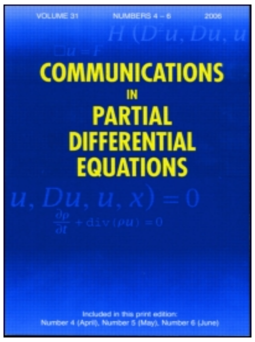

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Eigenfunctions for Partially Rectangular Billiards

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In this note, we further develop the methods of Burq and Zworski (2005) to study eigenfunctions for billiards which have rectangular components: these include the Bunimovich billiard, the Sinai billiard, and the recently popular pseudointegrable billiards (Bogomolny et al., 1999). The results are an application of a "black-box" point of view as presented in Burq and Zworski (2004).

Keywords Microlocal defect measures; Nonconcentration; Pseudointegrable billiards; Semiclassical analysis.

Mathematics Subject Classification Primary 35P20; Secondary 35B37.

1. Introduction

By a partially rectangular billiard, we mean a connected planar domain, Ω, with a piecewise smooth boundary, which contains a rectangle, $R \subset \Omega$, such that if we decompose the boundary of R, into pairs of parallel segments, $\partial R = \Gamma_1 \cup \Gamma_2$, then $\Gamma_i \subset \partial\Omega$, for at least one i.

We show that for such billiards, the eigenfunctions of the Dirichlet, Neumann, or periodic Laplacian cannot concentrate in closed sets in the interior of the rectangular part. A combination of this elementary result with the now standard, but highly nonelementary, propagation results of Melrose and Sjöstrand (1978/1982) and Bardos et al. (1992), can give further improvements—see Burq and Zworski (2004, 2005).

Here, we prove further nonconcentration results, away from the obstacle in the Sinai billiard (see Figure 1 and Theorem 3), and along certain trajectories in pseudointegrable billiards (see Figure 5 and Theorem 4). For recent motivation

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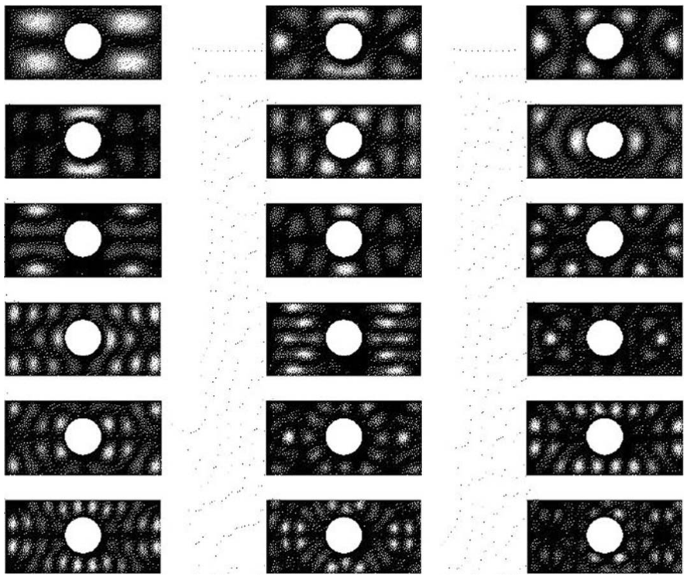

Figure 1. Experimental images of eigenfunctions in a Sinai billiard microwave cavity—see `http://sagar.physics.neu.edu`. We see that there is always a non-vanishing presence near the boundary of the obstacle as predicted by Theorem 3 below.

coming from the study of *quantum chaos* we suggest Bogomolny et al. (1999), Burq and Zworski (2005), Donnelly (2003), Zelditch (2004), and references given there.

2. Semiclassical Pseudodifferential Operators on a Torus

In this section, we discuss properties of Pseudodifferential Operators (PDOs) on a torus. In $\mathbb{R}^n$, we define the Weyl quantization of an operator $a(x, hD)$ where $a \in \mathscr{S}(\mathbb{R}^{2n})$, $a = a(x, \xi)$ by

$$a(x, hD)u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y, \xi\rangle} a(x, \xi) u(y) dy\, d\xi,$$

for $u \in \mathscr{S}$ and a symbol class by

$$S^k_\delta(m) = \{a \in C^\infty(\mathbb{R}^{2n}) | \partial^\alpha a| \leq C_\alpha h^{-\delta|\alpha|-k} m \text{ for all multi-indices } \alpha\},$$

where $m : \mathbb{R}^{2n} \to (0, \infty)$ is is an order function, i.e., there exist constants C, N such that

$$m(z) \leq C\langle z - w\rangle^N m(w).$$

We also define

$$S^{-\infty}_\delta(m) = \bigcap_{k=-\infty}^{\infty} S^k_\delta(m).$$

For $k, \delta = 0$ we write simply $S(m)$. On a torus, however, a and all its derivatives are bounded in the x variable, thus for h small and k positive, we need not worry about the derivatives in x, only those in ξ. Also, for k negative, provided that we have the proper local regularity for our symbol a, this definition still works perfectly on a torus.

Note also that we need only work with symbols that are periodic in the x-variable with period determined by the dimensions of the torus. In other words, $a(x, \xi) = a(x + \gamma, \xi)$ for $\gamma \in (a\mathbb{Z}) \times (b\mathbb{Z})$, where $a, b \in \mathbb{R}$. With this relation, we have the following proposition.

Proposition 2.1. *If $a(x, \xi)$ is a periodic symbol in x with period γ, then $a(x, D)T_\gamma = T_\gamma a(x, D)$ where $T_\gamma u(x) = u(x - \gamma)$.*

Proof. We calculate:

$$a(x, hD)T_\gamma(u) = \int_{\mathbb{R}^n} a(x, \xi) e^{\frac{i}{h}\langle x-y, \xi\rangle} \hat{u}(y - \gamma) dy\, d\xi \tag{2.1}$$

$$= \int_{\mathbb{R}^n} a(x - \gamma, \xi) e^{\frac{i}{h}\langle x-\gamma-\tilde{y}, \xi\rangle} \hat{u}(\tilde{y}) d\tilde{y}\, d\xi \tag{2.2}$$

$$= T_\gamma(a(x, D)u). \tag{2.3}$$

□

From the above proposition, it becomes clear that the properties of symbol classes in Euclidean space translate directly to properties of similarly defined symbol classes on a torus. For instance, we have the following result.

