STRICHARTZ ESTIMATES ON EXTERIOR POLYGONAL DOMAINS

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Abstract. Using a new local smoothing estimate of the first and third authors, we prove local-in-time Strichartz and smoothing estimates without a loss exterior to a large class of polygonal obstacles with arbitrary boundary conditions and global-in-time Strichartz estimates without a loss exterior to a large class of polygonal obstacles with Dirichlet boundary conditions. In addition, we prove a global-in-time local smoothing estimate in exterior wedge domains with Dirichlet boundary conditions and discuss some nonlinear applications.

1. Introduction

In this paper we prove a family of local- and global-in-time Strichartz estimates for solutions to the Schrödinger equation

$$\begin{cases} (D_t + \Delta)\, u(t,x) = 0 \\ \qquad\qquad u(0,x) = f(x), \end{cases} \tag{1}$$

where $D_t = i^{-1}\partial_t$, Δ is the negative definite Laplace-Beltrami operator on domains of the form $X = \mathbb{R}^2 \setminus P$ for P any non-trapping polygonal region such that no three vertices are collinear (as defined in the recent work of the first and last author [2]), and where we take either Dirichlet or Neumann boundary conditions for the Laplacian for the local result and only Dirichlet boundary conditions for the global result.[1] These assumptions and the resulting restrictions on allowed obstacles P are discussed in detail as Assumptions 1, 2 and 3 in Section 2 of [2], to which we refer the reader for more details. The main tools we require for the proof are the local smoothing estimate on such domains (due to the first two authors) and the Strichartz estimates on wedge domains (due to Ford [18]).

We note here that to define the Laplacian, we use the standard Friedrichs extension, which is the canonical self-adjoint extension of a non-negative densely defined symmetric operator as defined in for instance [16, 4]. The Neumann Laplacian is taken to be the usual Friedrichs extension of the Laplace operator acting on smooth

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The first author acknowledges the support of NSF postdoctoral fellowship DMS-1103436, and the third author was partially supported by NSF grant DMS-1001463. The second author was partially supported by an IBM Junior Faculty Development Award through the University of North Carolina and would like to thank the University of Chicago for hosting him during the completion of part of this work. The third author is grateful to Fabrice Planchon and John Stalker for a series of discussions of smoothing and Strichartz estimates on cones over a period of several years. In addition, we thank Fabrice Planchon for a careful reading of the result and providing several references for the scattering theory discussion in the final section.

[1] The essential distinction between the Dirichlet and Neumann cases for our purposes is that low-energy resolvent estimates seem to be readily available in the literature in the Dirichlet case (e.g. in [7]) but not in the Neumann case.

functions which vanish in a neighborhood of the vertices. The Dirichlet Laplacian is taken to be the typical Friedrichs extension of the Laplace operator acting on smooth functions which have compact support contained in X.

We now briefly discuss the geometric restrictions on P needed to apply the results of [2]. In particular, we review the non-trapping assumption on the exterior of P as it is an important restriction in all exterior domain results. Let P be a polygonal domain in $\mathbb{R}^2$, not necessarily connected. A *geometric geodesic* on $\mathbb{R}^2\backslash P$ that does not pass through the vertices of P is defined as a continuous curve that is a concatenation of maximally extended straight line segments in $\mathbb{R}^2\backslash P$, such that on ∂P, successive segments make equal angles with the boundary ("specular reflection"). More generally, a geometric geodesic is one that may pass through the vertices of P in such a way that it is locally a uniform limit of geometric geodesics missing the vertices. This means that in general such a geodesic has two possible continuations each time it hits a vertex, corresponding to taking the limit of families approaching the vertex from the left and right sides.

We let B be a closed ball containing P in its interior. We say that P is *non-trapping* if there exists $T > 0$ such that every geometric geodesic starting in B leaves B in time less than T (this condition is of course independent of the choice of B). We assume henceforth that P is non-trapping.

In addition to assuming that P is non-trapping, we also require the assumption that no three vertices of P are collinear along geometric geodesics. We further remark that the third assumption of [2], requiring that cone points be pairwise non-conjugate, is automatically satisfied for the Euclidean domains under consideration here.

We recall that admissible Strichartz exponents for the Schrödinger equation in dimension $n = 2$ are given by the following:

$$\frac{2}{p} + \frac{2}{q} = 1, \quad p, q \geq 2, \quad (p, q) \neq (2, \infty). \tag{2}$$

We are now ready to state the main result of this note.

Theorem 1. *For any (p, q) satisfying equation (2), there is a constant $C_{p,q,T}$ so that on $I = [0, T]$*

$$\left\| e^{-it\Delta} f \right\|_{L^p(I, L^q(X))} \leq C_{p,q,T} \left\| f \right\|_{L^2(X)}$$

for all $u_0 \in L^2(X)$. If X has Dirichlet boundary conditions, we can take $I = \mathbb{R}$.

Remark 1. Using a now standard application of the Christ-Kiselev lemma [12], we can conclude that for a solution u to the inhomogeneous Schrödinger IBVP

$$\begin{cases} (D_t + \Delta)\, u(t, x) = F(t, x) \\ u(0, x) = f(x) \end{cases} \tag{3}$$

satisfying either Dirichlet or Neumann homogeneous boundary conditions, the estimate

$$\|u\|_{L^{p_1}(I; L^{q_1}(X))} \leq C \left(\|f\|_{L^2(X)} + \|F\|_{L^{p_2'}(I; L^{q_2'}(X))} \right) \tag{4}$$

holds for $\frac{2}{p_j} + \frac{2}{q_j} = 1$ for $j = 1, 2$. Here, $(\cdot)'$ denotes the dual exponent, e.g. $\frac{1}{p_1} + \frac{1}{p_1'} = 1$.

2. Global Strichartz Estimates for the Model Problems

The proof of the theorem relies on Strichartz estimates on $\mathbb{R}^2$, as well as Strichartz estimates on a two-dimensional cone, which we recall the result here for completeness. Here and in what follows, we denote by $C(\mathbb{S}^1_\rho)$ the cone over the circle of circumference ρ, equipped with the conic metric $dr^2 + r^2\, d\theta^2$.

