S' \in \mathcal{S}(2n)$ then $S' = S'_\epsilon$ where S is the union of $S' \cap [0, n-1]$ and $-1 + (S \cap [n+1, \infty))$, and ϵ equals 0 if $n \notin S'$ and 1 if $n \in S'$.

It is not difficult to see that an integer x is a small atom for S with $x > g(S)/2 = n - 1/2$ if and only if $1 + x$ is a small atom for S'_ϵ with $1 + x > g(S'_\epsilon)/2 = n$. By lemma 3 this implies that a numerical set $S \in \mathcal{S}(2n-1)$ is in $\mathcal{G}(2n-1)$ if and only if S'_ϵ is in $\mathcal{G}(2n)$. To complete the proof, we note that $\gamma_{2n} = |\mathcal{G}(2n)|/2^{2n-1} = 2|\mathcal{G}(2n-1)|/2^{2n-1} = \gamma_{2n-1}$. □

For integers g and k with $g > k > 0$, let

$$\mathcal{B}(g, k) \;=\; \{S \in \mathcal{S}(g) \mid g - k \text{ is the largest small atom of } S\} \;.$$

Note that $\mathcal{B}(g, k)$ is a subset of $\mathcal{B}(g)$ and that $\mathcal{B}(g, k)$ is empty whenever $k \geq g/2$ by lemma 3. In order to describe $\mathcal{B}(g, k)$ we are led to the next definition. An ordered pair (L, M) of subsets of $(0, k)$ is *admissible* if it satisfies two conditions:

(ad1) $L \subseteq M$, and
(ad2) for every $x \in M$ there exists $y \in L$ with $x + y \leq k$ and $x + y \notin M$.

Let $\mathcal{A}(k)$ be the set of all admissible pairs of subsets of $(0, k)$, and let $A_k = |\mathcal{A}(k)|$ denote the cardinality of this set. The power set $\mathcal{P}(k, g-k)$ of the set $(k, g-k)$ consists of all subsets of $(k, g-k)$ and has cardinality 2^{g-2k-1}.

Theorem 5. *For integers g and k with $g > 2k > 0$ the set $\mathcal{B}(g, k)$ is in one-to-one correspondence with $\mathcal{A}(k) \times \mathcal{P}(k, g-k)$. In particular, the cardinality of $\mathcal{B}(g, k)$ equals $A_k\, 2^{g-2k-1}$.*

Proof. Suppose that $(L, M) \in \mathcal{A}(k)$ and $P \in \mathcal{P}(k, g-k)$. Then

$$S(L, M, P) = \mathbb{N}_g \cup L \cup P \cup \{g - k\} \cup \big(g - k + M\big) \qquad (1)$$

is a numerical set with Frobenius number g. (See figure 2.) Since $g - k + L \subseteq g - k + M$ by (ad1) and each nonzero element of $S(L, M, P) - L$ is larger than k, $g - k$ is a small atom for $S(L, M, P)$. Suppose that $x \in (g - k, g) \cap S(L, M, P)$.

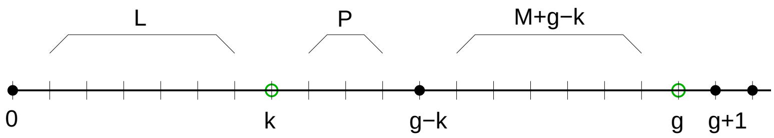

FIGURE 2. *The numerical set* $S(L,M,P)$

Then $x-g+k \in M$ and by (ad2) there is an integer $y \in L$ such that $y+x<g$ and $y+x \notin g-x+M$. This shows that $y+x \notin S(L,M,P)$ and that x is not an atom for $S(L,M,P)$. Thus $g-k$ is the largest small atom for $S(L,M,P)$ and $(L,M,P) \mapsto S(L,M,P)$ describes a function θ from $\mathcal{A}(k) \times \mathcal{P}(k,g-k)$ into $\mathcal{B}(g,k)$.

Now assume that $S \in \mathcal{B}(g,k)$, and define $L_S \subseteq (0,k)$, $M_S \subseteq (0,k)$ and $P_S \subseteq (g-k,g)$ by

$$L_S = S \cap (0,k),\ M_S = k-g+\big(S \cap (g-k,g)\big),\ P_S = S \cap (k,g+k)\ .$$

Since $g-k$ is a small atom of S then $\ell+g-k \in S \cap (g-k,g)$ for each $\ell \in L_S$, which implies that $\ell \in M_S$ and that $L_S \subseteq M_S$. Suppose that $x \in M_S$. Then $g-k+x$ is an element of S but not an atom of S (since $g-k$ is the largest small atom of S) and so there exists $y \in S$ with $g-k+x+y \leq g$ and $g-k+x+y \notin S$. It follows that $x+y \leq k$, $y \in L_S$ and $x+y \notin M_S$. Thus the pair (L_S, M_S) satisfies (ad1) and (ad2) and $(L_S,M_S) \in \mathcal{A}(k)$. Let Φ be the function from $\mathcal{B}(g,k)$ to $\mathcal{A}(k) \times \mathcal{P}(k,g-k)$ given by $S \mapsto (L_S,M_S,P_S)$. The proof is completed by observing that θ and Φ are inverses of each other. □

By lemma 3 the set $\mathcal{B}(g)$ can be expressed as the disjoint union of the sets $\mathcal{B}(g,k)$ where k ranges from 1 to $\lfloor (g-1)/2 \rfloor$. Thus, the cardinality of $\mathcal{B}(g)$ is the sum of the cardinalities of $\mathcal{B}(g,k)$ for $1 \leq k \leq \lfloor (g-1)/2 \rfloor$.

Corollary 6. *For each positive integer* g, $\beta_g = \displaystyle\sum_{k=1}^{\lfloor (g-1)/2 \rfloor} A_k\, 4^{-k}$.

Proof. Because of theorem 5 and the comment above, we have

$$\beta_g = \frac{|\mathcal{B}(g)|}{2^{g-1}} = \sum_{k=1}^{\lfloor (g-1)/2 \rfloor} \frac{|\mathcal{B}(g,k)|}{2^{g-1}} = \sum_{k=1}^{\lfloor (g-1)/2 \rfloor} \frac{A_k\, 2^{g-2k-1}}{2^{g-1}} = \sum_{k=1}^{\lfloor (g-1)/2 \rfloor} A_k\, 4^{-k}.$$

□

By corollary 6 the sequence $\{\beta_g\}$ is increasing, and being bounded above by 1, it must have a limit

$$\beta_\infty = \lim_{g\to\infty} \beta_g = \sum_{k=1}^{\infty} A_k 4^{-k}\ .$$

As a consequence the sequence $\{\gamma_g\} = \{1-\beta_g\}$ is decreasing with limit

$$\gamma_\infty = \lim_{g\to\infty} \gamma_g = 1-\beta_\infty\ .$$

By the next lemma, it is also possible to express γ_∞ as the sum of a positive series $\gamma_\infty = \sum_{k=1}^{\infty}(3^{k-1}-A_k)4^{-k}$.

