A MORSE INDEX THEOREM FOR ELLIPTIC OPERATORS ON BOUNDED DOMAINS

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Abstract. We consider a second-order, selfadjoint elliptic operator L on a smooth one-parameter family of domains $\{\Omega_t\}_{t\in[a,b]}$ with no assumptions on the geometry of the Ω_t's. It is shown that the Morse index of L can be equated with the Maslov index of an appropriately defined path in a symplectic Hilbert space constructed on the boundary of Ω_b. Our result is valid for a wide variety of boundary conditions, including (but not limited to) Dirichlet, Neumann and Robin.

Specifically, the Maslov index of the path we define relates the Morse index of L on Ω_b to the Morse index of L on Ω_a. This is particularly useful when Ω_a is a domain for which the spectrum is more readily understood, e.g. a region with very small volume. In other words, the Maslov index exactly computes the discrepancy between the Morse indices for the "original" problem (on Ω_b) and the "simplified" problem (on Ω_a). This generalizes previous results that were only available on star-shaped domains, or for Dirichlet boundary conditions.

We then discuss how one can practically compute the Maslov index using crossing forms, and present some applications to the spectral theory of Dirichlet and Neumann boundary value problems.

1. Introduction

Let L be a second-order, selfadjoint elliptic operator on a bounded domain $\Omega \subset \mathbb{R}^n$. Abstractly, the spectral theory of such operators is very well understood. However, it is still not known in general how to relate the spectrum of L to underlying geometric features of either the operator or the domain.

For instance, if $\bar{u}$ is a steady state for the reaction-diffusion equation $u_t + f(u) = \Delta u$, then the linear stability of $\bar{u}$ is determined by the spectrum of the operator $L = -\Delta + f'(\bar{u})$. This operator depends explicitly on the steady state through the potential $f'(\bar{u})$; it would be useful if in practice one could relate spectral properties of L (e.g. the number of negative eigenvalues) to the structure of $\bar{u}$.

A motivating example comes from the classic Sturm–Liouville theory for ordinary differential equations. If $\bar{u}$ is a steady state of $u_t + f(u) = u_{xx}$, then its Morse index can be found by counting the zeroes of the derivative $\bar{u}_x$. In a more geometric vein, the Morse index theorem shows that the number of unstable (i.e. length decreasing) directions in which a Riemannian geodesic can be perturbed is equal to the number of conjugate points along that geodesic [16]. This makes it possible to relate the index to the curvature of the manifold, which affects the existence and nonexistence of conjugate points in a fundamental way.

The first multi-dimensional version of Sturm–Liouville theory appeared in [24], for the case of a selfadjoint, elliptic operator L on a bounded domain in a Riemannian manifold, with Dirichlet boundary conditions. Assuming that the domain Ω could be deformed smoothly though a family $\{\Omega_t\}$ with $\mathrm{Vol}(\Omega_t) \to 0$, Smale showed that the Morse index of L is equal to the total number of times t (counting multiplicity) for which the problem

$$Lu = 0 \text{ in } \Omega_t, \quad u = 0 \text{ on } \partial\Omega_t$$

has a nontrivial solution. These times are analogous to conjugate points in the Riemannian case (which correspond to solutions of the Jacobi equation with Dirichlet boundary conditions) and so Smale's result can be viewed as a Morse index theorem for elliptic boundary value problems.

In [3] Arnol′d gave a symplectic interpretation of the Sturm–Liouville theory, equating the Morse index to the Maslov index—a topological invariant assigned to a path of Lagrangian subspaces in a fixed symplectic vector space [2]. This interpretation was extended to the multi-dimensional setting by Deng and Jones in [7], for a Schrödinger operator $L = -\Delta + V$ on a bounded, star-shaped domain Ω. Their idea was to shrink the domain to a point through the one-parameter family $\Omega_t := \{tx : x \in \Omega\}$, then for each $t \in (0, 1]$ define a pair of Lagrangian subspaces in $H^{1/2}(\partial\Omega) \oplus H^{-1/2}(\partial\Omega)$ that correspond to the given boundary condition, and the boundary data of weak solutions to $Lu = 0$, respectively. By construction these two subspaces intersect nontrivially when there exists a nontrivial solution to $Lu = 0$, with the prescribed boundary conditions, on Ω_t. This fact was used to relate the Maslov index of the path obtained by shrinking Ω to the Morse index of L on Ω.

The symplectic approach of Deng and Jones recovers the result of Smale (in the case of a star-shaped domain), but also allows one to consider more general boundary conditions. This generalization is significant because eigenvalues of general selfadjoint boundary value problems can exhibit more complicated behavior, with respect to domain variations, than in the Dirichlet case. For instance, in the Neumann problem the eigenvalues are not necessarily monotone increasing for a shrinking family of domains, as was recently observed in [17].

The main shortcoming of [7] is the star-shaped assumption regarding the domain. Of course stability problems on more general domains are of great interest, and we need effective tools for computing the Morse index in such cases. However, there is a more subtle (and perhaps more important) motivation for the consideration of general domains. If $\bar{u}$ is a function in the kernel of L, it is desirable to relate the Morse index of L to the geometric structure of $\bar{u}$ (analogous to the classic Sturm–Liouville theory, and Courant's nodal domain theorem). In this case a reasonable family of domains would be the sublevel sets

$$\Omega_t = \{x \in \Omega : \bar{u}(x) < t\},$$

which remain diffeomorphic as long as t does not pass through a critical value of $\bar{u}$. In general there is no reason for these Ω_t to be star-shaped, even when $\Omega \subset \mathbb{R}^n$ is a ball and the coefficients of L are radially symmetric.