Theorem 2 (Strichartz estimates on $\mathbb{R}^2$, $C(\mathbb{S}^1_\rho)$; see, e.g., Keel–Tao [22] and Ford [18])**.** *Suppose that (p, q) and $(\tilde{p}, \tilde{q})$ are admissible Strichartz exponents in the sense of equation (2). If u is a solution to the Schrödinger equation*

$$\begin{cases} (D_t + \Delta_Y)\, u(t, x) = F(t, x), \\ \qquad\qquad u(0, x) = f(x), \end{cases}$$

on $Y = \mathbb{R}^2$ or $C(\mathbb{S}^1_\rho)$, then

$$\|u\|_{L^p(\mathbb{R}; L^q(Y))} \leq C(\|f\|_{L^2} + \|F\|_{L^{\tilde{p}'}(\mathbb{R}; L^{\tilde{q}'}(Y))}),$$

where $\tilde{p}'$ and $\tilde{q}'$ are the conjugate exponents to $\tilde{p}$ and $\tilde{q}$, respectively.

Remark 2. Our results are closely related to the work on smoothing and Strichartz estimates for potentials with inverse-square singularities by Burq–Planchon–Stalker–Tahvildar-Zadeh and Planchon–Stalker–Tahvildar-Zadeh in [10] and [28]. For work on smoothing estimates for the Schrödinger equation in *smooth* exterior domains, we refer the reader to the early works of Burq [8], as well as Burq–Gerard–Tzvetkov [9] and Anton [1] who constructed parametrices for exterior domain problems that proved Strichartz estimates with errors controlled by local smoothing estimates. Local smoothing results were later extended by Robbiano–Zuily [31] to include quadratic potential wells. Scale invariant Strichartz estimates for exterior domains first appeared in Planchon–Vega [29] and Blair–Smith–Sogge [6], though not for the full range of admissible Strichartz pairs. For Strichartz estimates exterior to a smooth, convex obstacle however, scale invariant estimates have been established in the full range of estimates in Ivanovici [20], Ivanovici–Planchon [21], and Blair [3].

Remark 3. Strichartz estimates exist for the wave equation on both $\mathbb{R}^2$ and $C(\mathbb{S}^1_\rho)$, but the analog of Theorem 1 for the wave equation on exterior domains can be directly computed from the analysis done by Blair, Ford and the second author in [5] due to the finite speed of propagation. Hence, quantifying the effects of diffraction as in [2] plays a much larger role in Schrödinger dynamics than in the corresponding wave dynamics on such domains.

3. Local Smoothing Estimates for X and on Euclidean Cones

The proof of Theorem 1 will also rely upon local smoothing estimates for $\mathbb{R}^2$, $C(\mathbb{S}^1_\rho)$ as well as on the space X in order to glue together similar dispersive results on model problems. We begin with the local result that is independent of choice of boundary conditions for X as a consequence of being local-in-time (and thus requiring no low-energy resolvent estimates):

Theorem 3 ("Local" local smoothing estimate; see B.–W. [2])**.** *If X is a domain exterior to a non-trapping polygon, u is a solution of the Schrödinger equation*

$$\begin{cases} D_t u(t, z) + \Delta u(t, z) = 0, \\ \qquad u(0, z) = f(z), \end{cases}$$

with Dirichlet or Neumann boundary conditions and $\chi \in C^\infty(X)$ *is a smooth, compactly supported function, then* u *satisfies a local smoothing estimate*

$$\|\chi u\|_{L^2([0,T];\mathcal{D}_{1/2})} \leq C_T \|f\|_{L^2},$$

where $\mathcal{D}_{1/2}$ *is the domain of* $(-\Delta)^{1/4}$*.*

In the case of Dirichlet boundary conditions, we can significantly strengthen the above result to gain global control over the local smoothing norm.

Theorem 4 ("Global" local smoothing estimate)**.** *If* X *is a domain exterior to a non-trapping polygon,* u *is a solution of the Schrödinger equation*

$$\begin{cases} D_t u(t,z) + \Delta u(t,z) = 0, \\ u(0,z) = f(z), \end{cases}$$

with Dirichlet boundary conditions and $\chi \in C^\infty(X)$ *is a smooth, compactly supported function, then* u *satisfies a local smoothing estimate*

$$\|\chi u\|_{L^2(\mathbb{R};\mathcal{D}_{1/2})} \leq C \|f\|_{L^2},$$

where $\mathcal{D}_{1/2}$ *is the domain of* $(-\Delta)^{1/4}$*.*

Remark 4. Note that Theorems 3, 4 imply the dual estimate

$$\left\| \int_I e^{is\Delta} \chi F(s)\, ds \right\|_{L^2} \leq C \|F\|_{L^2(\mathbb{R};\mathcal{D}_{-1/2})}, \tag{5}$$

for I either $[0,T]$ or $\mathbb{R}$ respectively.

Proof. We rely on the high-frequency resolvent estimates of the first and third authors [2], estimates due to Morawetz [26] for intermediate frequencies and Burq [7] for small frequencies, then apply a now-standard TT^* argument.

Consider the operator $Tu_0 = \chi e^{-it\Delta} u_0$. We wish to show that T is a bounded operator from $L^2(X)$ to $L^2(\mathbb{R};\mathcal{D}_{1/2})$. It suffices to show that TT^* is bounded from $L^2(\mathbb{R};\mathcal{D}_{-1/2})$ to $L^2(\mathbb{R};\mathcal{D}_{1/2})$. The operator TT^* is given by

$$\begin{aligned} TT^* f &= \chi \int_{\mathbb{R}} e^{-i(t-s)\Delta} \chi f(s)\, ds \\ &= \chi \int_{s<t} e^{-i(t-s)\Delta} \chi f(s)\, ds + \chi \int_{s>t} e^{-i(t-s)\Delta} \chi f(s)\, ds = \chi T_+ f + \chi T_- f. \end{aligned}$$

Observe that $T_\pm f$ are both solutions of the inhomogeneous Schrödinger equation

$$D_t u + \Delta u = \frac{1}{i} \chi f.$$

Suppose for now that f is compactly supported in time, i.e., $f(t,x) = 0$ for $t \notin [-t_0, t_0]$. In this case, $T_+ f$ vanishes for $t < -t_0$ and $T_- f$ vansishes for $t > t_0$.