Lemma 7. *For each integer $k > 0$, $2^{\lfloor (k-1)/2 \rfloor} \le A_k \le 3^{k-1}$. Moreover $\gamma_{2k-1} - \gamma_\infty$ is positive and $\gamma_{2k-1} - \gamma_\infty = \beta_\infty - \beta_{2k-1} \le (3/4)^{k-1}$.*

Proof. Let L be an arbitrary nonempty subset of $(0, \lfloor (k+1)/2 \rfloor) \subset (0, k)$ with maximal element ℓ. For any element $x \in L$, $\ell + x \le 2\ell \le k$ and $\ell + x \notin L$. This shows that (L, L) is an admissible pair of subsets of $(0, k)$. Since there are $2^{\lfloor (k-1)/2 \rfloor}$ distinct subsets of $(0, \lfloor (k+1)/2 \rfloor)$, this verifies the inequality $2^{\lfloor (k-1)/2 \rfloor} \le A_k$.

Suppose (L, M) is an element of $\mathcal{A}(k)$. Then for each $x \in (0, k)$ we have three distinct possibilities: (1) $x \notin M$, (2) $x \in M$ and $x \notin L$, or (3) $x \in L$. Therefore there are 3^{k-1} pairs of subsets (L, M) in $(0, k)$ which satisfy $(ad1)$, and it follows that $A_k \le 3^{k-1}$. Now by definition and corollary 6

$$\gamma_{2k-1} - \gamma_\infty = \beta_\infty - \beta_{2k-1} = \sum_{i=k}^{\infty} A_i \, 4^{-i} \le \sum_{i=k}^{\infty} 3^{i-1} \, 4^{-i} = (3/4)^{k-1} .$$

□

Notice that $(\emptyset, \emptyset)$ is the only ordered pair of subsets of $(0, 1) = \emptyset$, and as it is admissible, this shows that $A_1 = 1$. Among ordered pairs of subsets of $(0, 2)$, condition (ad1) fails for $(\{1\}, \emptyset)$ and condition (ad2) fails for $(\emptyset, \{1\})$ while the two remaining ordered pairs $(\emptyset, \emptyset)$ and $(\{1\}, \{1\})$ are in $\mathcal{A}(2)$, and so $A_2 = 2$. With lemma 7 and these values of A_1 and A_2,

$$\beta_\infty \le \beta_5 + (3/4)^2 = \left(\frac{1}{4} + \frac{2}{16} \right) + \frac{9}{16} = \frac{15}{16},$$

which shows that both γ_∞ and β_∞ are strictly between 0 and 1. Using this approach with the more extensive data compiled in table 1, we see that $\beta_{33} = .510538\ldots$ approximates β_∞ to within $(3/4)^{16} = .0100226\ldots$. Taking midpoints gives the approximation $\beta_\infty \approx .515549$ accurate to within .005011, and subtracting from 1 gives $\gamma_\infty \approx .484451$ with the same degree of accuracy. This approximation can be rephrased as saying that $|\mathcal{G}(g)| \approx .484451 \times 2^{g-1}$ for large values of g.

If (L, M) is an admissible ordered pair of subsets of $(0, k)$ and L' and M' are subsets satisfying $L \subseteq L' \subseteq M' \subseteq M$ then (L', M') is also admissible. In particular, both (L, L) and (M, M) are elements of $\mathcal{A}(k)$ whenever $(L, M) \in \mathcal{A}(k)$. The computer routine that was used to generate the data in table 1 starts by first determining the collection of subsets $L \subseteq (0, k)$ for which (L, L) is admissible. (The cardinality of this collection is denoted by A'_k in the table. These numbers are important in their own right as we shall explain in the next section.) The routine then isolates nested pairs of sets in this collection and tests only these pairs for condition (ad2). Even with this, the algorithm has exponential complexity and slows down quite rapidly.

From the results of this section we may draw some further conclusions which directly address the Anti-Atom Problem for an arbitrary numerical monoid M.

Theorem 8. *Let $M \ne \mathbb{N}_g$ be a numerical monoid with Frobenius number g and let $g - k$ be the largest element of $M \cap (0, g)$. Then $|\mathcal{G}(M)| \le A_k 4^{-k} \, 2^{g-1} \le \frac{1}{3} \, (3/4)^k \times 2^{g-1}$.*

n	A'_n	A_n	β_{2n+1}	$\beta_{2n+1} + (3/4)^n$	A_{n-1}/A_n
1	1	1	.250000	1.000000	-
2	2	2	.375000	.937500	.5000
3	3	3	.421875	.843750	.6667
4	6	8	.453125	.769531	.3750
5	10	18	.470703	.708008	.4444
6	20	50	.482910	.660889	.3600
7	37	135	.491150	.624634	.3704
8	74	385	.497025	.597137	.3506
9	140	1065	.501087	.576172	.3615
10	280	3053	.503999	.560312	.3488
11	542	8701	.506073	.548308	.3509
12	1084	25579	.507598	.539274	.3402
13	2118	73693	.508696	.532453	.3471
14	4236	217718	.509507	.527325	.3385
15	8337	635220	.510090	.523462	.3427
16	16674	1888802	.510538	.520561	.3363

TABLE 1. Bounds for β_∞.

Proof. If $M \neq \mathbb{N}_g$ is a numerical monoid in $\mathcal{S}(g)$ and $g-k$ is the largest element in $M \cap (0,g)$ then $g-k$ is the largest small atom of every numerical set S with $A(S) = M$. Thus $\mathcal{G}(M) \subseteq \mathcal{B}(g,k)$ and $|\mathcal{G}(M)| \leq |\mathcal{B}(g,k)| = A_k 2^{g-2k-1}$. The last inequality follows from lemma 7. □

The value of k in theorem 8 satisfies $0 < k < g/2$. Since $\frac{1}{3}\left(3/4\right)^k \times 2^{g-1} \leq .25 \times 2^{g-1}$ is less than $.484451 \times 2^{g-1}$ for all k, we see that among all monoids in $\mathcal{S}(g)$ the one with largest anti-atom set is always $\mathbb{N}_g$ (which is not too surprising since $\mathcal{G}(\mathbb{N}_g)$ contains more than 48% of the elements of $\mathcal{S}(g)$).

As k increases from 1 to $\lfloor (g-1)/2 \rfloor$ the cardinality of $\mathcal{B}(g,k)$ decreases but the number of monoids in $\mathcal{B}(g,k)$ decreases as well. For example, $\mathcal{B}(g,1)$ contains all of the symmetric and pseudosymmetric monoids in $\mathcal{S}(g)$, while, at the other extreme, $\mathcal{B}(g, \lfloor (g-1)/2 \rfloor)$ contains only one monoid $\mathbb{D}_g$ which is defined by

$$\mathbb{D}_g = \mathbb{N}_g \cup \{\lfloor (g+2)/2 \rfloor\}. \tag{2}$$

Corollary 9. *For each nonnegative integer n, we have $|\mathcal{G}(\mathbb{D}_{2n+1})| = A_n$ and $|\mathcal{G}(\mathbb{D}_{2n+2})| = 2A_n$.*

Proof. It is not difficult to show that $\mathbb{D}_g$ is the only monoid in $\mathcal{B}(g, \lfloor (g-1)/2 \rfloor)$. (If M is a monoid in $\mathcal{B}(g, \lfloor (g-1)/2 \rfloor)$ then $M \cap (\lfloor (g+2)/2 \rfloor, g) = \emptyset$.) Thus $\mathbb{D}_g$ is the atom monoid of every numerical set in $\mathcal{B}(g, \lfloor (g-1)/2 \rfloor)$, and this implies that $|\mathcal{G}(\mathbb{D}_g)| = |\mathcal{B}(g, \lfloor (g-1)/2 \rfloor)| = A_{\lfloor (g-1)/2 \rfloor} 2^{g-2\lfloor (g-1)/2 \rfloor - 1}$, from which the corollary follows. □

When k is less than $\lfloor (g-1)/2 \rfloor$ the set $\mathcal{B}(g,k)$ will always contain at least two distinct numerical monoids (for example, $\mathbb{N}_g \cup \{g-k\}$ and $\mathbb{N}_g \cup \{g-k-1, g-k\}$). Thus $\mathbb{D}_g$ is the only monoid in $\mathcal{S}(g)$ for which the first inequality of theorem 8 is sharp.