In the current paper we show that, through a careful rescaling of the relevant operators and boundary conditions, it is possible to preserve the symplectic structure on the boundary as the domain is rescaled, with no assumptions on the geometry of Ω. This allows us to define a Maslov index, which can be interpreted as a signed enumeration of conjugate times, and relate it to the Morse index of the boundary value problem on Ω. In particular, given a family of domains $\{\Omega_t\}_{a \le t \le b}$, our main result is that the difference in Morse indices

$$M(L|_{\Omega_a}) - M(L|_{\Omega_b})$$

can be exactly computed by the Maslov index of a certain path of Lagrangian subspaces in the symplectic Hilbert space $H^{1/2}(\partial\Omega) \oplus H^{-1/2}(\partial\Omega)$. We also describe in detail how one can compute the relevant Maslov index in practice, and use the resulting formulas to determine Morse indices for a variety of boundary value problems.

Outline of the paper. In Section 2 we make precise our assumptions on the domains, operators and boundary conditions under consideration; our main results are stated in Section 2.5. The precise construction of the paths for which the Maslov index will be computed is given in Section 3, and the main theorem is proved in Section 4. In Section 5 we describe the explicit computation of the Maslov index via crossing forms, and give some applications to spectral problems with Dirichlet and Neumann boundary conditions.

Appendix A contains a brief summary of the relation between symmetric, bilinear forms and selfadjoint, unbounded operators that lies at the heart of our presentation. A review of the Fredholm–Lagrangian–Grassmannian and Maslov index for symplectic Hilbert spaces is given in Appendix B. In Appendix C we prove some technical regularity results for smooth families of bilinear forms that are are needed in the constructions of Section 3.

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2. Definitions and statement of results

2.1. The Morse index. Throughout we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Let L be a strongly elliptic operator of the form

$$Lu := -\partial_i(a^{ij}\partial_j u) + cu \tag{1}$$

with a^{ij} and c are smooth, real-valued functions, and $a^{ij} = a^{ji}$. Suppose D is a Dirichlet form for L, i.e. a symmetric, bilinear form such that

$$D(u,v) = \langle Lu, v\rangle_{L^2(\Omega)}$$

for all $u, v \in C_0^\infty(\Omega)$. Letting $\mathcal{X}$ be a closed subspace of $H^1(\Omega)$ that contains $H_0^1(\Omega)$, we say that a function $u \in \mathcal{X}$ is an eigenfunction for the $(D, \mathcal{X})$ problem, with eigenvalue λ, if

$$D(u,v) = \lambda \langle u, v\rangle_{L^2(\Omega)}$$

for all $v \in \mathcal{X}$. The connection between D and L is standard (see [1, 8, 14, 19] or Appendix A for details). Before proceeding, we define

$$\gamma u = u|_{\partial\Omega} \tag{2}$$

to be the Dirichlet trace operator, the precise domain and regularity properties of which will be recalled in Lemma 3.2.

Proposition 2.1. *There exists an unbounded, selfadjoint operator $L_{\mathcal{X}}$ with dense domain $\mathcal{D}(L_{\mathcal{X}}) \subset \mathcal{X}$ such that*

$$D(u,v) = \langle L_{\mathcal{X}} u, v\rangle_{L^2(\Omega)}$$

for all $u \in \mathcal{D}(L_{\mathcal{X}})$ and $v \in \mathcal{X}$, and a first-order differential operator B defined near $\partial\Omega$ such that

$$D(u,v) = \langle Lu, v\rangle_{L^2(\Omega)} + \int_{\partial\Omega} (Bu)(\gamma v)\,d\mu$$

whenever $u, v \in H^1(\Omega)$ and $Lu \in L^2(\Omega)$. Moreover, there exists an orthonormal basis for $L^2(\Omega)$ consisting of smooth eigenfunctions $\{u_i\}$ for $L_{\mathcal{X}}$, with discrete eigenvalues $\{\lambda_i\}$ tending to ∞.

It follows from the proof (which is sketched in Appendix A) that L is an extension of $L_{\mathcal{X}}$, and in fact

$$\mathcal{D}(L_{\mathcal{X}}) = \left\{ u \in \mathcal{X} : Lu \in L^2(\Omega) \text{ and } \int_{\partial\Omega} (Bu)(\gamma v)\,d\mu = 0 \text{ for all } v \in \mathcal{X} \right\}.$$

It is an immediate consequence of Proposition 2.1 and Theorem XIII.2 in [20] that the eigenvalues of $L_{\mathcal{X}}$ satisfy the minimax principle

$$\lambda_n = \sup_{\substack{V \subset L^2(\Omega) \\ \dim(V) = n}} \inf \left\{ \frac{D(u,u)}{\|u\|^2_{L^2(\Omega)}} : u \in \mathcal{X} \cap V^\perp \right\}$$

and the Morse index of $L_{\mathcal{X}}$ can be computed as

$$M(L_{\mathcal{X}}) = \sup\{\dim(U) : U \subset \mathcal{X}, D(u,u) < 0 \text{ for all } u \in U\}.$$

It is important to note that the boundary operator B depends only on D, and not on the space $\mathcal{X}$. In general the boundary conditions (and hence the domain of $L_{\mathcal{X}}$) will depend on both D and $\mathcal{X}$. To illustrate this dependence, we consider the form

$$D(u,v) = \int_\Omega [\nabla u \cdot \nabla v + Vuv] \tag{3}$$

on the following closed subspaces of $H^1(\Omega)$

$$\begin{aligned}
\mathcal{X}^0 &= H_0^1(\Omega),\\
\mathcal{X}^1 &= H^1(\Omega),\\
\mathcal{X}^2 &= \left\{u \in H^1(\Omega) : u|_{\Sigma_i} \text{ is constant for each } i\right\},\\
\mathcal{X}^3 &= \left\{u \in H^1(\Omega) : \int_{\Sigma_i} (\gamma u) d\mu = 0 \text{ for each } i\right\},
\end{aligned}$$

where $\{\Sigma_i\}$ are the connected components of $\partial\Omega$ and $d\mu$ is the induced volume form on $\partial\Omega$. Integrating by parts, we find that $L = -\Delta + V(x)$ and