We wish to show that there is a constant C, independent of t_0, so that

$$\int_{\mathbb{R}} \|\chi T_\pm f(t,x)\|^2_{\mathcal{D}_{1/2}}\, dt \leq C \int_{\mathbb{R}} \|f(t,x)\|^2_{\mathcal{D}_{-1/2}}\, dt.$$

By Plancherel's theorem, it suffices to show that

$$\int_{\mathbb{R}} \left\| \chi \widehat{T_\pm f}(E,x) \right\|^2_{\mathcal{D}_{1/2}} dE \leq C \int_{\mathbb{R}} \left\| \hat{f}(E,x) \right\|^2_{\mathcal{D}_{-1/2}} dE,$$

where $\hat{f}$ denotes the Fourier transform of f in t.

Observe that $\widehat{T_\pm f}(E,x)$ solves

$$(\Delta + E)\widehat{T_\pm f} = \frac{1}{i}\chi \hat{f}.$$

Moreover, the condition on the support of f implies that $\widehat{T_+ f}$ is holomorphic in the lower half-plane, while $\widehat{T_- f}$ is holomorphic in the upper half-plane. In particular, if $R(z) = (\Delta + z)^{-1}$ where it is invertible,

$$\widehat{T_\pm f}(E,x) = \lim_{\beta \downarrow 0} R(E \mp i\beta)\left(\frac{1}{i}\chi \hat{f}(E,x)\right).$$

We must thus estimate $\chi R(E \mp i0)\chi$ as an operator $\mathcal{D}_{-1/2} \to \mathcal{D}_{1/2}$. The high-frequency estimates from [2] imply that there is some E_0 so that for $E \geq E_0$,

$$\|\chi R(E \mp i0)\chi\|_{L^2 \to L^2} \leq \frac{C}{\sqrt{E}}.$$

Using this bound and the identity

$$\Delta \chi R(E \pm i\beta) = \chi - (E \pm i\beta)\chi R(E \pm i\beta) + [\Delta, \chi]R(E \pm i\beta)$$

yields the following high-energy estimate for the resolvent:

$$\|\chi R(E \pm i0)\chi\|_{L^2 \to \mathcal{D}_2} \leq C\sqrt{E}.$$

Interpolating the two estimates shows that $R(E \pm i0)$ is bounded (with uniform bound for $E \geq E_0$) as an operator from L^2 to $\mathcal{D}_1$ and thus from $\mathcal{D}_{-1/2}$ to $\mathcal{D}_{1/2}$.

The argument of Morawetz [26, Lemmas 15 and 16], which remains valid in our setting, shows that the same bound holds at intermediate energies as well. For uniform bounds down to $E = 0$, we rely upon an argument of Burq [7, Appendix B.2].

We now apply the resolvent estimates, which shows that there is a constant C independent of λ and t_0 so that

$$\left\| \chi \widehat{T_\pm f(E}, x) \right\|^2_{\mathcal{D}_{1/2}} \leq C \|f\|^2_{\mathcal{D}_{-1/2}}.$$

Integrating in E then finishes the proof in the compactly supported setting. For the general setting, we simply note that the constant is independent of the support and that compactly supported functions are dense in L^2_t. □

We will need one result that we have not been able to find explicitly in the literature, but whose proof uses standard methods. This result concerns global-in-time local smoothing for the Schrödinger equation on an infinite wedge domain, which of course serves as a local model for our polygon near a vertex and is equivalent to $C(\mathbb{S}^1_\rho)$ (Cf. [19], for instance). Let $X_\rho = \{\theta \in [0, \rho/2]\} \subset \mathbb{R}^2$ for $\rho \in [0, 4\pi)$.

Lemma 1. *A solution to the Schrödinger equation on X_ρ with Dirichlet or Neumann conditions satisfies the local smoothing estimates of Theorem 4. Consequently, the dual estimate* (5) *is satisfied on X_ρ as well.*

Proof. We first note that solutions to the Schrödinger equation on X_ρ with Dirichlet boundary conditions are equivalent, by extending in an odd manner to the cone over the circle of circumference ρ obtained by "doubling" the wedge X_ρ. That is to say, we may identify solutions to the Schrödinger equation on $\mathbb{R} \times X_\rho$ to solutions on the "edge manifold" $\mathbb{R} \times C(\mathbb{S}^1_\rho)$, where as usual $C(\mathbb{S}^1_\rho)$ has the metric $ds^2 = dr^2 + r^2 d\theta^2$

with $\theta \in \mathbb{S}^1_\rho$. We make this identification by extending the solutions to be odd or even under the involution

$$\mathbb{S}^1_\rho \ni \theta \to \rho - \theta$$

according to the choice of boundary condition. Thus, it will suffice to consider solutions to the Schrödinger equation on the cone,

$$\begin{cases} \left(D_t + \partial_r^2 + r^{-1}\partial_r + r^{-2}\Delta_{\mathbb{S}^1_\rho}\right) u(t,x) = 0, \\ u(0,x) = f(x). \end{cases} \tag{6}$$