The generating function for $\{\gamma_k\}$

For each integer $k > 0$ let $\mathcal{A}(k)'$ denote the collection of all subsets $L \subseteq (0,k)$ for which (L, L) is admissible. Thus $\mathcal{A}(k)'$ consists of those subsets L which satisfy the condition that for each $x \in L$ there is $y \in L$ such that $x + y \leq k$ and $x + y \notin L$. The cardinality of $\mathcal{A}(k)'$ will be denoted by A'_k.

Theorem 10. *There is a one-to-one correspondence between $\mathcal{G}(g)$ and $\mathcal{A}(g)'$. In particular, $|\mathcal{G}(g)| = A'_g$ and $\gamma_g = A'_g / 2^{g-1}$.*

Proof. For each $L \in \mathcal{A}(g)'$ consider the numerical set $\mathbb{N}_g \cup L \in \mathcal{S}(g)$. If $x \in L$ then there is an integer $y \in L$ such that $0 < x + y \leq g$ and $x + y \notin L$, which implies that $x + y \notin \mathbb{N}_g \cup L$. This shows that x is not an atom of $\mathbb{N}_g \cup L$, and that $\mathbb{N}_g \cup L$ has no small atoms. Thus the correspondence $L \mapsto \mathbb{N}_g \cup L$ is a function from $\mathcal{A}(g)'$ to $\mathcal{G}(g)$, and clearly this function is injective. Now suppose $S \in \mathcal{G}(g)$ and let $x \in S \cap (0,g)$. Since S has no small atoms, there is an integer $y \in S$ such that $x + y \notin S$. Thus $x + y \leq g$ (since the Frobenius number of S is g), $y \in S \cap (0,g)$ and $x + y \notin S \cap (0,g)$. It follows that $S \cap (0,g)$ is an element of $\mathcal{A}(g)'$, and the function $L \mapsto \mathbb{N}_g \cup L$ is surjective. □

Theorem 11. *For each $k \geq 1$, $A'_{2k} = 2\ A'_{2k-1}$ and $A'_{2k+1} = 2\ A'_{2k} - A_k$.*

Proof. The first equation follows immediately from lemma 4 and theorem 10. For the second equation we have

$$A'_{2k+1} = |\mathcal{G}(2k+1)| = |\mathcal{S}(2k+1)| - |\mathcal{B}(2k+1)| = 2^{2k} - \sum_{\ell=1}^{k} |\mathcal{B}(2k+1,\ell)|,$$

and by theorem 5

$$2^{2k} - \sum_{\ell=1}^{k} |\mathcal{B}(2k+1,\ell| = 4^k - \sum_{\ell=1}^{k} A_\ell \, 4^{k-\ell} \ .$$

A similar computation shows that

$$A'_{2k} = \frac{1}{2}\Big(4^k - \sum_{\ell=1}^{k-1} A_\ell \, 4^{k-\ell}\Big) \ .$$

Combining these gives $A'_{2k+1} - 2A'_{2k} = -A_k$. □

The set $\mathcal{G}(2k+1)$ can be constructed from $\mathcal{G}(2k-1)$ by a process in which each element of $\mathcal{G}(2k-1)$ will spawn either three or four elements of $\mathcal{G}(2k+1)$ as follows. If $S \in \mathcal{G}(2k-1)$ and Q is one of the four subsets of $\{k, k+1\}$ then let $S(Q) \in \mathcal{S}(2k+1)$ be given by

$$S(Q) = \mathbb{N}_{2k+1} \cup \Big(S \cap [1, k-1]\Big) \cup Q \cup \Big(2 + \big(S \cap [k, 2k-2]\big)\Big) \ . \qquad (3)$$

Since S has no small atoms, $S(Q)$ will not have any small atoms larger than $k+1$. Furthermore, if Q is one of $\emptyset$, $\{k\}$ or $\{k, k+1\}$ then $k+1$ is not an atom of $S(Q)$,

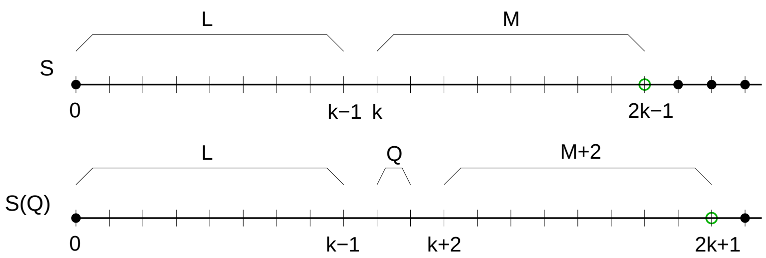

FIGURE 3. *The numerical set $S(Q)$*

and $S(Q)$ is an element of $\mathcal{G}(2k+1)$. But if Q equals the singleton set $\{k+1\}$ then sometimes $k+1$ will be an atom for $S(Q)$, in which case $S(Q)$ is not an element of $\mathcal{G}(2k+1)$. From this we see that $|\mathcal{G}(2k+1)| = A'_{2k+1}$ is four times $|\mathcal{G}(2k-1)| = A'_{2k-1}$ minus the number of elements of $\mathcal{G}(2k-1)$ which spawn only three elements of $\mathcal{G}(2k+1)$, and, since $A'_{2k+1} = 4A'_{2k-1} - A_k$ by theorem 11, the number of elements of $\mathcal{G}(2k-1)$ which spawn only three elements of $\mathcal{G}(2k+1)$ equals A_k.[7] As a result of these comments we can view the union of all the sets $\mathcal{G}(2k+1)$ as the vertices of a downward opening rooted tree in which each vertex is directly above the 3 or 4 vertices that it spawns, as pictured in figure 4. In the illustration the vertex labeled by a $2\times k$ matrix $\alpha = \left(\begin{smallmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ \alpha_{2k} & \alpha_{2k-1} & \cdots & \alpha_{k+1}\end{smallmatrix}\right)$ with entries in $\mathbb{Z}_2$ corresponds to the numerical set

$$S(\alpha) = \mathbb{N}_{2k+1} \cup \{i \mid \alpha_i = 1\}$$

in $\mathcal{G}(2k+1)$. Although we will not use it here, one can specify conditions on the matrix α which guarantee that $S(\alpha)$ is in $\mathcal{G}(2k+1)$: Call the $2 \times k$ matrix α *quadrivalent* if there is an integer ℓ with $1 \le \ell \le k$ such that the ℓth column of α is $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ and the $(k+1-\ell)$th column is $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$.[8] Then $S(\alpha) \in \mathcal{G}(2k+1)$ if and only if whenever the ith column of α equals $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ then the $2 \times (i-1)$ submatrix of α to the left of that column is quadrivalent. We also note that as one moves down the tree the ratio A_k/A'_{2k+1} of the number of vertices at level $2k+1$ which spawn three vertices by the total number of vertices at that level limits to 0. Indeed A_k is bounded above by the number of $2 \times k$ matrices α which are not quadrivalent and that number is easily seen to equal 3^k. Thus

$$\frac{A_k}{A'_{2k+1}} = \frac{A_k}{|\mathcal{G}(2k+1)|} = \frac{A_k}{\gamma_{2k+1}4^k} \le \frac{1}{\gamma_{2k+1}}\left(\frac{3}{4}\right)^k$$

and the latter limits to 0.