$$Bu = \left.\frac{\partial u}{\partial N}\right|_{\partial\Omega}.$$

The selfadjoint operators $L_{\mathcal{X}^0}, \ldots, L_{\mathcal{X}^3}$ given by Proposition 2.1 are all restrictions of L, with domains

$$\begin{aligned}
\mathcal{D}(L_{\mathcal{X}^0}) &= H^2(\Omega) \cap H_0^1(\Omega),\\
\mathcal{D}(L_{\mathcal{X}^1}) &= \left\{u \in H^2(\Omega) : \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0\right\},\\
\mathcal{D}(L_{\mathcal{X}^2}) &= \Big\{u \in H^2(\Omega) : u|_{\Sigma_i} \text{ is constant and}\\
&\qquad \int_{\Sigma_i} \frac{\partial u}{\partial N}\, d\mu = 0 \text{ for each } i\Big\},\\
\mathcal{D}(L_{\mathcal{X}^3}) &= \Big\{u \in H^2(\Omega) : \int_{\Sigma_i} (\gamma u) d\mu = 0 \text{ and}\\
&\qquad \left.\frac{\partial u}{\partial N}\right|_{\partial\Sigma_i} \text{ is constant for each } i\Big\}.
\end{aligned}$$

Note that $L_{\mathcal{X}^0}$ and $L_{\mathcal{X}^1}$ correspond to the Dirichlet and Neumann Laplacian, respectively. The boundary conditions for $L_{\mathcal{X}^2}$ arise in the study of inviscid fluid flow on multiply-connected domains (see e.g. Section 5 of [15]). One can also represent Robin boundary conditions through appropriate choices of D and $\mathcal{X}$—the reader is referred to [8] for further examples.

2.2. **Scaling of domains.** We now suppose that $\{\Omega_t\}_{a\le t\le b}$ is a smooth one-parameter family of domains, with diffeomorphisms $\varphi_t : \Omega_b \to \Omega_t$ given such that $\varphi_b = \mathrm{Id}$. For instance, if Ω is star-shaped, we can define $\Omega_t = \{tx : x \in \Omega\}$ and $\varphi_t(x) = tx$ for $t \in [\epsilon, 1]$. Another simple example comes from the gradient flow of a Morse function f: if $f^{-1}[a, b] \subset \mathbb{R}^n$ is compact and contains no critical points, it is easy to construct a smooth family $\{\varphi_t\}$ such that $\varphi_t(\Omega) = f^{-1}(-\infty, t]$ for $t \in [a, b]$.

It will be assumed that the Dirichlet form D is defined on a domain in $\mathbb{R}^n$ that contains $\cup_{a\le t\le b}\Omega_t$; the examples above both satisfy $\Omega_{t_1} \subset \Omega_{t_2}$ for $t_1 < t_2$, in which case it suffices to have D defined on Ω_b. We then define a family of Dirichlet forms $\{D_t\}$ on $\mathcal{X} \subset H^1(\Omega)$ by

$$D_t(u, v) = D|_{\Omega_t}(u \circ \varphi_t^{-1}, v \circ \varphi_t^{-1}). \tag{4}$$

By construction each D_t is symmetric and coercive, so there exists a corresponding family of unbounded, selfadjoint operators $\{L_{\mathcal{X},t}\}$ on $L^2(\Omega_b)$ (cf. Proposition 2.1). There also exist operators L_t and B_t such that

$$D_t(u, v) = \langle L_t u, v\rangle_{L^2(\Omega)} + \int_{\partial\Omega} (B_t u)(\gamma v)d\mu \tag{5}$$

whenever $u, v \in H^1(\Omega)$ and $L_t u \in L^2(\Omega)$.

Our main result, Theorem 1, relates the Morse indices of the operators $\{L_{\mathcal{X},a}\}$ and $\{L_{\mathcal{X},b}\}$. Both operators are defined on (dense subsets of) $L^2(\Omega_b)$. However, it follows from a change of variables that the $(D_t, \mathcal{X})$ eigenvalue problem is equivalent to the $(D|_{\Omega_t}, \mathcal{X}_t)$ problem, where $\mathcal{X}_t := \{u \circ \varphi_t^{-1} : u \in \mathcal{X}\}$. To determine the corresponding boundary conditions on $\partial\Omega_t$, it is necessary to identify the domain $\mathcal{X}_t$ of the Dirichlet form $D|_{\Omega_t}$. For the examples considered above, we have

$$\mathcal{X}_t^0 = H_0^1(\Omega_t), \quad \mathcal{X}_t^1 = H^1(\Omega_t),$$
$$\mathcal{X}_t^2 = \left\{u \in H^1(\Omega_t) : u|_{\Sigma_{ti}} \text{ is constant for each } i\right\},$$
$$\mathcal{X}_t^3 = \left\{u \in H^1(\Omega_t) : \int_{\Sigma_{ti}} (\gamma u)(\varphi_t^{-1})^* d\mu = 0 \text{ for each } i\right\}.$$

In the first three cases $\mathcal{X}_t^j$ depends on Ω_t, but not the particular diffeomorphism $\varphi_t : \Omega \to \Omega_t$. On the other hand, $\mathcal{X}_t^3$ is not, in general, equal to the space

$$\left\{u \in H^1(\Omega_t) : \int_{\Sigma_{ti}} (\gamma u)d\mu_t = 0 \text{ for each } i\right\},$$

because the pulled back volume form $(\varphi_t^{-1})^* d\mu$ on $\partial\Omega_t$ does not necessarily agree with the induced form $d\mu_t$. Thus the interpretation of a conjugate time, i.e. a value of t for which the $(D|_{\Omega_t}, \mathcal{X}_t^3)$ problem has a nontrivial kernel, depends on the particular diffeomorphisms $\{\varphi_t\}$ and not just the family of domains $\{\Omega_t\}$.