We refer the reader to [18, 4, 5] as well as [24] for a discussion of Sobolev spaces on cones and the nature of the operator $\Delta_{C(\mathbb{S}^1_\rho)}$. In particular, we briefly recall the characterization given in [24] of the nature of the domains of powers of the Friedrichs Laplacian on the cone.[2] First we recall the definition of b-vector fields and operators. The space of *b-vector fields*, denoted $\mathcal{V}_b(C(\mathbb{S}^1_\rho))$ is the vector space of vector fields on $[0,\infty) \times \mathbb{S}^1_\rho$ tangent to $0 \times \mathbb{S}^1_\rho$. In local coordinates (r,θ) near ∂M, they are generated over $C^\infty([0,\infty) \times \mathbb{S}^1_\rho)$ by the vector fields $r\partial_r$ and ∂_θ. One easily verifies that $\mathcal{V}_b(C(\mathbb{S}^1_\rho))$ forms a Lie algebra. The set of b-differential operators, $\mathrm{Diff}^*_{\mathrm{b}}(C(\mathbb{S}^1_\rho))$, is the universal enveloping algebra of this Lie algebra: it is the filtered algebra consisting of operators of the form

$$A = \sum_{|\alpha|+j \leq m} a_{j,\alpha}(r,\theta)(rD_r)^j D_\theta^\alpha \in \mathrm{Diff}^m_{\mathrm{b}}(C(\mathbb{S}^1_\rho)). \tag{7}$$

Now let $L^2_b(C(\mathbb{S}^1_\rho))$ be the space of square-integrable functions with respect to the "b-density" $r^{-1}\,dr\,d\theta$. We define the b-Sobolev spaces $H^m_{\mathrm{b}}(C(\mathbb{S}^1_\rho))$ for $m \in \mathbb{N}$ as

$$\left\{u : Au \in L^2_b(C(\mathbb{S}^1_\rho)) \text{ for } A \in \mathrm{Diff}^m_{\mathrm{b}}(C(\mathbb{S}^1_\rho))\right\}.$$

This definition can be extended to a definition of $H^s_{\mathrm{b}}(C(\mathbb{S}^1_\rho))$ for $s \in \mathbb{R}$ by interpolation and duality, or, better yet, by developing the b-pseudodifferential calculus as in [25].

Finally, we recall that Lemma 3.2 of [24] tells us that if we let $\mathcal{D}_s$ denote the domain of $\Delta^{s/2}_{C(\mathbb{S}^1_\rho)}$ (again, with $\Delta_{C(\mathbb{S}^1_\rho)}$ denoting the Friedrichs Laplacian) then

$$\mathcal{D}_s = r^{-1+s} H^s_{\mathrm{b}}(C(\mathbb{S}^1_\rho)), \quad |s| < 1.$$

This identification does break down at $s = 1$—see §3 of [25] for details.

Finally we are ready to prove a local smoothing estimate. We will prove an estimate of the form

$$\int_0^\infty \left(\left\| \langle r\rangle^{-\frac{3}{2}} \partial_r u \right\|^2_{L^2} + \left\| r^{-\frac{3}{2}} \sqrt{-\Delta_{\mathbb{S}^1_\rho}}\, u \right\|^2_{L^2} \right) dt \leq C \left\| f \right\|^2_{\mathcal{D}_{1/2}}. \tag{8}$$

Taken together with its time-reversed version, this will yield the estimate

$$\|\chi u\|_{L^2(\mathbb{R};\mathcal{D}_1)} \leq C \|f\|_{\mathcal{D}_{1/2}}$$

[2] As discussed in Remark 1.2[4] we take the Neumann Laplacian on the original planar domain to be the Friedrichs extension from smooth functions vanishing at the vertex or vertices, satisfying Neumann conditions at edge; thus, upon doubling to a cone, we are working with the Friedrichs extension of the Laplacian on smooth functions compactly supported away from the cone tip.

(indeed it is somewhat stronger than the needed estimate, being global in space, with weights, rather than local). Now to obtain the estimate stated in the theorem, we simply shift Sobolev exponents by applying this estimate to the solution $\langle\Delta\rangle^{-1/4}u$. Thus to prove the lemma it will suffice to obtain (8).

To do this, we separate variables and treat the small and large angular frequencies separately.

Let us first introduce the commutants A_0 and B_0, given by

$$A_0 = \partial_r, \quad B_0 = \frac{r}{\langle r\rangle}\partial_r$$

and observe that acting on smooth functions compactly supported away from the cone tip,

$$\begin{aligned}
[A_0, -\Delta_{C(\mathbb{S}^1_\rho)}] &= \frac{1}{r^2}\partial_r + \frac{2}{r^3}\partial_\theta^2,\\
[B_0, -\Delta_{C(\mathbb{S}^1_\rho)}] &= 2\left(\frac{1}{\langle r\rangle^3}\partial_r^2 + \frac{1}{r^2\langle r\rangle}\partial_\theta^2\right) + \frac{1}{r\langle r\rangle}\left(1 + \frac{1}{\langle r\rangle^2} - \frac{3r^2}{\langle r\rangle^4}\right)\partial_r.
\end{aligned}$$

Note that the formal adjoints (with respect to the usual volume form) are given by

$$\begin{aligned}
A_0^* &= -\partial_r - \frac{1}{r},\\
B_0^* &= -\frac{r}{\langle r\rangle}\partial_r - \frac{2+r^2}{\langle r\rangle^3}.
\end{aligned}$$

Now setting $A = (A_0 - A_0^*)/2$ and $B = (B_0 - B_0^*)/2$, we have

$$\begin{aligned}
\left[A, -\Delta_{C(\mathbb{S}^1_\rho)}\right] &= \frac{2}{r^3}\partial_\theta^2 + \frac{1}{2r^3},\\
\left[B, -\Delta_{C(\mathbb{S}^1_\rho)}\right] &= -2\partial_r^*\langle r\rangle^{-3}\partial_r + 2\langle r\rangle^{-1}r^{-2}\partial_\theta^2 + g,
\end{aligned} \tag{9}$$

with

$$g(r) = \frac{r^4 + 8r^2 - 8}{2\langle r\rangle^7}.$$

Now consider a function $u \in \mathcal{C}^\infty(\mathbb{R}; \mathcal{C}_c^\infty((0,\infty)\times\mathbb{S}^1_\rho)$ with

$$(D_t + \Delta_{C(\mathbb{S}^1_\rho)})u = F, \ u(0,x) = f. \tag{10}$$

We will separate $u(t)$ in angular modes, preserving a single high-angular-frequency component and separating out all low-angular-frequency components: given J, we let $e_j(\theta) = e^{2\pi i j\theta/\rho}$ and we Fourier analyze u into

$$u = \sum_j a_j(t,r)e_j(\theta) \equiv \sum u_j(t,r,\theta).$$

We then split

$$u = \tilde{u} + \sum_{|j|<J} u_j(t,r)$$

with

$$\tilde{u} = \sum_{|j|\geq J} a_j(t,r)e_j(\theta)$$

denoting the high angular frequencies. Let $\widetilde{F}$, $\tilde{f}$ etc. denote the corresponding decompositions of F, f.