[7]This can also be verified by a combinatorial argument. If $(L, M) \in \mathcal{A}(k)$ then (L', L') is an element of $\mathcal{A}(2k-1)' \cong \mathcal{G}(2k-1)$ which spawns only three elements of $\mathcal{G}(2k+1)$, where $L' = L \cup (M + k - 1)$.

[8]The reason for this terminology is that if $S(\alpha)$ is an element of $\mathcal{G}(2k+1)$, then the matrix α is quadrivalent if and only if $S(\alpha)$ spawns four elements of $\mathcal{G}(2k+3)$.

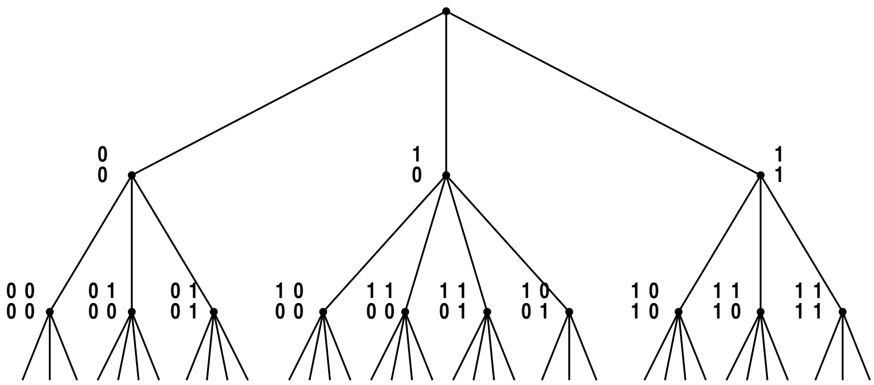

FIGURE 4. *Rooted tree for* $\bigcup\{\mathcal{G}(2k+1) \mid k \in \mathbb{N}\}$.

Let $g(z)$ and $f(z)$ be the analytic functions defined by

$$g(z) = \sum_{k=1}^{\infty} A_k z^k \qquad \text{and} \qquad f(z) = \sum_{k=1}^{\infty} A'_k z^k \ .$$

Corollary 12. *The functions $f(z)$ and $g(z)$ satisfy the relation*

$$\big(2z-1\big) f(z) = z\left(g(z^2) - 1\right) \ .$$

Proof. Using theorem 11 we have

$$f(z) = \sum_{k=1}^{\infty} A'_{2k-1} z^{2k-1} + \sum_{k=1}^{\infty} A'_{2k} z^{2k} = (2z+1) \sum_{k=1}^{\infty} A'_{2k-1} z^{2k-1} \ . \tag{4}$$

Also $A_k = 4A'_{2k-1} - A'_{2k+1}$ by theorem 11, and

$$\begin{aligned} g(z^2) &= \sum_{k=1}^{\infty} (4A'_{2k-1} - A'_{2k+1}) z^{2k} \\ &= 4z \sum_{k=1}^{\infty} A'_{2k-1} z^{2k-1} - \frac{1}{z} \sum_{k=1}^{\infty} A'_{2k-1} z^{2k-1} + A'_1 \\ &= \frac{(2z-1)(2z+1)}{z} \sum_{k=1}^{\infty} A'_{2k-1} z^{2k-1} + 1 = \frac{2z-1}{z} f(z) + 1 \ , \end{aligned}$$

where the last equality follows from (4). □

Corollary 13. *The analytic function $f(z)$ has a singularity at $z = 1/2$, and its radius of convergence at the origin equals $1/2$. Other than $z = 1/2$, the singularities of $f(z)$ coincide with those of $g(z^2)$ and $f(z)(z-1/2)$ is continuous on $|z| \leq 1/2$.*

Proof. Since $\sum_{k=1}^{\infty} A_k 4^{-k}$ sums to $\beta_\infty = g(1/4)$, $\sum_{k=1}^{\infty} A_k z^k$ converges absolutely and $f(z)$ is continuous on $|z| \leq 1/4$. Note that $f(z)$ has no singularities in $|z| < 1/2$ because $g(z)$ has none in $|z| < 1/4$. The remainder of the proof follows from corollary 12. □

By definition the integers A'_k and A_k satisfy $0 \leq A'_k \leq A_k$ for all $k > 0$, and by corollary 13 we know that the series $\sum_{k=1}^{\infty} A'_k z^k$ diverges when $z = 1/2$. Therefore $\sum_{k=1}^{\infty} A_k z^k$ must also diverge for $z = 1/2$ by the comparison test, and this shows that the radius of convergence for $g(z)$ at the origin is between $1/4$ and $1/2$. To find the precise value, the ratio test would lead one to examine the sequence A_{n-1}/A_n. Empirical evidence from the last column of table 1 perhaps suggests that this sequence has a limit infimum larger than $1/4$, but we have not been able to ascertain this. So the determination of the radius of convergence of $g(z)$ at the origin remains open. Notice that if this radius of convergence is larger than $1/4$ then $g(z^2)$ is analytic in a neighborhood of $z = 1/2$ and

$$\lim_{z\to 1/2} (z - 1/2)f(z) = \lim_{z\to 1/2} z\left(g(z^2) - 1\right)/2 = (\beta_\infty - 1)/4 = -\gamma_\infty/4,$$

which would imply that $f(z)$ has a simple pole with residue $-\gamma_\infty/4$ at $z = 1/2$.

Now the generating function for the sequence $\{\gamma_k\}$ is

$$h(z) = \sum_{k=1}^{\infty} \gamma_k\, z^k$$

which satisfies

$$h(z) = \sum_{k=1}^{\infty} \frac{A'_k}{2^{k-1}}\, z^k = 2f\left(z/2\right) = \left(\frac{z}{z-1}\right)\left(g(z^2/4) - 1\right).$$

By corollary 13 the radius of convergence of this series equals 1. If the radius of convergence at the origin for $g(z)$ is larger than $1/4$ then $g(z^2/4) - 1$ is analytic in a disk with radius larger than 1 centered at the origin, and $h(z)$ has a simple pole at $z = 1$ whose residue is $-\gamma_\infty$ (and this is the only pole on $|z| = 1$).

Symmetric numerical sets with no small atoms

A numerical set S with Frobenius number g is *negative semisymmetric* provided that $g - x \notin S$ whenever $x \in S$. For a positive integer g, define

$$\mathcal{S}^{\sigma}(g) \;=\; \{S \in \mathcal{S}(g) \mid S \text{ is maximal negative semisymmetric in } \mathcal{S}(g)\}$$

where maximality is measured with respect to subset inclusion. Then $\mathcal{S}^{\sigma}(g)$ consists of the symmetric numerical sets in $\mathcal{S}(g)$ when g is odd, and the pseudosymmetric numerical sets in $\mathcal{S}(g)$ when g is even. Each element of $\mathcal{S}^{\sigma}(g)$ is the union of $\mathbb{N}_g$ with a subset of $(0, g) - \{g/2\}$ that is carried onto its complement by $x \mapsto g - x$, and hence the cardinality of $\mathcal{S}^{\sigma}(g)$ equals $2^{\lfloor (g-1)/2 \rfloor}$.