Proposition 2.2. *Given $u(x) = u(x + \gamma)$ where γ is as above, and u is L^2 on a torus, then $a(x, hD)u(x)$ is L^2 on the torus, when $a \in \mathcal{S}_\delta(1)$, $0 \le \delta \le \frac{1}{2}$.*

Proof. Note that the condition on a implies that it is L^2 bounded. Given a function $u(x)$ which is periodic on a torus, we can write it as $\sum_\gamma T_\gamma u_0$ where $u_0 = \chi(x)u(x)$ and $\chi(x)$ is equal to 1 on a single copy of the torus in the plane and zero otherwise. Note that no assumptions about the smoothness of $\chi(x)$ are made. Hence, $u_0 \in L^2$ and therefore, so is $a(x, hD)u_0$. Then, $a(x, hD)u(x) = \sum_\gamma T_\gamma a(x, hD)u_0$. The sum converges for each x since $a(x, D)u_0$ will have compact support and we have $a(x, hD)u$ a periodic function that is L^2 on the torus. □

Using similar techniques, we would like to develop the concept of a microlocal defect measure in this setting. Consider a collection of functions $\{u(h)\}_{0<h\le h_0}$ such that

$$\sup_{0<h\le h_0} \|u(h)\|_{L^2} < \infty.$$

As shown in Evans and Zworski (2003), we have the following theorem in Euclidean space.

Theorem 1. *There exists a Radon measure μ on $\mathbb{R}^n$ and a sequence $h_j \to 0$ such that*

$$\langle a^w(x, h_jD)u(h_j), u(h_j)\rangle \to \int_{\mathbb{R}^{2n}} a(x, \xi)d\mu \tag{2.4}$$

for all symbols $a \in S(1)$.

We call μ a microlocal defect measure associated with the family $\{u(h)\}_{0<h\leq h_0}$. Note that an $S(1)$ symbol on the torus corresponds to an $S(1)$ symbol on the plane, therefore this result proves the existence of microlocal defect measures on a torus as well.

Proof. **1.** Let $\{a_k\} \in C_c^\infty$ be dense in $C_0(\mathbb{R}^{2n})$. Select a sequence $h_j^1 \to 0$ such that

$$\langle a_1^w(x, h_j^1D)u(h_j^1), u(h_j^1)\rangle \to \alpha_1.$$

Then, select a subsequence $\{h_j^2\} \subset \{h_j^1\}$ such that

$$\langle a_2^w(x, h_j^2D)u(h_j^2), u(h_j^2)\rangle \to \alpha_2.$$

Continue such that at the kth step, you take a subsequence $\{h_j^k\} \subset \{h_j^{k-1}\}$ such that

$$\langle a_k^w(x, h_j^kD)u(h_j^k), u(h_j^k)\rangle \to \alpha_k.$$

Then by a diagonal argument, arrive at a sequence h_j such that

$$\langle a_k^w(x, h_jD)u(h_j), u(h_j)\rangle \to \alpha_k$$

for all $k = 1, 2, \ldots$.

2. Define $\Phi(a_k) = \alpha_k$. By a standard theorem on operator norms, we have for each k that

$$|\Phi(a_k)| = |\alpha_k| = \lim_{h_j\to\infty} |\langle a_k^w u(h_j), u(h_j)\rangle| \leq \limsup_{h_j\to\infty} C\|a_k^w\|_{L^2\to L^2} \leq C\sup|a_k|.$$

The mapping Φ is bounded, linear, and densely defined, therefore uniquely extends to a bounded linear functional on $S(1)$, with the estimate

$$|\Phi(a)| \leq C\sup|a|$$

for all $a \in S(1)$. The Riesz Representation Theorem therefore implies the existence of a (possibly complex valued) Radon measure on $\mathbb{R}^{2n}$ such that

$$\Phi(a) = \int_{\mathbb{R}^{2n}} a(x, \xi)d\mu. \qquad \square$$

We now quote a general theorem about microlocal defect measures on Euclidean space which we can then apply to a torus. To state the propagation theorem in the form sufficient for our applications, we follow Burq (2002).

Let us consider a Riemannian manifold without boundary, M. By partitions of unity we can define semi-classical pseudodifferential operators $a(x, hD_x)$ associated to symbols $a(x, \xi) \in \mathscr{C}_c^\infty(T^*M)$.

Now we consider a sequence (u_n) bounded in $L^2(M)$. satisfying

$$(-h_n^2\Delta_g - 1)u_n = 0, \tag{2.5}$$

where Δ_g is the Laplace–Beltrami operator. Using (2.5), as in Gérard and Leichtnam (1993) (see also Burq, 2002) we can prove the following result.

Proposition 2.3. *There exist a subsequence (n_k) and a positive Radon measure on T^*M, μ (a semi-classical measure for the sequence (u_n)), such that for any $a \in \mathscr{C}_c^\infty(T^*M)$*

$$\lim_{k\to+\infty}(a^w(x, h_{n_k}D_x)u_{n_k}, u_{n_k})_{L^2(M)} = \langle \mu, a(x, \xi)\rangle. \tag{2.6}$$

Furthermore, this measure satisfies:

(1) *The support of μ is included in the characteristic manifold*

$$\Sigma \stackrel{\text{def}}{=} \{(x, \xi) \in T^*M; p(x, \xi) = \|\xi\|_x = 1\} \tag{2.7}$$

where $\|\cdot\|_x$ is the norm for the metric at the point x;

(2) *The measure μ is invariant under the bicharacteristic flow (the flow of the Hamilton vector field of p)*

$$H_p\mu = 0; \tag{2.8}$$

(3) *For any $\varphi \in \mathscr{C}_c^\infty(T^*M)$,*

$$\lim_{k\to+\infty}\|\varphi u_{n_k}\|^2 = \langle \mu, |\varphi|^2\rangle. \tag{2.9}$$

The first two properties above are weak forms of the elliptic regularity and propagation of singularities results, whereas the last one states that there is no loss of L^2-mass at infinity in the ξ variable.