Now for j sufficiently large,

$$\left\langle -\partial_\theta^2 u_j, u_j \right\rangle \gg 1,$$

and hence by (9), there exists $c > 0$ with

$$\left\langle [B, \Delta_{C(\mathbb{S}^1_\rho)}]\tilde{u}, \tilde{u} \right\rangle \geq c\big(\left\| \langle r \rangle^{-\frac{3}{2}} \partial_r u \right\|^2_{L^2} + \left\| r^{-\frac{3}{2}} \sqrt{-\Delta_{\mathbb{S}^1_\rho}} u \right\|^2_{L^2} \big). \tag{11}$$

Thus, we compute

$$i^{-1}\partial_t \langle B\tilde{u}, \tilde{u} \rangle = \left\langle [B, -\Delta_{C(\mathbb{S}^1_\rho)}]\tilde{u}, \tilde{u} \right\rangle + 2\,\mathrm{Im} \left\langle \widetilde{F}, B\tilde{u} \right\rangle.$$

Integration then yields

$$\int_0^T \langle [B, \Delta_{C(\mathbb{S}^1_\rho)}]\tilde{u}, \tilde{u} \rangle dt \leq |\langle B\tilde{u}(T), \tilde{u}(T) \rangle| + |\langle B\tilde{u}(0), \tilde{u}(0) \rangle| + 2 \int_0^T \left| \left\langle \widetilde{F}, B\tilde{u} \right\rangle \right| dt.$$

Since $B \in \mathrm{Diff}^1_{\mathrm{b}}(C(\mathbb{S}^1_\rho))$, we certainly have

$$B : \mathcal{D}_{1/2} \to \mathcal{D}_{-1/2} = \mathcal{D}^*_{1/2},$$

by the identification $\mathcal{D}_{\pm 1/2} = r^{-1 \pm 1/2} H^{\pm 1/2}_{\mathrm{b}}(C(\mathbb{S}^1_\rho))$. Thus, for all T,

$$\int_0^T \big(\left\| \langle r \rangle^{-\frac{3}{2}} \partial_r u \right\|^2_{L^2} + \left\| r^{-\frac{3}{2}} \sqrt{-\Delta_{\mathbb{S}^1_\rho}} u \right\|^2_{L^2} \big)\, dt \leq c \int_0^T \|u\|_{\mathcal{D}_{1/2}} \|F\|_{\mathcal{D}_{1/2}}\, dt + \|f\|^2_{\mathcal{D}_{1/2}} + \|u(T)\|^2_{\mathcal{D}_{1/2}}.$$

An elementary energy estimate for the inhomogeneous equation shows that since $e^{-it\Delta} : \mathcal{D}_s \to \mathcal{D}_s$ for all s, for all $t > 0$ we have

$$\|u(t)\|_{\mathcal{D}_{1/2}} \leq \|f\|_{\mathcal{D}_{1/2}} + \int_0^\infty \|F(s)\|_{\mathcal{D}_{1/2}}\, ds.$$

Thus for all $T > 0$,

$$\int_0^T \big(\left\| \langle r \rangle^{-\frac{3}{2}} \partial_r u \right\|^2_{L^2} + \left\| r^{-\frac{3}{2}} \sqrt{-\Delta_{\mathbb{S}^1_\rho}} u \right\|^2_{L^2} \big)\, dt \leq c\big(\int_0^\infty \|F\|_{\mathcal{D}_{1/2}} \big)^2 + c\|f\|^2_{\mathcal{D}_{1/2}}, \tag{12}$$

with the constant c independent of T. Thus, the map

$$(f, F) \in \mathcal{D}_{1/2} \oplus L^1(\mathbb{R}; \mathcal{D}_{1/2}) \to u$$

extends by continuity to yield (12) in particular[3] with any $f \in \mathcal{D}_{1/2}$ and $F = 0$. Letting $T \to \infty$, this is the desired estimate for the $\tilde{u}$ term.

We now turn to the u_j terms, for which we will need the commutant A as well. We treat the cases $j \neq 0$ and $j = 0$ separately. For $j \neq 0$ we note that $\left\langle \partial_\theta^2 e_j, e_j \right\rangle < -1/4$ owing to our assumption that $\rho < 4\pi$. Thus

$$\left\langle [A, -\Delta_{C(\mathbb{S}^1_\rho)}]u_j, u_j \right\rangle \geq c(\left\| r^{-3/2} \partial_\theta u_j \right\|^2_{L^2} + \left\| r^{-3/2} u_j \right\|^2_{L^2}).$$

[3]The reader may note that in obtaining these estimates we have not pursued optimality in F : the solution u should be one derivative more regular. We have avoided this issue owing to the breakdown of the identification of the domain $\mathcal{D}_1$ with a weighted b-Sobolev space (principally relevant in our analysis of the zero-angular-mode below). We of course obtain the correct mapping properties of the inhomogeneous Schrödinger equation from $L^2\mathcal{D}_{-1/2}$ to $L^2\mathcal{D}_{1/2}$ ex post facto by duality.