It turns out one can always modify $\{\varphi_t\}$ to obtain a new family $\{\widehat{\varphi}_t\}$ such that $\widehat{\varphi}_t(\partial\Omega) = \varphi_t(\partial\Omega)$ for all t, and

$$\widehat{\mathcal{X}}_t^3 = \left\{ u \in H^1(\Omega_t) : \int_{\Sigma_{ti}} (\gamma u) d\mu_t = 0 \text{ for each } i \right\},$$

but we will not explore this issue any further in the current paper.

2.3. **A symplectic Hilbert space.** We define

$$\mathcal{H} = H^{1/2}(\partial\Omega) \oplus H^{-1/2}(\partial\Omega).$$

In Appendix B it is shown that $\mathcal{H}$ has the structure of a symplectic Hilbert space. Through a minor abuse of notation, we will henceforth denote the dual pairing between $H^{1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)^* \cong H^{-1/2}(\partial\Omega)$ by the integral notation

$$_{H^{1/2}(\partial\Omega)} \langle f, g \rangle_{H^{-1/2}(\partial\Omega)} = \int_{\partial\Omega} fg d\mu$$

for $f \in H^{1/2}(\partial\Omega)$ and $g \in H^{-1/2}(\partial\Omega)$.

We now construct two families of Lagrangian subspaces of $\mathcal{H}$, corresponding to the rescaled differential operators and boundary conditions, respectively. The space of weak solutions to $L_t u = \lambda u$, in the absence of boundary conditions, is defined by

$$K_{\lambda,t} = \left\{ u \in H^1(\Omega) : D_t(u,v) = \lambda \langle u, v \rangle_{L^2(\Omega)} \text{ for all } v \in H_0^1(\Omega) \right\} \tag{6}$$

for any $(\lambda, t) \in \mathbb{R} \times [a,b]$. We define a trace map $\operatorname{Tr}_t : C^1(\bar{\Omega}) \to C^0(\partial\Omega) \times C^0(\partial\Omega)$ by

$$\operatorname{Tr}_t u := (\gamma u, B_t u), \tag{7}$$

where γ as above is the Dirichlet trace operator (i.e. restriction to the boundary), and B_t is the rescaled boundary operator from (5). It is observed in Lemma 3.2 below that Tr_t can be extended to a bounded operator on $K_{\lambda,t}$, and so we can define

$$\mu(\lambda, t) := \operatorname{Tr}_t(K_{\lambda,t}). \tag{8}$$

We also define the space of admissible boundary values by

$$\nu := \left\{ (f,g) \in \mathcal{H} : f \in \gamma(\mathcal{X}), \int_{\partial\Omega} g(\gamma v) d\mu = 0 \text{ for all } v \in \mathcal{X} \right\}. \tag{9}$$

Again referring to the four examples above, we have

$$\nu^0 = \{0\} \oplus H^{-1/2}(\partial\Omega), \quad \nu^1 = H^{1/2}(\partial\Omega) \oplus \{0\},$$

$$\nu^2 = \left\{ (f,g) \in \mathcal{H} : f|_{\Sigma_i} \text{ is constant and } \int_{\Sigma_i} g \, d\mu = 0 \text{ for each } i \right\},$$

$$\nu^3 = \left\{ (f,g) \in \mathcal{H} : \int_{\Sigma_i} f \, d\mu = 0 \text{ and } g|_{\Sigma_i} \text{ is constant for each } i \right\}.$$

2.4. **Conjugate times.** The spaces $\mu(\lambda, t)$ and ν are defined such that a nontrivial intersection corresponds to an eigenvalue for $L_{\mathcal{X},t}$, as will be proved in Section 3.4 below.

Proposition 2.2. *The intersection $\mu(\lambda, t) \cap \nu$ is nontrivial if and only if there is a nonzero function $u \in \mathcal{D}(L_{\mathcal{X},t})$ with $L_{\mathcal{X},t}u = \lambda u$. Moreover,*

$$\dim [\mu(\lambda, t) \cap \nu] = \dim \ker (L_{\mathcal{X},t} - \lambda).$$

We say that $t_* \in [a, b]$ is a *conjugate time* if $\mu(0, t_*) \cap \nu \neq \{0\}$. By the above proposition, t_* is a conjugate time if and only if $L_{\mathcal{X},t_*}$ has a nontrivial kernel, which is true if and only if

$$\ker D_{t_*} := \{u \in \mathcal{X} : D_{t_*}(u, v) = 0 \text{ for all } v \in \mathcal{X}\}.$$

is nontrivial. By a change of coordinates we find that $\ker D_{t_*}$ is isomorphic to

$$\ker D|_{\Omega_{t_*}} := \{u \in \mathcal{X}_t : D|_{\Omega_{t_*}}(u, v) = 0 \text{ for all } v \in \mathcal{X}_{t_*}\}.$$

Referring to our model problem (3), it follows that $t_* \in [a, b]$ is a conjugate time for the $\mathcal{X}^0$ (Dirichlet) problem if there exists $u \in H^2(\Omega_{t_*})$ such that

$$-\Delta u + V(x)u = 0, \ \ u|_{\partial\Omega_{t_*}} = 0,$$

and is a conjugate time for the $\mathcal{X}^1$ (Neumann) problem if there exists $u \in H^2(\Omega_{t_*})$ such that

$$-\Delta u + V(x)u = 0, \ \ \left.\frac{\partial u}{\partial N_t}\right|_{\partial\Omega_{t_*}} = 0.$$

Analogous to (5), there is an operator $\widehat{B}_t$ such that

$$D|_{\Omega_t}(u, v) = \langle Lu, v\rangle_{L^2(\Omega_t)} + \int_{\partial\Omega_t} (\widehat{B}_t u)(\gamma v)\,d\mu \tag{10}$$

for all $u, v \in C^\infty(\overline{\Omega}_t)$ (and hence for all $u, v \in H^1(\Omega_t)$ such that $Lu \in L^2(\Omega_t)$, by a density argument—see the proof of Lemma 3.2 below for details). Note that in the example above we have $\widehat{B}_t = \partial/\partial N_t$ on $\partial\Omega_t$, whereas the rescaled boundary operator B_t (on $\partial\Omega$) is given by a somewhat complicated expression involving the Jacobian of the transformation φ_t.