Proof. We will prove this proposition only for the case of a torus, but the methods are applicable to any manifold.

(0) (Positivity) We need to show that $a \geq 0$ implies

$$\int_{\mathbb{T}^2\times\mathbb{R}^2} a(x, \xi)d\mu \geq 0.$$

Since $a \geq 0$, using the sharp Gårding inequality, we see that

$$a^w(x, hD) \geq -Ch.$$

Let $h = h_j \to 0$, to see

$$\int_{\mathbb{T}^2\times\mathbb{R}^2} a\, d\mu = \lim_{j\to\infty}\langle a^w(x, h_jD)u(h_j), u(h_j)\rangle \geq 0.$$

(1) (Support of μ) Let a be a smooth function such that $\mathrm{supp}(a) \cap p^{-1}(1) = \emptyset$. We must show

$$\int_{\mathbb{T}^2 \times \mathbb{R}^2} a \, d\mu = 0.$$

Select $\chi \in C_c^\infty(\mathbb{T}^2 \times \mathbb{R}^2)$ such that

$$\mathrm{supp}(a) \cap \mathrm{supp}(\chi) = 0.$$

Then,

$$a^w(x, hD)(((p-1)^w + i\chi^w)^{-1}(p-1)^w)(x, hD) = a^w(x, hD) + \mathscr{O}(h)_{L^2 \to L^2}.$$

Apply $a^w(x, hD)$ to $u(h)$ to see that $\|a^w(x, hD)u(h)\|_{L^2} = o(1)$ and thus $\langle a^w(x, hD)u(h), u(h)\rangle \to 0$. But,

$$\langle a^w(x, hD)u(h_j), u(h_j)\rangle \to \int_{\mathbb{T}^2 \times \mathbb{R}^2} a \, d\mu.$$

(2) (Flow Invariance) Select a as above, then

$$\langle [p^w, a^w]u(h), u(h)\rangle = \langle (p^w a^w - a^w p^w)u(h), u(h)\rangle \tag{2.10}$$

$$= \langle a^w u(h), p^w u(h)\rangle - \langle p^w u(h), (a^w)^* u(h)\rangle \tag{2.11}$$

$$= o(h) \quad \text{as } h \to 0. \tag{2.12}$$

However, $[p^w, a^w] = \frac{h}{i}\{p, a\}^w + O(h^2)$. Hence,

$$\langle [p^w, a^w]u(h), u(h)\rangle = \frac{h}{i}\langle \{p, a\}^w u(h), u(h)\rangle + \langle o(h)u(h), u(h)\rangle.$$

Dividing through by h and allowing $h_j \to 0$, we see

$$\int_{\mathbb{T}^2 \times \mathbb{R}^2} \{p, a\} d\mu = 0.$$

So, if Φ_t is the flow generated by the Hamiltonian vector field H_p, then

$$\frac{d}{dt}\int_{\mathbb{T}^2 \times \mathbb{R}^2} (\Phi_t^* a) d\mu = \int_{\mathbb{T}^2 \times \mathbb{R}^2} (H_p a)(\Phi_t) d\mu = \int_{\mathbb{T}^2 \times \mathbb{R}^2} \{p, a\} d\mu = 0.$$

Now, (3) follows easily by looking at the operator $|\varphi(x, \xi)|^2$ and applying the result about existence of a microlocal defect measure. $\square$

3. Partially Rectangular Billiards

In this section we will need to recall the basic control results (Burq, 1992; Burq and Zworski, 2004) for rectangles, and the propagation results (Bardos et al., 1992; Burq, 2002; Burq and Gérard, 1996; Melrose and Sjöstrand, 1978/1982) for billiards.

Since in the specific application presented in Section 4 we only use propagation away from the boundary, that is the only case we will review.

The following result from Burq (1992) is related to some earlier control results of Haraux (1989) and Jaffard (1990).[1]

Proposition 3.1. *Let Δ be the Dirichlet, Neumann, or periodic Laplace operator on the rectangle $R = [0, 1]_x \times [0, a]_y$. Let ω_x be a nonempty open subset of $[0, 1]$. Then for any nonempty $\omega \subset R$ of the form $\omega = \omega_x \times [0, a]_y$, there exists C such that for any solutions of*

$$(\Delta - z)u = f \quad \text{on } R, \quad u\restriction_{\partial R} = 0 \tag{3.1}$$

we have

$$\|u\|^2_{L^2(R)} \leq C\big(\|f\|^2_{H^{-1}([0,1]_x;L^2([0,a]_y))} + \|u\restriction_\omega\|^2_{L^2(\omega)}\big) \tag{3.2}$$

Proof. We will consider the Dirichlet case (the proof is the same in the other two cases) and decompose u, f in terms of the basis of $L^2([0, a])$ formed by the Dirichlet eigenfunctions $e_k(y) = \sqrt{2/a}\sin(2k\pi y/a)$,

$$u(x, y) = \sum_k e_k(y)u_k(x), \quad f(x, y) = \sum_k e_k(y)f_k(x). \tag{3.3}$$

We get for u_k, f_k the equation

$$(\Delta_x - \big(z + (2k\pi/a)^2\big))u_k = f_k, \quad u_k(0) = u_k(1) = 0. \tag{3.4}$$

We now claim that

$$\|u_k\|^2_{L^2([0,1]_x)} \leq C\big(\|f_k\|^2_{H^{-1}([0,1]_x)} + \|u_k\restriction_{\omega_x}\|^2_{L^2(\omega)}\big), \tag{3.5}$$

from which, by summing the squares in k, we get (3.2).

To see (3.5) we can use the propagation result above in Prop. 2.3 in dimension one, but in this case an elementary calculation is easily available—see Burq and Zworski (2005). $\square$

The following theorem is an easy consequence of Proposition 3.1.