Consequently, using the commutant $B + \epsilon^{-1}A$ in the argument above with ϵ sufficiently small formally yields the desired estimate on u_j, for $j \neq 0$. Note that it is essential that $A \in r^{-1}\mathrm{Diff}^1_{\mathrm{b}}(C(\mathbb{S}^1_\rho))$, hence $A : \mathcal{D}_{1/2} \to \mathcal{D}_{-1/2} = \mathcal{D}^*_{1/2}$ so that we may proceed just as above.

Finally, we deal with $j = 0$. Here we employ the commutant $B - \epsilon^{-1}A$, where we remark that on u_0, a crucial sign flips and we have

$$\left\langle [A, -\Delta_{C(\mathbb{S}^1_\rho)}]u_0, u_0 \right\rangle \leq -c(\left\| r^{-3/2}\partial_\theta u_0 \right\|^2_{L^2} + \left\| r^{-3/2}u_0 \right\|^2_{L^2})$$

(where of course we really have $\partial_\theta u_0 = 0$). □

Remark 5. The above proof, while appealingly simple, does not extend to the case of a slit obstacle, i.e. $\rho = 4\pi$ or indeed to any product cone in which $-1/4$ is in the spectrum of the Laplacian on the link. However, addition of a third operator,

$$W_0 = f(r)\partial_r,$$

with f to be determined allows these cases to be handled in the same fashion as above. We calculate in general:

$$W_0^* = -W_0 - \frac{f(r)}{r} - f'(r),$$

giving the multiplier

$$W = \frac{W_0 - W_0^*}{2} = f(r)\partial_r + \frac{f(r)}{2r} + \frac{f'(r)}{2}.$$

Thus,

$$\left\langle [W, -\Delta_{C(\mathbb{S}^1_\rho)}]u, u \right\rangle = -\langle 2f'(r)\partial_r u, \partial_r u\rangle + \left\langle \frac{f''(r)}{r}u, u \right\rangle + \left\langle \frac{f'''(r)}{2}u, u \right\rangle - \left\langle \frac{f'(r)}{2r^2}u, u \right\rangle.$$

Consequently, provided

- $f \in C^3$ with uniformly bounded derivatives,
- $f' < 0$,
- $f''(r)/r + f'''(r)/2 - f'(r)/(2r^2) > 0$

we obtain an estimate. In particular, taking $f(s) = (1 + r)^{-\frac{1}{2}}$ gives a positive operator satisfying bounds as in (8) with slightly different weights,

$$\int_0^\infty \left(\left\| (1+r)^{-\frac{3}{4}}\partial_r u \right\|^2_{L^2} + \left\| r^{-1}\langle r\rangle^{-\frac{3}{4}}\sqrt{-\Delta_{\mathbb{S}^1_\rho}}u \right\|^2_{L^2} \right) dt \leq C \left\| f \right\|^2_{\mathcal{D}_{1/2}} \tag{13}$$

on this mode. Note, we have made no attempt to optimize weights here.

4. Proof of Theorem 1

We are now ready to prove the main result, which follows from the same arguments whether $I = [0, T]$ or $\mathbb{R}$. Suppose that the polygon is contained in a ball of radius R_0 in $Z_0 = \mathbb{R}^2$ and let $U_0 = \mathbb{R}^2 \setminus \overline{B(0, R_0)}$. For each vertex of the polygon, we let U_j be a neighborhood of that vertex in X so that U_j can be considered as a neighborhood of the cone point in a wedge domain Z_j given by $\{\theta \in [0, \rho/2]\} \subset \mathbb{R}^2$. We may assume that the union of the U_j covers $X \setminus U_0$. Let $\chi_0, \chi_1, \ldots, \chi_N$ be a partition of unity subordinate to this cover of X.

Set u to be the solution of the Schrödinger equation with initial data f, i.e.,

$$\begin{cases} (D_t + \Delta)\, u = 0, \\ \qquad u(0, z) = f(z). \end{cases}$$

Consider now $u_j = \chi_j u_j$. Note that u_j solves the following inhomogeneous Schrödinger equation on Z_j:

$$\begin{cases} D_t u_j + \Delta u_j = [\Delta, \chi_j] u, \\ \qquad u_j(0, z) = \chi_j(z) f(z). \end{cases}$$

We write $u_j = u_j' + u_j''$, where u_j' is the solution of the homogeneous equation on Z_j with the same initial data and u_j'' is the solution of the inhomogeneous equation with zero initial data. We know by [22] (for Z_0) and by [18] (for Z_j, $j > 0$) that u_j satisfies the homogeneous Strichartz estimate.

We now set $v_j(t, z) = [\Delta, \chi_j] u$. Then by Duhamel's Principle,

$$u_j'' = \int_0^t e^{-i(t-s)\Delta_{Z_j}} v_j(s)\, ds.$$

Note that $[\Delta, \chi_j]$ is a compactly supported differential operator of order 1 supported away from the vertices and so the local smoothing estimate for X implies that there is a constant C so that

$$\|v_j\|_{L^2(\mathbb{R}, \mathcal{D}_{-1/2})} \leq C \|f\|_{L^2(X)}.$$

We wish to show that u_j'' obeys the Strichartz estimates. As we are assuming that $p > 2$, the Christ–Kiselev lemma [12] implies that it is enough to show the estimate for

$$\int_{\mathbb{R}} e^{-i(t-s)\Delta_{Z_j}} v_j(s)\, ds = e^{-it\Delta_{Z_j}} \int_{\mathbb{R}} e^{is\Delta_{Z_j}} v_j(s)\, ds.$$

By the dual local smoothing estimate for Z_j from Lemma 1,

$$\left\| \int_{\mathbb{R}} e^{is\Delta_{Z_j}} v_j(s) \right\|_{L^2} \leq C \|v_j\|_{L^2(\mathbb{R}; \mathcal{D}_{-1/2})} \leq C \|f\|_{L^2}.$$

Applying the homogeneous Strichartz estimate for the propagator then finishes the proof.