The set $\mathcal{S}^{\sigma}(g)$ is partitioned into two subsets

$$\mathcal{G}^{\sigma}(g) \;=\; \{S \in \mathcal{S}^{\sigma}(g) \mid S \text{ has no small atoms}\} \;=\; \mathcal{G}(g) \cap \mathcal{S}^{\sigma}(g)$$

and

$$\mathcal{B}^{\sigma}(g) \;=\; \{S \in \mathcal{S}^{\sigma}(g) \mid S \text{ has at least one small atom}\} \;=\; \mathcal{B}(g) \cap \mathcal{S}^{\sigma}(g)\;.$$

We define

$$\beta_g^{\sigma} = \frac{|\mathcal{B}^{\sigma}(g)|}{|\mathcal{S}^{\sigma}(g)|} = \frac{|\mathcal{B}^{\sigma}(g)|}{2^{\lfloor (g-1)/2 \rfloor}}$$

and

$$\gamma_g^\sigma = \frac{|\mathcal{G}^\sigma(g)|}{|\mathcal{S}^\sigma(g)|} = \frac{|\mathcal{G}^\sigma(g)|}{2^{\lfloor (g-1)/2 \rfloor}}.$$

The next lemma describes a direct connection between symmetric and pseudosymmetric numerical sets.

Lemma 14. *The correspondence $S \mapsto S_0'$ where*

$$S_0' = \Big(S \cap [0, n-1]\Big) \cup \Big(1 + S \cap [n, \infty)\Big)$$

defines a bijection from $\mathcal{S}^\sigma(2n-1)$ to $\mathcal{S}^\sigma(2n)$ which carries $\mathcal{G}^\sigma(2n-1)$ onto $\mathcal{G}^\sigma(2n)$. Therefore $\gamma_{2n}^\sigma = \gamma_{2n-1}^\sigma$ and $\beta_{2n}^\sigma = \beta_{2n-1}^\sigma$.

Proof. Notice that S_0' is the numerical set S_ϵ' where $\epsilon = 0$ as defined in lemma 4. The proof follows upon observing that S is symmetric if and only if S_0' is pseudosymmetric. □

For integers g and k with $g > 2k > 0$, let

$$\mathcal{B}^\sigma(g,k) = \{S \in \mathcal{B}^\sigma(g) \mid g-k \text{ is the largest small atom of } S\}$$

(i.e. $\mathcal{B}^\sigma(g,k) = \mathcal{B}(g,k) \cap \mathcal{S}^\sigma(g)$). Then $\mathcal{B}^\sigma(g)$ is the disjoint union of the sets $\mathcal{B}^\sigma(g,k)$ as k ranges between 1 and $\lfloor (g-1)/2 \rfloor$ by lemma 3. In order to describe $\mathcal{B}^\sigma(g,k)$ we are led to the next definitions.

Let M be a subset of $(0,k)$. Define

$$M_+ = \{m \in M \mid k - m \in M\} \ ,$$

$$M_- = \{x \in (0,k) \mid x \notin M \text{ and } k - x \notin M\} \ ,$$

and

$$M^* = \{x \in (0,k) \mid k - x \notin M\} \ .$$[9]

With these definitions, observe that $M^* = (M - M_+) \cup M_-$.

A subset $M \subseteq (0,k)$ is called *σ-admissible* if it satisfies the two conditions:

(σ-ad1) $M_- = \emptyset$, and
(σ-ad2) for each $x \in M$ there is $y \in M^*$ with $x + y < k$ and $x + y \notin M$.

Let $\mathcal{A}^\sigma(k)$ be the set of all σ-admissible subsets of $(0,k)$ and let A_k^σ denote the cardinality of this set. Also, for integers k and g with $g > 2k > 0$ let $\mathcal{P}^\sigma(k, g-k)$ be the collection of all subsets of $(k, g-k) - \{g/2\}$ that are carried onto their complement by the reflection $x \mapsto g - x$. The cardinality of this collection is $2^{\lfloor (g-2k-1)/2 \rfloor}$. Recall the definition of the numerical set $S(L,M,P)$ as

$$S(L,M,P) = \mathbb{N}_g \cup L \cup P \cup \{g-k\} \cup \big(M + g - k\big)$$

from equation (1) in theorem 5.

Theorem 15. *Let g and k be integers with $g > 2k > 0$. The correspondence $(M,P) \mapsto S(M^*, M, P)$ defines a bijection from $\mathcal{A}^\sigma(k) \times \mathcal{P}^\sigma(k, g-k)$ onto $\mathcal{B}^\sigma(g,k)$. In particular, the cardinality of $\mathcal{B}^\sigma(g,k)$ is $A_k^\sigma\, 2^{\lfloor (g-2k-1)/2 \rfloor}$.*

[9]This last definition is closely related to the definition of the dual S^* of a numerical set S. If $S = \mathbb{N}_k \cup M \in \mathcal{S}(k)$ then $S^* = \mathbb{N}_k \cup M^*$.

Proof. Let $(M,P) \in \mathcal{A}^\sigma(k) \times \mathcal{P}^\sigma(k, g-k)$. An integer $x \in (0,k)$ is an element of M^* if and only if $k - x \notin M$, which is equivalent to asserting that $g - x \notin M + g - k$. This together with the fact that P is an element of $\mathcal{P}^\sigma(k, g-k)$ implies that $S(M^*, M, P) \in \mathcal{S}^\sigma(g)$. Since $M_- = \emptyset$, $M^* = M - M_+ \subseteq M$. If $x \in M^*$ then $x \in M$ and $g - k + x \in M + g - k \subset S(M^*, M, P)$, and if x is an integer larger than k then $g - k + x > g$ and $g - k + x \in S(M^*, M, P)$. This shows that $g - k$ is an atom for $S(M^*, M, P)$. An element of $S(M^*, M, P)$ in the interval $(g-k, k)$ has the form $x + g - k$ for some $x \in M$. By the definition of $\mathcal{A}^\sigma(k)$ there is $y \in M^*$ with $y < k - x$ and $x + y \notin M$. Thus $y + (x + g - k)$ is an element of $(g-k, g)$ which is not an element of $M + g - k$, and $x + g - k$ is not an atom of $S(M^*, M, P)$. It follows that $g - k$ is the largest small atom of $S(M^*, M, P)$ and $S(M^*, M, P) \in \mathcal{B}^\sigma(g,k)$.