2.5. **Main results.** By construction, $\{\mu(\lambda, t)\}$ is a smooth family of Lagrangian subspaces of $\mathcal{H}$, and has a well-defined Maslov index with respect to ν. Our main result is the following.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth domain, and $\varphi_t : \Omega \to \Omega_t$ a smooth family of diffeomorphisms for $t \in [a, b]$. Suppose D is an elliptic Dirichlet form with smooth coefficients, defined on a closed subspace $\mathcal{X}$ of $H^1(\Omega)$ that contains $H^1_0(\Omega)$, and define $L_{\mathcal{X},t}$, $\mu(\lambda, t)$ and ν as above. Then the Maslov index of $\mu(\lambda, t)$ with respect to ν is well-defined, and satisfies*

$$\textbf{\textit{Mas}}(\mu(0, t)|_{a\le t\le b}; \nu) = M(L_{\mathcal{X},a}) - M(L_{\mathcal{X},b}). \tag{11}$$

The Maslov index can be interpreted as a signed count of the number of conjugate times in $[a, b]$, and it is thus natural to wonder when the difference in Morse indices is in fact equal to the number of conjugate times. Such a result requires monotonicity of the Maslov index (with respect to t), in the sense that all crossings of $\mu(0, t)$ and ν occur with the same orientation. This is satisfied for the Dirichlet problem as long as the domains $\{\Omega_t\}$ satisfy a nesting property.

Corollary 2.3. *Assume $\Omega_{t_1} \subset \Omega_{t_2}$ for $t_1 < t_2$, and let $\mathcal{X} = H_0^1(\Omega)$. Then the number of conjugate times in $[a, b]$ is finite, and*

$$M(L_{\mathcal{X},b}) = M(L_{\mathcal{X},a}) + \sum_{t \in [a,b)} \dim \ker D_t. \tag{12}$$

This is precisely the index theorem proved by Smale in [24]. A symplectic interpretation was given by Swanson in [25, 26]; our method differs in its ability to handle much more general boundary conditions. Note that the sum includes $t = a$, but not $t = b$, so it is not relevant if $t = b$ is a conjugate time. This makes intuitive sense because the Dirichlet eigenvalues are monotone decreasing with respect to t, so a zero eigenvalue at $t = b$ will correspond to a positive eigenvalue for $t < b$, which does not contribute to the Morse index. More generally (i.e. for other boundary conditions), we have that a conjugate time $t = b$ can only contribute nonpositively to the Morse index.

While such monotonicity should not always be expected, the Maslov index allows one to compute the direction of intersection between μ and ν using crossing forms (as defined in Appendix B) and hence determine the net contribution to the Morse index from each conjugate time. A conjugate time by definition corresponds to a zero eigenvalue for D_t, with multiplicity $\dim \ker D_t$—the crossing form determines how many of these eigenvalues are increasing, and how many are decreasing, with respect to t. Related formulas for the motion of simple eigenvalues can be found in [4, 10, 12].

In the star-shaped case, where $\Omega_t := \{tx : x \in \Omega\}$ for $t \in (0, 1]$, the rescaled Dirichlet form D_t can be computed easily, and one obtains more explicit expressions for the crossing form than are available in the general case. In particular, we are able to deduce monotonicity results for the spectrum of the *Neumann Laplacian* $-\Delta_N$, which we define to be the unbounded, self-adjoint operator corresponding (via Proposition 2.1) to the Dirichlet form $D(u, v) = \int_\Omega \nabla u \cdot \nabla v$ with domain $\mathcal{X} = H^1(\Omega)$, and similarly for the rescaled operators $-\Delta_{N,t}$ on Ω_t. For ease of exposition we assume here (as in the rest of the paper) that $V(x)$ is smooth on $\overline{\Omega}$.

Corollary 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a smooth, star-shaped domain, and suppose λ is an eigenvalue of multiplicity k for $L_t := -\Delta_{N,t} + V(x)$, for some $t \in (0, 1)$. If*

$$\lambda > V(x) + \frac{1}{2} x \cdot \nabla V(x) \tag{13}$$

for all $x \in \Omega_t$, then the Morse index satisfies

$$M(L_{t+\delta} - \lambda) = M(L_{t-\delta} - \lambda) + k$$

provided $\delta > 0$ is sufficiently small.

In other words, as the domain expands from $\Omega_{t-\delta}$ to $\Omega_{t+\delta}$, the number of Neumann eigenvalues smaller than λ increases precisely by k, the multiplicity of λ at time t, assuming the eigenvalue λ is sufficiently large.

In particular, setting $V = 0$ we find that any positive eigenvalue of the Neumann Laplacian satisfies

$$M(-\Delta_{N,t+\delta} - \lambda) = M(-\Delta_{N,t-\delta} - \lambda) + k, \tag{14}$$

under the hypotheses of Corollary 2.4. While seemingly elementary, this result is actually rather subtle, because the monotonicity of the eigenvalues (or Morse index) for the Neumann Laplacian is known to fail for domains that are not star-shaped, even in the radially symmetric case. For instance, it was shown in [17] that the first (nonzero) Neumann eigenvalue on the annulus

$$A_{r,R} := \{x \in \mathbb{R}^n : r \leq |x| \leq R\}$$

is monotone decreasing with respect to both r and R. This is very different from the behavior of the first Dirichlet eigenvalue, which is decreasing in R but *increasing* in r (because the domain is shrinking at r increases).