5. A brief comment on nonlinear applications

Let $X = \mathbb{R}^2 \setminus P$, for P a non-trapping polygonal domain with either Dirichlet or Neumann boundary conditions on each edge. We use our loss-less local in time Strichartz estimates to extend well-posedness results to problems of the form

$$D_t u + \Delta u = \beta(|u|^2) u, \quad u_0 \in H^s(X) \tag{14}$$

for some $s \geq 0$, where β is any polynomial such that $\beta(0) = 0$. For a more general set of assumptions on β, see for instance the treatments in the books of Cazenave [11] and Tao [35] and the references therein, as well as a succinct exposition in the recent survey article of D'Ancona [15]. Largely, what follows will mirror the discussions in $\mathbb{R}^2$ from [11]. The arguments throughout implicitly rely upon Sobolev embeddings and Gagliardo-Nirenberg inequalities extending from $\mathbb{R}^2$ to X. Note, the nonlinear Schrödinger equation (14) has conservation of mass

$$M(u) = \|u\|_{L^2(X)} \tag{15}$$

$$= M(u_0) \tag{16}$$

and conservation of energy

$$E(u) = \|\nabla u\|_{L^2(X)} + \int_X B(u^2) d\mathrm{Vol}_X \tag{17}$$
$$= E(u_0),$$

where

$$B(z) = \int_0^z \beta(y) dy.$$

These conservation laws that are quite useful for the study of well-posedness in that they allow one to control the L^2 and H^1 norms of the solution for a variety of nonlinearities β.

5.1. **Local Nonlinear Results.** The main results are of the following form.

Theorem 5. *Let $0 \leq \sigma < 1$. Given $\|u_0\|_{H^\sigma} < M$, there exists a $T_{max}(M)$, $T_{min}(M) \leq \infty$ such that (14) has a unique solution $u \in C((-T_{min}, T_{max}), H^s) \cap L^q_{(-T_{min},T_{max})} L^r(X)$ with continuous dependence upon the initial data. In addition, if T_{max} (T_{min}) $< \infty$, then $\|u\|_{H^s} \to \infty$ as $t \to T_{max}$ (T_{min}). In particular, for L^2-subcritical nonlinearities, $T_{\max} = T_{\min} = \infty$.*

Proof. We discuss the proof in some special cases, citing proper references for further details.

Given the conservation laws, if $\sigma = 1$, the results follow from careful estimates using the density of smooth functions in $H^1(X)$ (see Theorem 3.3.5 in [11]). Hence, the primary contribution to well-posedness theory easily derived from the Strichartz estimates is the component of uniqueness of the evolution (see Theorems 4.3.1 and 3.3.9 in [11]).

Lemma 2. *Given the assumptions on β, there exist ρ_1, $\rho_2 \in [2, \infty)$ such that for each $0 < M < \infty$, there exists $C(M) < \infty$ such that*

$$\|\beta(|u|^2)u - \beta(|v|)^2 v\|_{L^{\rho_1'}} \leq C(M)\|u - v\|_{L^{\rho_2}}$$

for all $u, v \in H^1(X)$ such that $\|u\|_{H^1}, \|v\|_{H^1} \leq M$. As a result, if u and v are weak solutions of (14) on a time interval I with initial data $u_0 \in H^1$, then $u = v$.

The lemma follows by applying the Strichartz estimates to the equation for $u-v$ to obtain an estimate

$$\|u - v\|_{L^q L^r} \leq C\|u - v\|_{L^{q'} L^r} \tag{18}$$

for (q, r) an admissible Strichartz pair and the observation using Hölder's inequality that if for $0 \in I$, a finite interval,

$$\|f\|_{L^b(I)} \leq C\|f\|_{L^a(I)}.$$

with $b > a$, then $f = 0$ almost everywhere in I.

Let us now consider $\sigma = 0$. Let us take

$$\beta(z^2) = g z^{\rho-1}, \tag{19}$$

for $\rho > 1$, $g \in \mathbb{R}$. Since we are in two dimensions, for $1 < \rho < 3$, using Theorem 1 and (4), we are able to apply standard bootstrapping arguments to such nonlinear problems, which we include here for completeness. We take instead of (14) the corresponding integral equation

$$S(u) = e^{-it\Delta}u_0 + i\int_0^t e^{-i(t-s)\Delta}\beta(|u|^2)u(s)ds,$$

which we show serves as a contraction mapping $S : Y_T \to Y_T$, for

$$Y_T = C([-T,T], L^2(X)) \cap L^p([-T,T], L^q(X)$$

with p, q an allowed Strichartz pair. Indeed, from (4) and Hölder's inequality as the equation is by scaling invariance computed to be sub-L^2-critical there exists p_1, q_1 Strichartz pairs such that

$$\begin{cases} p_1'\rho < p, \ q_1'\rho = q, \\ \frac{2}{p} + \frac{2}{q} = 1, \ \frac{2}{p_1} + \frac{2}{q_1} = 1, \\ p, p_1 \in [2,\infty], \ q, q_1 \in [2,\infty) \end{cases}$$

and

$$\|S(u)\|_{Y_T} \leq C(\|u_0\|_{L^2} + T^\rho \|u\|_{Y_T}),$$

$$\|S(u) - S(v)\|_{Y_T} \leq \tilde{C}T^\lambda \|u - v\|_{Y_T} \||u| + |v|\|_{Y_T}^{\rho-1}.$$

See, for instance, the discussion in [11], Section 4.6 for a detailed example of how to apply the bootstrapping principle once such bounds of the solution map are proved. Hence, for each $u_0 \in L^2$, there exists a time interval T and an upper bound U such that given $u, v \in B(0,U) \subset Y_T$, we have

$$S : B(0,U) \to B(0,U),$$

$$\|S(u) - S(v)\|_{Y_T} \leq \frac{1}{2}\|u - v\|_{Y_T}.$$

For $0 < \sigma < 1$, in $\mathbb{R}^2$, interpolative results up the H^s critical exponent $\rho = 1 + 2/(1-\sigma)$ hold in Besov type spaces in which it is simple to take advantage of the Sobolev embeddings in H^s, see for instance Section 4.9 of [11]. Such spaces can be defined on $C(\mathbb{S}^1_\rho)$, but doing so goes beyond the scope of this note. □

5.2. **Global Nonlinear Results.** Let us take β as in (19) for simplicity and assume Dirichlet boundary conditions. A remaining open problem is to determine how much of the above linear analysis can be extended to domains X with Neumann boundary conditions; this would permit us to address global nonlinear questions in the Neumann case as well.