To complete the proof it only remains to show that each numerical set in $\mathcal{B}^\sigma(g,k)$ equals $S(M^*, M, P)$ for some $(M,P) \in \mathcal{A}^\sigma(k) \times \mathcal{P}^\sigma(k, g-k)$. Let $T \in \mathcal{B}^\sigma(g,k)$, and set $M = (T \cap (g-k, g)) - g + k \subseteq (0,k)$ and $P = T \cap (k, g-k)$. If $x \in T \cap (0,k)$ then $x + g - k \in T \cap (g-k, g)$, since $g - k$ is an atom of T, and hence $x \in M$. Note further that $k - x$ is not in T because if it were then both $k - x$ and $g - (k - x) = x + g - k$ would be elements of T contradicting the negative semisymmetry of T. This shows that $x \in M^*$ and $T \cap (0,k) \subseteq M^*$. Moreover if $x \in M^*$ then $k - x \notin M$ which means that $g - k + (k - x) = g - x \notin T$ and that $x \in T$ since T is maximally negative semisymmetric. Thus $M^* = T \cap (0,k)$ and $S(M^*, M, P) = T$. Clearly $P \in \mathcal{P}^\sigma(k, g-k)$ so to complete the proof it must be shown that $M \in \mathcal{A}^\sigma(k)$. If $M_- \neq \emptyset$ then there is $x \in (0,k)$ such that $x \notin M$ and $k - x \notin M$, and then $(g-k) + k - x = g - x \notin T$ and $x \notin T$ which contradicts the maximality of T. This verifies that M_- is empty. If $x \in M$ then $g - k + x$ is an element of $T \cap (0,g)$ larger than $g - k$ so that $g - k + x$ is not an atom of T since $g - x$ is the largest atom of T. It follows that there is an element $y \in T$ with $y < k - x$ such that $y + g - k + x \notin T$ (note that y cannot equal $k - x$ because otherwise both $k - x$ and $g - k + x$ would be elements of T contradicting the assumption that T is negative semisymmetric), and $M \in \mathcal{A}^\sigma(k)$. $\square$

By lemma 3 and the theorem we have

$$|\mathcal{B}^\sigma(g)| = \sum_{k=1}^{\lfloor (g-1)/2 \rfloor} |\mathcal{B}^\sigma(g,k)| = \sum_{k=1}^{\lfloor (g-1)/2 \rfloor} A_k^\sigma \, 2^{\lfloor (g-2k-1)/2 \rfloor} \tag{5}$$

and dividing by $2^{\lfloor (g-1)/2 \rfloor}$ produces the next result.

Corollary 16. *For each $g > 0$, $\beta_g^\sigma = \sum_{k=1}^{\lfloor (g-1)/2 \rfloor} A_k^\sigma \, 2^{-k}$.* $\square$

Thus $\{\beta_g^\sigma\}$ is an increasing sequence, and it has a limit

$$\beta_\infty^\sigma = \sum_{k=1}^{\infty} A_k^\sigma \, 2^{-k} \ .$$

It follows immediately that $\{\gamma_g^\sigma\}$ is a decreasing sequence which converges to $\gamma_\infty^\sigma = 1 - \beta_\infty^\sigma$.

Corollary 17. *For each positive integer* n, $A_n^\sigma \le 3^{\lfloor (n-3)/2 \rfloor}$ *and*

$$\gamma_{2n-1}^\sigma - \gamma_\infty^\sigma = \beta_\infty^\sigma - \beta_{2n-1}^\sigma \le \left(\frac{\sqrt{3}}{2}\right)^{n-1} .$$

Proof. Let $M \subseteq (0, n)$ be an element of $\mathcal{A}^\sigma(n)$. Suppose that $n = 2k + 1$ is odd. Then $(0, n)$ is partitioned into k doubletons $\{i, n - i\}$ where $1 \le i \le k$. Since $M_- = \emptyset$ the intersection of $\{i, n - i\}$ with M must be nonempty, and so there are three possibilities for each of these intersections. Notice also that $n-1$ cannot be an element of M by condition (σ-ad2), and the intersection of M with the doubleton $\{1, n - 1\}$ must be $\{1\}$. Thus there are at most $3^{k-1} = 3^{\lfloor (n-3)/2 \rfloor}$ possibilities for M. When $n = 2k$ is even, $(0, n)$ can be partitioned into $(k - 1)$ doubletons $\{i, n - i\}$ where $1 \le i \le k - 1$ and a singleton $\{k\}$. The intersection of M with $\{k\}$ must equal $\{k\}$ since $M_- = \emptyset$. By a similar argument as before there are at most $3^{k-2} = 3^{\lfloor (n-3)/2 \rfloor}$ possibilities for M. Now $\beta_\infty^\sigma - \beta_{2n-1}^\sigma = \sum_{k=n}^\infty A_k^\sigma 2^{-k} \le \sum_{k=n}^\infty 3^{\lfloor (k-3)/2 \rfloor} 2^{-k} \le \left(\frac{\sqrt{3}}{2}\right)^{n-1}$. □

Some values of β_{2n-1}^σ are given in table 2. Note that $\beta_{63}^\sigma = .76356\ldots$ approximates β_∞^σ to within $(\sqrt{3}/2)^{31} = .0115731\ldots$ by corollary 17. Subtracting from 1 gives $\gamma_{63}^\sigma = .23644\ldots$, which approximates γ_∞^σ to within .0115731. Taking midpoints gives the approximation $\gamma_\infty^\sigma \approx .230653$ accurate to within .00579.

n	$A_{2n-1}^{\sigma\prime}$	A_n^σ	β_{2n-1}^σ	R_n
1	1	1	0	1
2	1	0	.5	∞
3	2	1	.5	1
4	3	0	.625	∞
5	6	2	.625	.871
6	10	0	.6875	∞
7	20	3	.6875	.855
8	37	1	.71094	1
9	73	7	.71484	.806
10	139	3	.72852	.896
11	275	17	.73145	.773
12	533	7	.73975	.850
13	1059	43	.74146	.749
14	2075	24	.74670	.797
15	4126	118	.74817	.728
16	8134	74	.75177	.764

n	$A_{2n-1}^{\sigma\prime}$	A_n^σ	β_{2n-1}^σ	R_n
17	16194	330	.75290	.711
18	32058	206	.75542	.744
19	63910	888	.75620	.700
20	126932	612	.75790	.725
21	253252	2571	.75848	.688
22	503933	1810	.75971	.711
23	1006056	7274	.76014	.679
24	2004838	5552	.76100	.698
25	4004124	21099	.76134	.671
26	7987149	16334	.76196	.689
27	15957964	61252	.76221	.665
28	31854676	49025	.76266	.680
29	63660327	179239	.76285	.659
30	127141415	146048	.76318	.673
31	254136782	523455	.76332	.654
32	507750109	440980	.76356	.666

TABLE 2. Approximating β_∞^σ, where $R_n = 1/\sqrt[n]{A_n^\sigma}$.

For a numerical monoid $M \in \mathcal{S}(g)$ let

$$\mathcal{G}^\sigma(M) = \mathcal{G}(M) \cap \mathcal{S}^\sigma(g) = \{S \in \mathcal{S}^\sigma(g) \mid A(S) = M\} .$$

Note that M will not be an element of $\mathcal{G}^\sigma(M)$ unless M is symmetric or pseudosymmetric, and that $\mathcal{G}^\sigma(M)$ may be empty. If $M \in \mathcal{S}(g)$ is a numerical monoid in $\mathcal{B}(g,k)$ (which means that $g-k$ is the largest integer in $M \cap (0,g)$) then $\mathcal{G}^\sigma(M) \subseteq \mathcal{B}^\sigma(g,k)$. Therefore

$$|\mathcal{G}^\sigma(M)| \leq |\mathcal{B}^\sigma(g,k)| = A_k^\sigma\, 2^{\lfloor (g-2k-1)/2 \rfloor} \leq \frac{1}{3\sqrt{3}} \left(\frac{\sqrt{3}}{2}\right)^k \times 2^{\lfloor (g-1)/2 \rfloor}$$

by corollary 17, and this is the symmetric analogue of the inequality in theorem 8. In particular we see that $|\mathcal{G}^\sigma(\mathbb{N}_g)| = |\mathcal{G}^\sigma(g)|$ is larger than $|\mathcal{G}^\sigma(M)|$ for every numerical monoid $M \in \mathcal{S}(g)$ other than $\mathbb{N}_g$. Taking $k = \lfloor (g-1)/2 \rfloor$ in the above shows that $|\mathcal{G}^\sigma(\mathbb{D}_{2k+1})| = |\mathcal{B}^\sigma(g,k)| = A_k^\sigma$ and that $|\mathcal{G}^\sigma(\mathbb{D}_{2k+2})| = 2A_k^\sigma$, where $\mathbb{D}_n$ is defined in equation (2).