By a unique continuation argument it in fact suffices to have

$$\lambda \geq V(x) + \frac{1}{2} x \cdot \nabla V(x)$$

for all x, with strict inequality on some (nonempty) open set in Ω_t. Since V and ∇V are uniformly bounded on Ω, there are only a finite number of eigenvalues (at each time t) for which this condition could potentially fail. If the potential is radial, $V(x) = f(|x|)$, this is equivalent to

$$\lambda \geq f(r) + \frac{r}{2} f'(r)$$

for $r \leq t$.

As a final example, suppose the potential satisfies

$$0 > V(x) + \frac{1}{2} x \cdot \nabla V(x)$$

for all $x \in \Omega$ (which in particular implies $V(0) < 0$). Then the Morse index of $L = -\Delta_N + V(x)$, with Neumann boundary conditions, can be related to the number of conjugate times $t \in (0,1)$, as in Corollary 2.3. Specifically, letting $c(t)$ denote the dimension of the solution space of

$$-\Delta u + V(x)u \text{ in } \Omega_t, \quad \frac{\partial u}{\partial N_t} = 0 \text{ on } \partial\Omega_t$$

for each $t \in (0,1)$, we have that

$$M(-\Delta_N + V) = \sum_{t \in (0,1)} c(t) + 1. \tag{15}$$

3. Construction of the symplectic path

In this section we give in detail the construction of the subspaces $\mu(\lambda, t)$ and ν outlined in Section 2. Throughout we consider the symplectic Hilbert space $\mathcal{H} := H^{1/2}(\partial\Omega) \oplus H^{-1/2}(\partial\Omega)$ with symplectic form ω defined by

$$\omega\left((f_1, g_1), (f_2, g_2)\right) = \int_{\partial\Omega} (f_1 g_2 - f_2 g_1) d\mu, \tag{16}$$

where $d\mu$ denotes the induced area form on $\partial\Omega$. We denote by $J : \mathcal{H} \to \mathcal{H}$ the almost complex structure on $\mathcal{H}$, given by

$$J(f, g) = \left(R^{-1} g, -Rf\right) \tag{17}$$

for $(f, g) \in \mathcal{H}$, where $R : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega) \cong H^{1/2}(\partial\Omega)^*$ is the Riesz duality isomorphism.

The main definitions and properties of symplectic Hilbert spaces are given in Appendix B; for now we simply recall that $\Lambda(\mathcal{H})$ denotes the *Lagrangian–Grassmannian* of $\mathcal{H}$, and $\mathcal{F}_\nu(\Lambda(\mathcal{H}))$ the *Fredholm–Lagrangian–Grassmannian* of $\mathcal{H}$ with respect to a fixed Lagrangian subspace $\nu \in \Lambda(\mathcal{H})$. The following proposition, which is the main result of this section, summarizes many of the properties of $\mu(\lambda, t)$ and ν needed in the proof of Theorem 1.

Proposition 3.1. *Assuming the hypotheses of Theorem 1, we have that* $\mu \in C^\infty\left(\mathbb{R} \times [a, b]; \mathcal{F}_\nu(\Lambda(\mathcal{H}))\right)$.

In particular, for each $(\lambda, t) \in \mathbb{R} \times [a, b]$ the subspaces $\mu(\lambda, t)$ and ν are Lagrangian, and comprise a Fredholm pair. Moreover, μ varies smoothly with respect to both λ and t. As described in Appendix B, the Maslov index is defined for any continuous path in the Fredholm–Lagrangian–Grassmannian, but its computation via crossing forms requires some degree of differentiability.

We assume for the remainder of the section that the hypotheses of Theorem 1 are satisfied.

3.1. **The trace map.**

Recall that for each $t \in [a, b]$, there exist operators L_t and B_t such that

$$D_t(u, v) = \langle L_t u, v \rangle_{L^2(\Omega)} + \int_{\partial\Omega} (B_t u)(\gamma v) d\mu$$

for all $u, v \in H^1(\Omega)$ and $L_t u \in L^2(\Omega)$.

We let A_t denote the principle part of L_t (written in divergence form), and define the space $H^{1,0}_{A_t}(\Omega)$ by

$$H^{1,0}_{A_t}(\Omega) := \{u \in H^1(\Omega) : A_t u \in L^2(\Omega)\},$$

with the graph norm $\|u\|^2_{A_t} := \|u\|^2_{H^1(\Omega)} + \|A_t u\|^2_{L^2(\Omega)}$. Note that $K_{\lambda,t} \subset H^{1,0}_{A_t}$, and in fact each $u \in K_{\lambda,t}$ satisfies

$$\|u\|_{A_t} \leq C\|u\|_{H^1(\Omega)}$$

for some constant C that depends on both λ and t. The following lemma shows that $H^{1,0}_{A_t}(\Omega)$ is an appropriate domain for the generalized trace operator.

Lemma 3.2. *For each $t \in [a,b]$ the map Tr_t defined in (7) extends to a bounded map*

$$\operatorname{Tr}_t \colon H^{1,0}_{A_t}(\Omega) \longrightarrow H^{1/2}(\partial\Omega) \oplus H^{-1/2}(\partial\Omega),$$

and the identity

$$D_t(u,v) = \langle L_t u, v\rangle_{L^2(\Omega)} + \int_{\partial\Omega} (B_t u)(\gamma v)\, d\mu \tag{18}$$

holds for all $u \in H^{1,0}_{A_t}(\Omega)$ and $v \in H^1(\Omega)$. Moreover, if $U \subset \mathbb{R} \times [a,b]$ is open and $u_{\lambda,t} \in C^k(U, H^1(\Omega))$ satisfies $u_{\lambda,t} \in K_{\lambda,t}$ for all $(\lambda,t) \in U$, then $\operatorname{Tr}_t(u_{\lambda,t}) \in C^k(U,\mathcal{H})$.