Theorem 2. *Let Ω be a partially rectangular billiard with the rectangular part $R \subset \Omega$, $\partial R = \Gamma_1 \cup \Gamma_2$, a decomposition into parallel components satisfying $\Gamma_2 \subset \partial\Omega$. Let Δ be the Dirichlet or Neumann Laplacian on Ω. Then for any neighbourhood of Γ_1 in Ω, V, there exists C such that*

$$-\Delta u = \lambda u \Longrightarrow \int_V |u(x)|^2 dx \geq \frac{1}{C}\int_R |u(x)|^2 dx, \tag{3.6}$$

that is, no eigenfuction can concentrate in R and away from Γ_1.

[1]We remark that as noted in Burq (1992) the result holds for any product manifold $M = M_x \times M_y$, and the proof is essentially the same.

Proof. Let us take x, y as the coordinates on the stadium, so that x parametrizes $\Gamma_2 \subset \partial\Omega$ and y parametrizes Γ_1, then

$$R = [0, 1]_x \times [0, a]_y.$$

Let $\chi \in \mathscr{C}_c^\infty((0, 1))$ be equal to 1 on $[\varepsilon, 1 - \varepsilon]$. Then $\chi(x)u(x, y)$ is a solution of

$$(\Delta - z)\chi u = [\Delta, \chi]u \quad \text{in } R \tag{3.7}$$

with the boundary conditions satisfied on ∂R. Applying Proposition 3.1, we get

$$\|\chi u\|_{L^2(R)} \leq C\big(\|[\Delta, \chi]u\|_{H_x^{-1};L_y^2} + \|u\restriction_{\omega_\varepsilon}\|_{L^2(\omega_\varepsilon)}\big) \leq C'\|u\restriction_{\omega_\varepsilon}\|_{L^2(\omega_\varepsilon)}, \tag{3.8}$$

where ω_ε is a neighbourhood of the support of $\nabla\chi$. Since a neighbourhood of Γ_1 in Ω has to contain ω_ε for some ε, (3.6) follows. □

4. Applications

In Burq and Zworski (2004, 2005), Proposition 3.1 is used to prove that in the case of the Bunimovich billiard shown in Figure 2, the states have nonvanishing density near the vertical boundaries of the rectangle. That follows from Theorem 2 which shows that we must have positive density in the wings of the billiard, and the propagation result (in the boundary case) based on the fact that any diagonal controls a disc geometrically (see Burq and Zworski, 2004, Section 6.1; in fact we can use other control regions as shown in Figure 2). Here we consider another case which accidentally generalizes a control theory result of Jaffard (1990).

The Sinai billiard (see Figure 1) is defined by removing a strictly convex open set, $\mathscr{O}$, with a $\mathscr{C}^\infty$ boundary, from a flat torus, $\mathbb{T}_{a,b}^2 \stackrel{\text{def}}{=} (a\mathbb{S}^1) \times (b\mathbb{S}^1)$:

$$S \stackrel{\text{def}}{=} \mathbb{T}_{a,b}^2 \backslash \mathscr{O}.$$

The following theorem results by applying Theorem 1 to a torus with sides of arbitrary length.

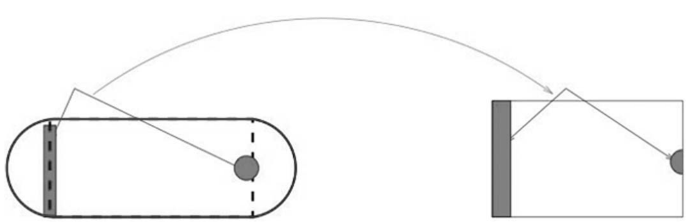

Figure 2. Control regions in which eigenfunctions have positive mass and the rectangular part for the Bunimovich stadium.

Theorem 3. *Let V be any open neighbourhood of the convex boundary, $\partial\mathcal{O}$, in a Sinai billiard, S. If Δ is the Dirichlet or Neumann Laplace operator on S then there exists a constant, $C = C(V)$, such that*

$$-h^2\Delta u = E(h)u \Longrightarrow \int_V |u(x)|^2 dx \geq \frac{1}{C}\int_S |u(x)|^2 dx, \tag{4.1}$$

for any h and $|E(h) - 1| < \frac{1}{2}$.

Proof. First note that we can easily limit ourselves to the case where our flat torus has one side of length 1 and one side of length a. Suppose that the result is not true, in other words, there exists a sequence of eigenfunctions u_n, $\|u_n\| = 1$, with the corresponding eigenvalues $\lambda_n \to \infty$, such that $\int_V |u_n(x)|^2 dx \to 0$.

We first observe that the only directions in the support of the corresponding semi-classical defect measure, μ, have to be "rational", in other words, the trajectory must travel along a line of slope $\frac{ma}{n}$ where $m, n \in \mathbb{N}$. The projection of a trajectory with an irrational direction is dense on the torus and hence must encounter the obstacle $\partial\mathcal{O}$ (and consequently V). The propagation result recalled in Proposition 2.3, part (3), gives a contradiction by choosing a proper test function ϕ which is non-zero on the support of the measure μ resulting from our sequence of eigenfunctions (we remark that we apply this result as long as the trajectory does not encounter the obstacle and consequently we need only the *interior* propagation).

Hence let us assume that there exists a rational direction in the support of the measure which then contains the periodic trajectory in that direction. As shown in Figure 3, we can find a maximal rectangular neighbourhood of the projection of that trajectory which avoids the obstacle.