The generating function for $\{\gamma_k^\sigma\}$

We define $\mathcal{A}^\sigma(k)'$ to be the subset of $\mathcal{A}^\sigma(k)$ consisting of all σ-admissible sets $M \subseteq (0,k)$ for which M_+ has at most one element. If $M \in \mathcal{A}^\sigma(k)'$ then there are two possibilities: either k is odd and $M_+ = \emptyset$ (because M_+ has an even number of elements whenever k is odd) or k is even and $M_+ = \{k/2\}$ (because $k/2$ must be in M_- or M_+ whenever k is even, and $M_- = \emptyset$). The cardinality of $\mathcal{A}^\sigma(k)'$ will be denoted by $A_k^{\sigma\,\prime}$.

Theorem 18. *There is a one-to-one correspondence between the sets $\mathcal{G}^\sigma(g)$ and $\mathcal{A}^\sigma(g)'$. In particular, $|\mathcal{G}^\sigma(g)| = A_g^{\sigma\,\prime}$ and $\gamma_g^\sigma = A_g^{\sigma\,\prime}\, 2^{-\lfloor (g-1)/2 \rfloor}$.*

Proof. Given $M \in \mathcal{A}^\sigma(g)'$ define $S = \mathbb{N}_g \cup M^*$. If g is odd then $M_- = M_+ = \emptyset$ so that $M^* = M$ and S is symmetric. If g is even then $M_- = \emptyset$, $M_+ = \{g/2\}$ and $M^* = M - \{g/2\}$ which implies that S is pseudosymmetric. In either case, S is an element of $\mathcal{S}^\sigma(g)$ and we can write $M^* = M - \{g/2\}$. Suppose that S has a small atom. By lemma 3 there is a small atom s in $A(S)$ with $s > g/2$. Then $s \in M^* \subseteq M$ and by (σ-ad2) there exists $y \in M^*$ such that $s + y < g$ and $s + y \notin M$. Note that $s + y \notin M - \{g/2\} = M^*$ since $s > g/2$. Thus $s + y \notin S$ in contradiction of the assumption that $s \in A(S)$. This shows that $S \in \mathcal{G}^\sigma(g)$, and thus $M \mapsto S$ is an injective function from $\mathcal{A}^\sigma(g)'$ to $\mathcal{G}^\sigma(g)$. If $S \in \mathcal{G}^\sigma(g)$ then it is not hard to check that $S = \mathbb{N}_g \cup M^*$ where $M = (S \cap (0,g))^* \in \mathcal{A}^\sigma(g)'$, completing the proof. □

Theorem 19. *For each $k \geq 1$, $A_{2k}^{\sigma\,\prime} = A_{2k-1}^{\sigma\,\prime}$ and $A_{2k+1}^{\sigma\,\prime} = 2A_{2k}^{\sigma\,\prime} - A_k^\sigma$.*

Proof. The first statement follows from lemma 14 and theorem 18. For the second we have

$$A_{2k+1}^{\sigma\,\prime} = |\mathcal{G}^\sigma(2k+1)| = |\mathcal{S}^\sigma(2k+1)| - |\mathcal{B}^\sigma(2k+1)| = 2^k - \sum_{\ell=1}^{k} A_\ell^\sigma 2^{k-\ell}$$

using equation (5), and $A_{2k}^{\sigma\,\prime} = 2^{k-1} - \sum_{\ell=1}^{k-1} A_\ell^\sigma 2^{k-\ell-1}$ by a similar computation. Combining these two equations gives $2A_{2k}^{\sigma\,\prime} - A_{2k-1}^{\sigma\,\prime} = A_k^\sigma$. □

As in figure 4 we may view $\bigcup\{\mathcal{G}^{\sigma}(2k+1) \mid k \in \mathbb{N}\}$ as the vertices of a downward opening rooted tree. Here each element of $\mathcal{G}^{\sigma}(2k-1)$ (represented by a vertex in the kth level of the tree) will spawn either one or two elements of $\mathcal{G}^{\sigma}(2k+1)$ (represented by vertices at the $(k+1)$st level). If $S \in \mathcal{G}^{\sigma}(2k-1)$ and Q is either $\{k\}$ or $\{k+1\}$ then let $S(Q)$ be defined by equation (3) and observe that $S(Q) \in \mathcal{S}^{\sigma}(2k+1)$. As before $S(Q)$ has no small atoms when $Q = \{k\}$ but $k+1$ may be a small atom for $S(Q)$ when $Q = \{k+1\}$. From this we see that $|\mathcal{G}^{\sigma}(2k+1)| = A^{\sigma\prime}_{2k+1}$ is two times $|\mathcal{G}^{\sigma}(2k-1)| = A^{\sigma\prime}_{2k-1}$ minus the number of elements of $\mathcal{G}^{\sigma}(2k-1)$ which spawn only one element of $\mathcal{G}^{\sigma}(2k+1)$, and it follows from theorem 19 that the number of elements of $\mathcal{G}^{\sigma}(2k-1)$ which spawn only one element of $\mathcal{G}^{\sigma}(2k+1)$ equals A^{σ}_{k}. Figure 5 shows the first few levels of the rooted tree. In this illustration a labeling sequence $\alpha = (\alpha_1, \ldots, \alpha_k)$ with entries in $\mathbb{Z}_2$ represents the same numerical set in $\mathcal{G}^{\sigma}(2k+1)$as the $2 \times k$ matrix $\left(\begin{smallmatrix} \alpha_1 & \cdots & \alpha_k \\ \alpha_1^* & \cdots & \alpha_k^* \end{smallmatrix}\right)$ represented in figure 4, where $\alpha_i^* = 1 - \alpha_i$. Call the sequence α *bivalent* if there is an integer ℓ with $1 \leq \ell \leq k$ such that $\alpha_\ell = \alpha_{k+1-\ell} = 1$. Then α represents an element of $\mathcal{G}^{\sigma}(2k+1)$ if and only if $(\alpha_1, \ldots, \alpha_{i-1})$ is bivalent whenever $\alpha_i = 0$.

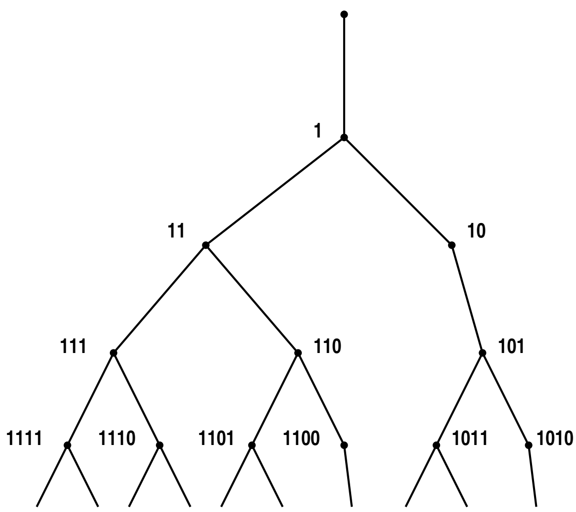

FIGURE 5. *Rooted tree for* $\bigcup\{\mathcal{G}^{\sigma}(2k+1) \mid k \in \mathbb{N}\}$.