For the L^2 sub-critical case $1 < \rho < 3$, the natural L^2 conservation gives global well-posedness in L^2 for such equations by simply iteration of the argument over this uniform time interval. Up to this point, our analysis has cared very little about the leading order sign in the nonlinearity, which is generally irrelevant to finite time results. For the L^2 critical/supercritical case ($3 \leq \rho < \infty$), however, one must rely upon the natural H^1 conservation laws of such a system; such results are known to hold provided the nonlinearity is defocusing ($g > 0$)—see [11], Theorem 6.1.1, taking $\sigma = 1$.

For the defocusing case ($g > 0$), global well-posedness in H^1 holds immediately for any initial data for all powers of the nonlinearity since the positive conserved energy in (17) means that the H^1 norm can bounded for all time at any scale.

For the focusing case ($g < 0$), global well-posedness is a subtle phenomenon owing to the potential existence of nonlinear bound states. With sufficient regularity in weighted spaces, a calculation by Weinstein [36] showed that in the case $\rho = 3$ on $\mathbb{R}^2$ there is a finite threshold of L^2 mass below which a solution exists for all time. The threshold is related to a nonlinear bound state that gives an optimal constant for the Gagliardo-Nirenberg inequality

$$\|f\|_{L^p(X)} \leq C_{opt}(X) \|u\|^{\alpha(p)}_{\dot{H}^1(X)} \|u\|^{1-\alpha(p)}_{L^2(X)},$$

where $p < \infty$ in two dimensions.[4] Following [11], Theorem 6.1.1, if one can use the conservation of energy and mass to provide an a priori bound on the H^1 norm throughout the evolution, then the methods sketched above yield the following theorem about Schrödinger evolution on X, where we employ the notation from Theorem 5:

Theorem 6. *Assume that there exists $0 < \epsilon < 1$, $M > 0$ and $0 < C(M)$ such that*

$$\left| \int_X B(u^2) d\, Vol_X \right| \leq (1-\epsilon) \|u\|^2_{H^1(X)} + C(M) \tag{20}$$

for $u \in H^1(X)$ such that $\|u\|_{L^2(X)} \leq M$. Then, given $\|u_0\|_{L^2} \leq M$ in (14), we have $T_{min} = T_{max} = \infty$ in Theorem 5.

As we have *global*-in-time Strichartz estimates, it is natural to pursue the question of *scattering* of solutions with general critical or supercritical nonlinearities in two settings:

- In the focusing case, for small enough data in H^1 (possibly with the condition of finite variance or $\|xu\|^2_{L^2(X)} < \infty$),
- In the defocusing case, for general data.

Note that a scattering state, say u_+, can be easily seen to depend upon global dispersive results in the sense that global existence implies that in the H^1 norm we construct

$$\begin{aligned} u_+ &= \lim_{t \to \infty} e^{-it\Delta} u(x,t) \\ &= u_0 - i \int_0^\infty e^{-is\Delta} \beta(|u|^2) u(s) ds, \end{aligned}$$

provided the integral is bounded. See for instance Tao [35], Chapter 3.6 for a discussion. We consequently propose the following:

Conjecture 1. *Given $\epsilon > 0$ sufficiently small and $u_0 \in H^1$, $\|u_0\|_{H^1(X)} < \epsilon$, there exist u_+, $u_- \in H^1$ such that given u a global solution to (14), we have*

$$\|u(t) - e^{it\Delta} u_\pm\|_{H^1} \to 0$$

as $t \to \pm\infty$ with

$$M(u_+) = M(u_-) = M(u_0) \quad \text{and} \quad \|\nabla u_+\|^2_{L^2(X)} = \|\nabla u_-\|^2_{L^2(X)} = E(u_0).$$

[4]It would be of interest to see how such a calculation translates to the settings of exterior polygonal domains and in particular if the relevant Gagliardo-Nirenberg constant changes at all on product cones or polygonal exterior domains.

For more details,[5] we refer the reader to the treatment of scattering theory in higher dimensions of Strauss [32], [33] and in particular to the treatment by Nakanishi in [27], where the result was obtained in $\mathbb{R}^2$. See also [14] for a one-dimensional scattering result and [13], [29] for two- and three-dimensional scattering results using Interaction Morawetz style estimates, which have potential for applying to exterior domains. In two dimensions with a *smooth* star-shaped obstacle, scattering is proved in [30]; as the techniques there rely only upon integration by parts, they should extend to star-shaped polygonal domains (and possibly a broader class of non-trapping polygons) as well. In the case of finite variance on $\mathbb{R}^2$, this result follows from an extension of the pseudoconformal transformation, which is based upon a natural commuting vector field constructed from the Hamilton flow defined by the Schrödinger operator globally on $\mathbb{R}^2$; thus we do not currently have a version of this conservation law for exterior domains X.

Scattering can be shown in the defocussing case for any initial data given that the nonlinearity is critical/supercritical using a variant of the Morawetz estimates, which give local energy decay in the form of an estimate

$$\int_X \frac{|u|^p}{|x|^q} dx \leq \left(\frac{p}{2-q} \right)^q \|u\|_{L^p}^{p-q} \|\nabla u\|_{L^p}^q \tag{21}$$

for $u \in W^{1,p}(X)$ where $q \leq 2$ and $0 \leq q \leq p$, $1 \leq p < \infty$. In recent works, such questions have been approached on $\mathbb{R}^2$ for less regularity using concentration compactness techniques and interaction Morawetz estimates in the works of [17, 23]; one might hope to generalize these results to product cones and exterior domains.

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