The rectangle can be described as $R = [0, a_1]_{x_1} \times [0, b_1]_{y_1}$ with the y_1 coordinate parametrizing the trajectory. Let $\epsilon, \delta > 0$ be small. Let u be an eigenfunction in our sequence and define $\chi(x_1) \in \mathscr{C}_c^\infty(\mathbb{T}^2)$ such that

$$\chi(x_1) = \begin{cases} \chi(x_1) = 1 & \text{for all } x_1 \in (\epsilon, a_1 - \epsilon), \\ \chi(x_1) = 0 & \text{for all } x_1 \in \mathbb{R}\backslash\left(\frac{\epsilon}{2}, a_1 - \frac{\epsilon}{2}\right). \end{cases}$$

Note that we can then write $\chi = \chi(x, y)$ for $(x, y) \in \mathbb{T}^2_{1,a}$ as x_1 is simply a rotation and translation of the standard coordinates. Then $\chi(x, y)u(x, y)$ is a function on R satisfying the periodicity condition. Let $\Phi_\xi(v) = \Phi(\xi - v)$, where we define $\Phi(\xi) \in \mathscr{C}_c^\infty(\mathbb{R}^2)$ such that

$$\Phi(\xi) = \begin{cases} \Phi(\xi) = 1 & \text{for } \xi \in B(0, \delta), \\ \Phi(\xi) = 0 & \text{for } \xi \in \mathbb{R}^2\backslash B(0, 2\delta), \end{cases}$$

where $B(0, \delta)$ is a ball centered at zero of radius δ. Note, due to the compact support of this function in ξ, $\Phi_\xi(D)$ is in the symbol class $S(\langle\xi\rangle^{-N})$ for any N. Let Δ_R be the (periodic) Laplacian on R. Using Fourier decomposition we can arrange that $[\Delta_R, \Phi_\xi(D)] = \{\mathcal{O}\}(h^\infty)$. Let U be a neighborhood of the obstacle $\mathcal{O}$ such that $U \subset V$, where V is as above. Since our eigenfunction is only defined on $\mathbb{T}^2_{1,a}\backslash\mathcal{O}$, let us

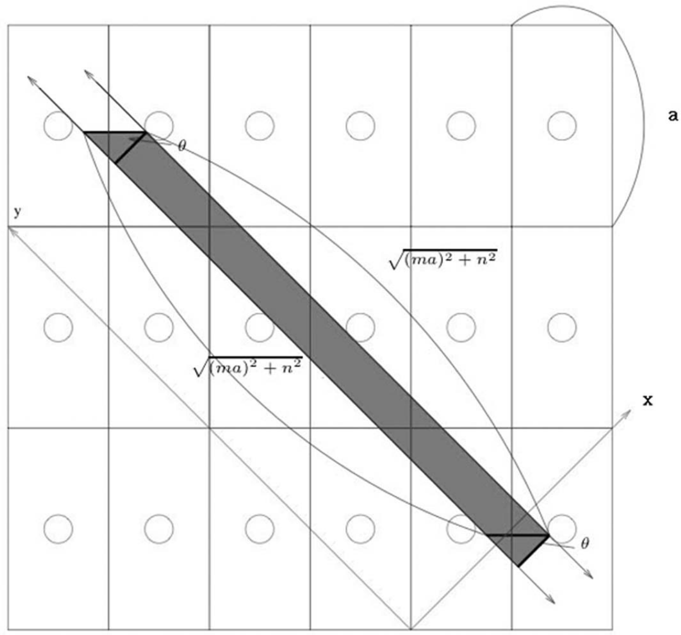

Figure 3. A maximal rectangle in a rational direction, avoiding the obstacle. Because the parrallelogram is certainly periodic and our region has uniform width, it is clear that the resulting rectangle is periodic.

introduce a smooth function χ_0 which is zero on U and 1 on $\mathbb{T}^2_{1,a}\backslash V$. Choose U, ϵ such that $\chi\chi_0 = \chi$. Hence,

$$\begin{aligned}(-h^2\Delta_R - E(h))\Phi_\xi(D)\chi\chi_0 u &= [-h^2\Delta_R, \Phi_\xi(D)\chi]u \\ &= \Phi_\xi(D)[-h^2\Delta_R, \chi]\chi_0 u + \mathcal{O}(h^\infty), \quad \|u\|_{L^2} = 1,\end{aligned}$$

where we can interchange u and $\chi_0 u$ by the construction above.

As in the proof of Proposition 3.1, we now see that

$$\|\Phi_\xi \chi\chi_0 u\|_{L^2} \leq C \int_\omega |\chi_0 u|^2 + \mathcal{O}(h^\infty), \tag{4.2}$$

where ω is a neighbourhood of $\operatorname{supp} \nabla\chi$ (in the calculus of semi-classical pseudodifferential operators). Since the semi-classical defect measure of $\Phi_\xi\chi\chi_0 u$ (which is $|\Phi_\xi\chi\chi_0|^2 \times \mu$) was assumed to be non-zero, (4.2) shows that the measure of $\chi_0 u\restriction_\omega$ is non-zero and consequently there is a point in the intersection of the supports of μ and $\chi_0 u\restriction_\omega$. But μ is invariant by the flow (as long as it does not intersect the obstacle) and hence, once we choose ϵ, δ small enough such that all the cut-offs above are very close to the boundary of R, its support can be made to intersect any neighbourhood of $\partial\mathcal{O}$. $\square$

Now, from the above theorem, we see the following simple, but important observation.

Remark 1. Let $S = \mathbb{T}^2_{a,b} \backslash \mathcal{O}$ where $\mathcal{O}$ is sufficiently smooth in the case of Neumann boundary conditions, but otherwise lacking restrictions. Then, for V any open neighborhood of $\partial\mathcal{O}$, and u a solution of $-h^2\Delta u = E(h)u$ as above, then (4.1) is satisfied. This follows from the above argument as neither the convexity of the obstacle nor the fact that the obstacle was open ever appeared in the argument. Thus, the result holds for any obstacle (even connectedness is not assumed here) and is applicable to the special case of pseudointegrable billiards (see for instance Bogomolny et al., 1999 for motivation and description). In the next section, we use an argument similar to that above in order to say even more about concentration along trajectories in specific pseudointegrable billiards. By an elementary reflection principle, the result also holds for an obstacle inside a square with Dirichlet or Neumann conditions on the boundary of the square.

5. Pseudointegrable Billiards

We define a pseudointegrable billiard to be a plane polygonal billiard with corners whose angles are of the form $\frac{\pi}{n}$, for any integer n (see Bogomolny and Schmit, 2004). In particular, we will be working with the billiard $P = \mathbb{T}^2_{a,b} \backslash S$ where S is a slit that is parrallel to a side of the torus but not a closed loop. In Remark 1, we point out that Theorem 3 allows us to make statements about the L^2 mass of eigenfunctions in a neighborhood of the slit for pseudointegrable billiards. For this particular type of billiard, it would be ideal to state that every eigenfunction must have non-zero mass in a small neighborhood of the edges of the slit (see Figure 4). In this section, we prove a weaker result about nonconcentration along certain classical trajectories in P of semiclassical defect measures obtained from eigenfunctions u such that $(-\lambda - \Delta)u = 0$ on P.