If we associate $\alpha = (\alpha_1, \ldots, \alpha_k)$ with the finite set $F(\alpha) = \{0\} \cup \{i \mid \alpha_i = 1\}$ then α is bivalent if and only if $k+1 \in F(\alpha) + F(\alpha)$. Furthermore, α represents an element of $\mathcal{G}^{\sigma}(2k+1)$ precisely when $F(\alpha)$ is an 'additive 2-basis for k' (which means that $[0,k) \subseteq F(\alpha) + F(\alpha)$). Thus $A^{\sigma\prime}_{2k+1}$ equals the number of subsets of $[0,k)$ which are additive 2-bases for k. By theorem 18, $A^{\sigma\prime}_{2k+1} = \gamma^{\sigma}_{k} 2^k$ which is asymptotically equal to $\gamma^{\sigma}_{\infty} 2^k$ where $\gamma^{\sigma}_{\infty} \approx .230653$ as described above. The study of finite additive 2-bases for k has a long history, especially in relation to the determination of bounds for the smallest cardinality of such bases. The introduction of [GN] has a nice overview of this. The first 19 terms of the sequence $\{A^{\sigma\prime}_{2k-1}\}$ have been posted at [S] by M. Torelli as sequence number A008929. The paper [T] describes some related sequences. (In that paper a finite additive 2-basis is called a (finite) 'Goldbach sequence'.)

Let $g^\sigma(z)$ and $f^\sigma(z)$ be the analytic functions defined by

$$g^\sigma(z) = \sum_{k=1}^{\infty} A_k^\sigma z^k \quad \text{and} \quad f^\sigma(z) = \sum_{k=1}^{\infty} A_k^{\sigma\,\prime} z^k \ .$$

Corollary 20. *The functions $f^\sigma(z)$ and $g^\sigma(z)$ satisfy the relation*

$$\left(2z^2 - 1\right) f^\sigma(z) = z(z+1)\left(g^\sigma(z^2) - 1\right) \ .$$

Proof. First observe that

$$f^\sigma(z) = \sum_{k=1}^{\infty} A_{2k-1}^{\sigma\,\prime} z^{2k-1} + \sum_{k=1}^{\infty} A_{2k}^{\sigma\,\prime} z^{2k} = (z+1)\sum_{k=1}^{\infty} A_{2k-1}^{\sigma\,\prime} z^{2k-1} \tag{6}$$

by theorem 19. From the same theorem, $A_k^\sigma = 2A_{2k-1}^{\sigma\,\prime} - A_{2k+1}^{\sigma\,\prime}$ and

$$\begin{aligned} g^\sigma(z^2) &= \sum_{k=1}^{\infty} 2A_{2k-1}^{\sigma\,\prime} z^{2k} - \sum_{k=1}^{\infty} A_{2k+1}^{\sigma\,\prime} z^{2k} \\ &= 2z\sum_{k=1}^{\infty} A_{2k-1}^{\sigma\,\prime} z^{2k-1} - \frac{1}{z}\sum_{k=1}^{\infty} A_{2k-1}^{\sigma\,\prime} z^{2k-1} + A_1^{\sigma\,\prime} \\ &= \left(2z - \frac{1}{z}\right)\sum_{k=1}^{\infty} A_{2k-1}^{\sigma\,\prime} z^{2k-1} + 1 = \frac{2z^2-1}{z(z+1)} f^\sigma(z) + 1. \end{aligned}$$

□

Corollary 21. *The analytic function $f^\sigma(z)$ has singularities at $z = \pm 1/\sqrt{2}$ and its radius of convergence at the origin equals $1/\sqrt{2}$. Except for $z = \pm 1/\sqrt{2}$ and possibly for $z = -1$, the singularities of $f^\sigma(z)$ coincide with those of $g^\sigma(z^2)$ and $f^\sigma(z)(2z^2-1)$ is continuous on the closed disk $|z| \le 1/\sqrt{2}$.*

Proof. Since $g^\sigma(1/2) = \beta_\infty^\sigma$ the power series $\sum_{k=1}^{\infty} A_k^\sigma z^k$ converges absolutely and $g^\sigma(z)$ is continuous on $|z| \le 1/2$. By corollary 20 $f^\sigma(z)$ has singularities at $z = \pm 1/\sqrt{2}$ and $f^\sigma(z)$ has radius of convergence $1/\sqrt{2}$ at the origin. The last property also follows immediately from corollary 20. □

Since $0 < A_k^{\sigma\,\prime} < A_k^\sigma$, the series $\sum_{k=1}^{\infty} A_k^\sigma \left(1/\sqrt{2}\right)^k$ diverges by comparison with $\sum_{k=1}^{\infty} A_k^{\sigma\,\prime} \left(1/\sqrt{2}\right)^k$, and so the radius of convergence of $g^\sigma(z)$ at the origin must be between $1/2$ and $1/\sqrt{2}$. The root test would equate this radius of convergence with the limit infimum of $R_n = 1/\sqrt[n]{A_n^\sigma}$. This value seems to be larger than $1/2$ by the data in the last column of table 2, but we have not been able to ascertain this.

Let $h^\sigma(z) = \sum_{k=1}^\infty \gamma_k^\sigma z^k$ be the generating function for $\{\gamma_k^\sigma\}$. Using equation (6) and corollary 20 we have

$$\begin{aligned} h^\sigma(z) &= \sum_{k=1}^\infty \frac{A_k^{\sigma\prime}}{2^{\lfloor (k-1)/2 \rfloor}} z^k = \sum_{k=1}^\infty \frac{A_{2k-1}^{\sigma\prime}}{2^{k-1}} z^{2k-1} + \sum_{k=1}^\infty \frac{A_{2k}^{\sigma\prime}}{2^{k-1}} z^{2k} \\ &= \sqrt{2}\Big(z+1\Big) \sum_{k=1}^\infty A_{2k-1}^{\sigma\prime} \left(\frac{z}{\sqrt{2}}\right)^{2k-1} = 2\left(z + 1/z + \sqrt{2}\right) f^\sigma \left(z/\sqrt{2}\right) \\ &= \left(\frac{z}{z-1}\right)\left(g^\sigma\left(z^2/2\right) - 1\right). \end{aligned}$$

Therefore $h^\sigma(z)$ has radius of convergence 1 at the origin and has a singularity at $z = 1$. If the radius of convergence of $g^\sigma(z)$ at the origin is larger than $1/2$ then $z = 1$ is the only singularity of $h^\sigma(z)$ inside a circle with radius larger than 1 centered at the origin and $h^\sigma(z)$ would have a simple pole at $z = 1$ with residue

$$\lim_{z\to 1}(z-1)h^\sigma(z) = \lim_{z\to 1} z(g^\sigma(z^2/2) - 1) = g^\sigma(1/2) - 1 = -\gamma_\infty^\sigma\ .$$

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