Proof. It is well known that $\gamma\colon H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is bounded, and the boundedness of $B_t\colon H^{1,0}_{A_t}(\Omega) \to H^{-1/2}(\partial\Omega)$ is given in Proposition 2.4 of [11]. The identity (18) is true for all $u, v \in C^\infty(\overline{\Omega})$ by the definition of L_t and B_t, and so the general result follows from a density argument and the continuity of γ and B_t.

Since $K_{\lambda,t} \subset H^{1,0}_{A_t}$, it follows from (18) that

$$\int_{\partial\Omega} (B_t u_{\lambda,t})(\gamma v)\, d\mu = D_t(u_{\lambda,t}, v) - \lambda \langle u_{\lambda,t}, v\rangle_{L^2(\Omega)} \tag{19}$$

for all $v \in H^1(\Omega)$ and all $(\lambda,t) \in U$. Equivalently,

$$\int_{\partial\Omega} (B_t u_{\lambda,t}) g\, d\mu = D_t(u_{\lambda,t}, Eg) - \lambda \langle u_{\lambda,t}, Eg\rangle_{L^2(\Omega)}$$

for all $g \in H^{1/2}(\partial\Omega)$, where $E : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ is a bounded right inverse for the Dirichlet trace γ. By assumption $u_{\lambda,t} \in C^k(U, H^1(\Omega))$ and D_t is smooth, so we find that $B_t u_{\lambda,t} \in C^k(U, H^{-1/2}(\partial\Omega))$. □

The next lemma, which is a consequence of the unique continuation property for second-order elliptic operators, shows that Tr_t in fact gives an isomorphism from $K_{\lambda,t}$ onto its image, $\mu(\lambda,t)$.

Lemma 3.3. *For each $(\lambda,t) \in \mathbb{R} \times [a,b]$ there exists $C = C(\lambda,t)$ such that*

$$\|u\|_{H^1} \le C \|\operatorname{Tr}_t u\|_{\mathcal{H}} \tag{20}$$

for every $u \in K_{\lambda,t}$.

The constant C can in fact be chosen uniformly in t, and uniformly in λ for any compact interval $\lambda \in [\lambda_1, \lambda_2]$, but such generality will not be required in our construction.

Proof. It follows from the coercivity estimate for D_t that

$$\|u\|_{H^1(\Omega)} \leq C(\|\operatorname{Tr}_t u\|_{\mathcal{H}} + \|u\|_{L^2(\Omega)}) \tag{21}$$

for all $u \in K_{\lambda,t}$, where C depends on λ and t. To obtain the stronger estimate (20) we argue by contradiction, using a standard compactness argument.

Assuming the existence of a sequence $\{u_i\}$ in $K_{\lambda,t}$ with $\|u_i\|_{L^2(\Omega)} = 1$ and $\|u_i\|_{H^1} \geq i\|\operatorname{Tr}_t u_i\|_{\mathcal{H}}$ for each i, we conclude from (21) that $\{u_i\}$ is bounded in $H^1(\Omega)$. Therefore there is a function $\bar{u} \in H^1(\Omega)$ with $\|\bar{u}\|_{L^2(\Omega)} = 1$, and a subsequence $\{u_i\}$, such that $u_i \to \bar{u}$ in $L^2(\Omega)$ and $u_i \rightharpoonup \bar{u}$ in $H^1(\Omega)$. It follows that $\bar{u} \in K_{\lambda,t}$, and it can be shown (cf. the proof of Lemma 3.8 below) that $\operatorname{Tr}_t u_i \rightharpoonup \operatorname{Tr}_t \bar{u}$ in $\mathcal{H}$. Since $\{u_i\}$ is bounded in $H^1(\Omega)$, we necessarily have $\operatorname{Tr}_t u_i \to 0$ in $\mathcal{H}$, hence $\operatorname{Tr}_t \bar{u} = 0$.

By construction $\bar{u} \in H^1(\Omega)$ is a non vanishing weak solution to $L_t\bar{u} = \lambda\bar{u}$, with boundary data $\gamma\bar{u} = 0$ and $B_t\bar{u} = 0$. However, it follows from a unique continuation argument that this is only possible if $\bar{u} \equiv 0$, so we obtain a contradiction and the proof is complete. The required unique continuation argument is given in Proposition 2.5 of [5]; see also Theorem 3.2.2 in [13], and the general survey [27]. □

In particular, this implies $\dim \operatorname{Tr}_t(V) = \dim V$ for any finite dimensional subspace $V \subset K_{\lambda,t}$ (cf. Proposition 2.2 above).

3.2. **The solution space.** We now turn our attention to the space $\mu(\lambda, t) = \operatorname{Tr}_t(K_{\lambda,t})$.

Lemma 3.4. *For each $(\lambda, t) \in \mathbb{R} \times [a, b]$, $\mu(\lambda, t)$ is a closed, isotropic subspace of $\mathcal{H}$.*

Proof. That $\mu(\lambda, t)$ is closed in $\mathcal{H}$ follows immediately from Lemmas 3.2 and 3.3 and the fact that $K_{\lambda,t}$ is a closed subspace of $H^1(\Omega)$. To see that $\mu(\lambda, t)$ is isotropic, consider $u, v \in K_{\lambda,t}$. It follows from (18) that

$$\int_{\partial\Omega} (B_t u)(\gamma v) d\mu = \int_{\partial\Omega} (B_t v)(\gamma u) d\mu,$$

hence $\omega(\operatorname{Tr}_t u, \operatorname{Tr}_t v) = 0$ as required. □

We next analyze the regularity of $\mu(\lambda, t)$ in the Lagrangian–Grassmannian, first recalling that the topology on $\Lambda(\mathcal{H})$ is defined by identifying a subspace μ with the corresponding orthogonal projection P_μ in the space of bounded operators $B(\mathcal{H})$. If $\rho \in \mathcal{H}$ is a fixed Lagrangian subspace, and $A : \rho \to \rho$ a bounded, selfadjoint operator, then the graph of A over ρ, defined by

$$G_\rho(A) := \{x + JAx : x \in \rho\},$$

is a Lagrangian subspace of $\mathcal{H}$, with J as in (17). By Equation (2.16) of [9] the corresponding orthogonal projection is

$$P_{G_\rho(A)}(x + Jy) = (I + A^2)^{-1}(x + Ay) + JA\left[(I + A^2)^{-1}(x + Ay)\right] \tag{22}$$

for $x, y, \in \rho$. Thus it suffices to express the spaces $\{\mu(\lambda, t)\}$ as the graphs of a smooth family $\{A(\lambda, t)\}$ of selfadjoint operators on some fixed Lagrangian subspace ρ.