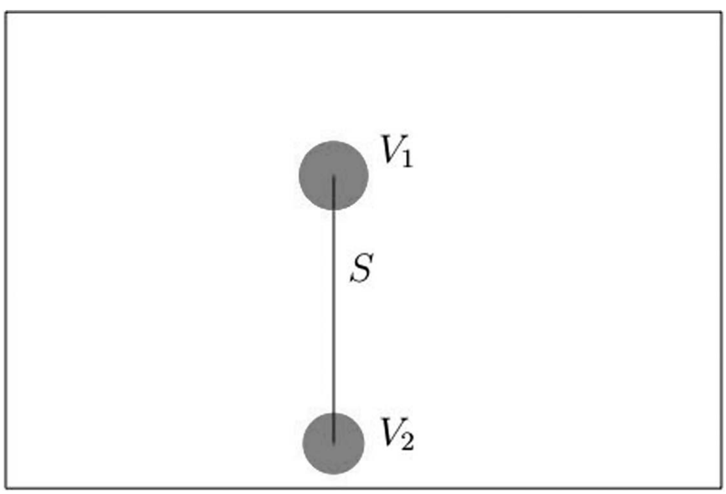

Figure 4. A pseudointegrable billiard P consisting of a torus with a slit, S along which we have Dirichlet boundary conditions. We would like to show that eigenfunctions of the Laplacian on this torus must have concentration in the shaded regions V_1 and V_2.

As with the Sinai billiard, the classical behavior of trajectories must be taken into account in our treatment of this problem. There cannot be concentration along trajectories that do not hit the slit as shown by Theorem 3. If a trajectory has irrational slope (i.e., the slope cannot be written in the form $\frac{ma}{nb}$, for $m, n \in \mathbb{N}$), it is dense in P, and thus has mass near the edges of the slit as in Section 4. Therefore, for our purpose, we concern ourselves only with rational trajectories which intersect the slit at some point. As we are dealing with periodic boundary conditions, let us consider the plane tiled with copies of the billiard P.

Assume that S is parrallel to the y-axis. Let $\gamma \in S^*(P)$ be a trajectory. Note that γ represents a solution to Hamilton's ode, or in other words, is a classical solution to the problem. Given the natural projection

$$\pi_1 : S^*(P) \to P,$$

we take $\gamma' = \pi_1(\gamma)$, or the physical path mapped out by the trajectory. Consider the projection

$$\tilde{\pi} : \mathbb{R}^2 \to \mathbb{T}^2_{a,b}.$$

We see that

$$\tilde{\pi} : \mathbb{R}^2 \backslash \widetilde{S} \to P, \quad \text{where } \widetilde{S} = \tilde{\pi}^{-1}(S).$$

Define

$$\pi_2 : S^*(\mathbb{R}^2 \backslash \widetilde{S}) \to S^*(P)$$

to be the natural projection. Let $\tilde{\gamma} = \pi_2^{-1}(\gamma)$. We can write

$$\tilde{\gamma} = \bigcup_{j=1}^{\infty} \gamma_j,$$

where each γ_j is a trajectory in $S^*(\mathbb{R}^2 \backslash \widetilde{S})$. We note that by construction, $\gamma_i \cap \gamma_j = \emptyset$ for $i \neq j$. To see this, assume that $\gamma_i \cap \gamma_j = (x, \xi)$. Then, $\gamma_i = \gamma_j$ as they would represent trajectories which travel through the same point in the same direction by ode uniqueness. Now, let

$$\pi_1^* : S^*(\mathbb{R}^2 \backslash \widetilde{S}) \to \mathbb{R}^2 \backslash \widetilde{S}.$$

Select one trajectory from the above union, say γ_1. Let $\gamma_1' = \pi_1^*(\gamma_1)$. We see that either γ_1' is bounded in the x-direction or γ_1' is unbounded in the x-direction. Note that this property then holds for all γ_j, $j \in \mathbb{N}$. For a trajectory γ, if the resulting path γ_1' is bounded in the x-direction, we say γ is x-bounded. We define γ as x-unbounded if γ_1' is unbounded in the x-direction. See Figure 5 for examples. Now, we are prepared to state our theorem concerning the billiard P.

Theorem 4. *Let γ be an x-bounded trajectory on $P = \mathbb{T}^2 \backslash S$. If Δ is the Dirichlet Laplace operator on P then there exists no microlocal defect measure obtained from the eigenfunctions on P such that* $\operatorname{supp}(d\mu) = \gamma$.

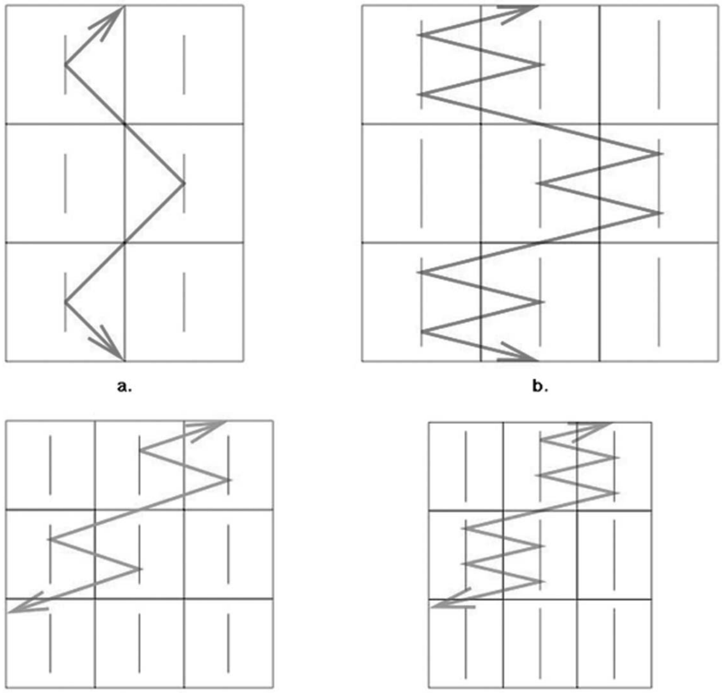

Figure 5. Some examples of x-bounded and x-unbounded trajectories. Above, a. and b. represent typical x-bounded trajectories, while c. and d. represent x-unbounded trajectories.

Proof. Let γ' be as above. Let V_ϵ be an ϵ neighborhood of γ'. Suppose false, we would have a sequence of eigenfunctions u_n, $\|u_n\|_{L^2} = 1$ with the property

$$\int_{P \setminus V_\epsilon} |u_n|^2 dx \to 0,$$

for any ϵ. We show this is impossible.

For each u_n, we have $(-\Delta - \lambda_n)u_n = 0$, $u_n|_S = 0$, $u_n \in L^2(P)$. Let $\tilde{\pi}$ be as above. We define the sequence $\tilde{u}_n = \tilde{\pi}^{-1}u_n$. We have $(-\Delta - \lambda_n)\tilde{u}_n = 0$, $\tilde{u}_n|_{\widetilde{S}} = 0$, and $\tilde{u}_n \in L^2_{\text{per}}(\mathbb{R}^2 \setminus \widetilde{S})$.

If $\pi_2 : S^*(\mathbb{R}^2 \setminus \widetilde{S}) \to S^*(P)$ is as above and $\tilde{\gamma} = \pi_2^{-1}(\gamma)$, then $\tilde{u}_n \to d\tilde{\mu}$ with

$$\text{supp}(d\tilde{\mu}) = \tilde{\gamma} \subset S^*(\mathbb{R}^2 \setminus \widetilde{S}).$$

Now, let $\pi_1^* : S^*(\mathbb{R}^2 \setminus \widetilde{S}) \to \mathbb{R}^2 \setminus \widetilde{S}$ be as above. Select one trajectory, say γ_1. As γ_1 is x-bounded, $\gamma_1' = \pi_1^*(\gamma_1)$ is contained in a strip in the plane which is infinite in the y-direction and bounded in the x-direction. Thus, γ_1' is contained in a strip, C_0, with minimal width in the x-direction. Then, $\tilde{u}_n$ satisfies $(-\Delta - \lambda_n)\tilde{u}_n = 0$ on the interior of C_0, is periodic in the y-direction, and satisfies the following boundary conditions in the x-direction: Dirichlet boundary conditions along the slits that intersect the boundary of C_0 and periodic boundary conditions otherwise.

Without loss of generality, we can choose the x-coordinates such that the boundaries of C_0 are $x = -R$ and $x = 0$. We can then reflect to a strip, say $\widetilde{C}_1$, with boundaries $x = -R$ and $x = R$, by defining a new function on $\widetilde{C}_1$ by

$$\tilde{u}_n^{(1)}(x, y) = \begin{cases} \tilde{u}_n(x, y) & x \in [-R, 0], \\ -\tilde{u}_n(-x, y) & x \in (0, R). \end{cases}$$

Note that $\tilde{u}_n^{(1)}$ is periodic with period $2R$. As a result, in the sense of distributions we have

$$(-\Delta - \lambda_n)\tilde{u}_n^{(1)} = f_n^{(1)}$$

on $\widetilde{C}_1$, where

$$f_n^{(1)} = 2u(0, y)\delta_0'(x) - 2u(R, y)\delta_R'(x).$$

We note that $f_n^{(1)}$ is supported away from the slits, $\widetilde{S}$.

Define

$$\pi_1^\sharp(x, y) = \begin{cases} (x, y) & -R \leq x \leq 0, \\ (-x, y) & 0 \leq x \leq R. \end{cases}$$

If $\pi_1^\sharp : \widetilde{C}_1 \to C_0$, then

$$(\pi_1^\sharp)^{-1}\Big(\bigcup_j \gamma_j'\Big)$$

is again a union of paths resulting from disjoint trajectories. Now, we iterate this procedure a finite number of times, stopping the iteration when the disjoint trajectories in the lift intersect each slit only once (see Figure 6).

After each reflection, we restrict to a new minimal width strip, say C_i. Let us call $\widetilde{C}_i$ the strip resulting from the ith reflection. We define $\pi_i^\sharp : \widetilde{C}_i \to C_{i-1}$ for $1 \leq i < N$ such that

$$\pi_i^\sharp(x, y) = \begin{cases} (x, y) & (x, y) \in C_{i-1}, \\ (2R_i - x, y) & (x, y) \in C_{i-1}'. \end{cases}$$

Here, C_{i-1}' is defined as the reflected strip and $x = R_{i-1}$ is the line of reflection for $\widetilde{C}_i$. We can subsequently define $f_n^{(i)}$ as a sum of delta functions resulting from jumps that occur after reflection, similar to $f_n^{(1)}$ above. We also have $\pi^N : \mathbb{R}^2 \to C_N$, the natural projection that results after we tile the plane with copies of C_N. So, we have:

$$\mathbb{R}^2 \xrightarrow{\pi_N} C_N \subset \widetilde{C}_N \xrightarrow{\pi_N^\sharp} C_{N-1} \subset \widetilde{C}_{N-1} \xrightarrow{\pi_{N-1}^\sharp} \cdots \xrightarrow{\pi_2^\sharp} C_1 \subset \widetilde{C}_1 \xrightarrow{\pi_1^\sharp} C.$$

Note that

$$\pi_N^{-1}(\gamma_1') = \bigcup_j \gamma_{1,j}',$$

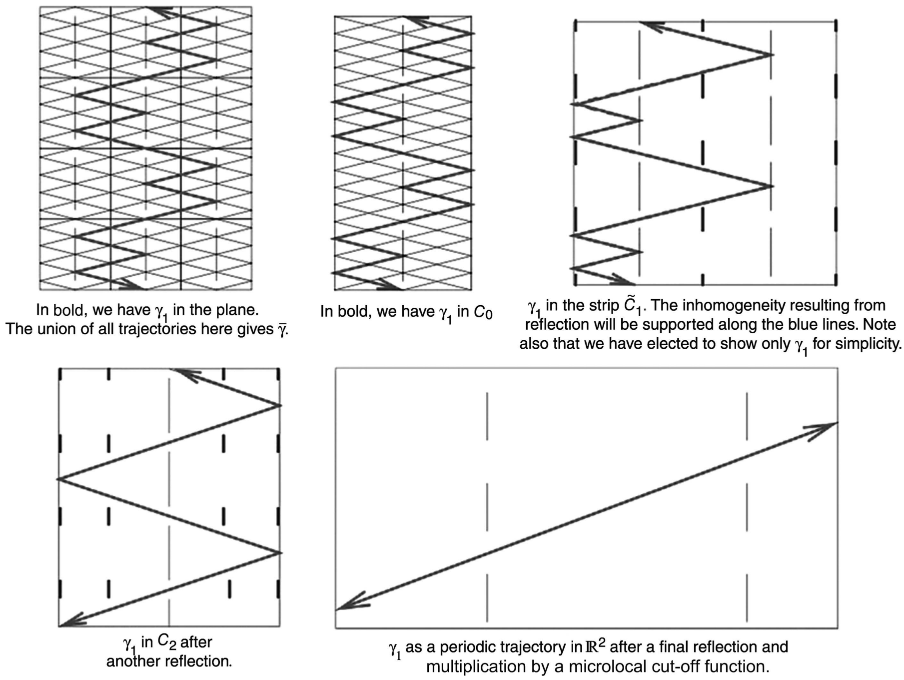

Figure 6. This diagram describes how we "unfold" the eigenfunctions in order to derive a contradiction.

where $\{\gamma'_{1,j}\}$ is the set of all paths in C_N generated by the trajectory γ_1 and the periodicity in y.

After a finite number of reflections, we "unfolded" γ'_1 to be a periodic line on a large strip, C_N, which does not intersect a slit anywhere. Now, let us choose Φ_ξ, χ, and χ_0 as above in order to cutoff microlocally on this strip around γ_1. Again, recall that we can set $\chi\chi_0 = \chi$. As $f_n^{(i)}$ is supported only in between the slits for each $i \in \mathbb{N}$, $1 \leq i \leq N$, by choosing Φ_ξ to commute with the periodic Laplacian, we have

$$\begin{aligned}(-h^2\Delta_R - E(h))\Phi_\xi\chi\chi_0 u_n &= \Phi_\xi\chi f_n + [-h^2\Delta_R, \Phi_\xi\chi]u \\ &= \Phi_\xi[-h^2\Delta_R, \chi]\chi_0 u + \mathcal{O}(h^\infty), \quad \|u\|_{L^2} = 1.\end{aligned}$$

Thus, the result follows by contradiction from the proof of Theorem 3. $\square$

Remark 2. Although this result only shows nonconcentration, the proof of Theorem 3 can be used to show that if γ is an x-bounded trajectory and u is an eigenfunction supported on $\gamma' = \pi_1(\gamma)$, then in fact there must be mass at the edges of the slits as desired.

Remark 3. If instead of a torus, we had Dirichlet boundary conditions on the boundary as well as the slit, then this nonconcentration result can also be applied by an elementary reflection principle argument. In this case, x-bounded trajectories are simple to define as they result in an odd number of reflections off either side of the slit before repeating periodically.

Remark 4. It is difficult to use this method on x-unbounded trajectories as the reflection principle is no longer applicable. If one could prove Theorem 3 for parrallelograms as well as rectangles or somehow apply the recent results about defect measures on boundaries from Miller (2000), one could possibly extend this result to include all trajectories in the above Pseudointegrable Billiards.

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References

Bardos, C., Lebeau, G., Rauch, J. (1992). Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.* 30:1024–1065.

Bogomolny, E., Schmit, C. (2004). Structure of wave functions of pseudointegrable billiards. *Phys. Rev. Lett.* 92:244102.

Bogomolny, E., Gerland, U., Schmit, C. (1999). Models of intermediate spectral statistics. *Phys. Rev. E* 59:1315–1318.

Burq, N. (1992). Control for Schrodinger equations on product manifolds. Unpublished.

Burq, N. (2002). Semi-classical estimates for the resolvent in non trapping geometries. *Int. Math. Res. Notices* 5:221–241.

Burq, N., Gérard, P. (1996). Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes. *Comptes Rendus de L'Académie des Sciences* 749–752, t.325, Série I.

Burq, N., Zworski, M. (2004). Geometric control in the presence of a black box. *JAMS* 17:443–471.

Burq, N., Zworski, M. (2005). Bouncing ball modes and quantum chaos. *SIAM Review* 47(5):43–49.

Donnelly, H. (2003). Quantum unique ergodicity. *Proc. Amer. Math. Soc.* 131:2945–2951.

Evans, L. C., Zworski, M. (2003). Lectures on semiclassical analysis. *Unpublished Lecture Series.*

Gérard, P., Leichtnam, E. (1993). Ergodic properties of eigenfunctions for the Dirichlet problem. *Duke Mathematical Journal* 71:559–607.

Haraux, A. (1989). Séries lacunaires et contrôle semi-interne des vibrations d'une plaque rectangulaire. *J. Math. Pures Appl.* 68(4):457–465.

Jaffard, S. (1990). Contrôle interne exact des vibrations d'une plaque rectangulaire. *Portugal. Math.* 47(4):423–429.

Melrose, R. B., Sjöstrand, J. (1978/1982). Singularities of boundary value problems I & II. *Communications in Pure Applied Mathematics* 31 & 35: 593–617 & 129–168.

Miller, L. (2000). Refraction of high-frequency waves density by sharp interfaces and semiclassical measures at the boundary. *J. Math. Pures Appl.* 79(3):227–269.

Zelditch, S. (2004). Quantum unique ergodicity. *Proc. AMS* 1869–1872.