If $L_t - \lambda$ has trivial Neumann kernel, then $\mu(\lambda, t)$ is just the graph of the Neumann-to-Dirichlet map over the Lagrangian subspace $\{0\} \oplus H^{-1/2}(\partial\Omega)$, which can be shown to vary smoothly in λ and t. More generally, in the proof of the following proposition we observe that one can always perturb to find a Robin-type boundary condition for which $L_t - \lambda$ is invertible, then express $\mu(\lambda, t)$ as the graph of the corresponding Robin-to-Robin map.

Proposition 3.5. *For each* $(\lambda, t) \in \mathbb{R} \times [a, b]$*,* $\mu(\lambda, t)$ *is a Lagrangian subspace of* $\mathcal{H}$*, and* $\mu \in C^\infty(\mathbb{R} \times [a, b], \Lambda(\mathcal{H}))$*.*

Proof. Let $(\lambda_0, t_0) \in \mathbb{R} \times [a, b]$. We will construct an open set $U \subset \mathbb{R} \times [a, b]$ containing (λ_0, t_0), a Lagrangian subspace $\rho \subset \mathcal{H}$, and a family of bounded, selfadjoint operators $A(\lambda, t) : \rho \to \rho$, such that $G_\rho(A(\lambda, t)) = \mu(\lambda, t)$ for all $(\lambda, t) \in U$. The family $\{A(\lambda, t)\}$ is smooth, in the sense that $A(\cdot, \cdot) \in C^\infty(U, B(\rho))$, so it follows from (22) that the map $(\lambda, t) \mapsto P_{\mu(\lambda,t)}$ is smooth, completing the proof.

To see that the claimed U and A exist, we define a perturbed Dirichlet form $D_{\beta,\lambda,t}$ by

$$D_{\beta,\lambda,t}(u, v) = D_t(u, v) - \lambda \langle u, v\rangle_{L^2(\Omega)} - \beta \int_{\partial\Omega} (R\gamma u)(\gamma v) d\mu$$

for $u, v \in H^1(\Omega)$ and $\beta \in \mathbb{R}$. It follows from Theorem 3.2 of [23] that $D_{\beta_0,\lambda_0,t_0}$ is invertible for some $\beta_0 = \beta_0(\lambda_0, t_0) \in \mathbb{R}$, and Lemma C.2 then implies $D_{\beta_0,\lambda,t}$ is invertible in some neighborhood U of (λ_0, t_0).

We define the subspace

$$\rho := \{(f, g) \in \mathcal{H} : f + \beta_0 R^{-1} g = 0\},$$

which is easily seen to be Lagrangian, with

$$J\rho = \{(f, g) \in \mathcal{H} : g - \beta_0 R f = 0\}.$$

Let $(f, g) \in \rho$. For each $(\lambda, t) \in U$ there exists a unique function $u_{\lambda,t} \in H^1(\Omega)$ such that

$$D_{\beta_0,\lambda,t}(u_{\lambda,t}, v) = \int_{\partial\Omega} (g - \beta_0 R f)(\gamma v) d\mu \tag{23}$$

for all $v \in H^1(\Omega)$. In particular this implies $D_t(u_{\lambda,t}, v) = \lambda \langle u_{\lambda,t}, v\rangle_{L^2(\Omega)}$ for any $v \in H^1_0(\Omega)$, so $u_{\lambda,t} \in K_{\lambda,t}$. Proposition C.1 implies $u_{\lambda,t} \in C^\infty(U, H^1(\Omega))$, and it follows from Lemma 3.2 that

$$B_t u_{\lambda,t} = g - \beta_0 R f + \beta_0 R \gamma(u_{\lambda,t}) \tag{24}$$

gives a path in $C^\infty(U, H^{-1/2}(\partial\Omega))$.

Since J is an isomorphism, we can implicitly define an operator $A(\lambda, t) : \rho \to \mathcal{H}$ by

$$JA(\lambda, t)(f, g) := (\gamma(u_{\lambda,t}) - f, \beta_0 R \gamma(u_{\lambda,t}) - \beta_0 R f)\,.$$

It follows that $JA(\lambda,t)(f,g) \in J\rho$, so we in fact have $A(\lambda,t) : \rho \to \rho$.

To see that A is selfadjoint, we take (f_1, g_1) and (f_2, g_2) in ρ, and let u_1 and u_2 denote the respective solutions to (23) (omitting the λ and t subscripts for convenience). Writing (23) for u_1 with the test function $v = u_2$, and vice versa, we have

$$D_t(u_1, u_2) - \lambda \langle u_1, u_2 \rangle_{L^2(\Omega)} = \int_{\partial\Omega} [\beta_0 R(\gamma u_1 - f_1) + g_1](\gamma u_2) d\mu$$

and

$$D_t(u_2, u_1) - \lambda \langle u_2, u_1 \rangle_{L^2(\Omega)} = \int_{\partial\Omega} [\beta_0 R(\gamma u_2 - f_2) + g_2](\gamma u_1) d\mu.$$

Subtracting and using the fact that $\int_{\partial\Omega}(Rh_1)h_2 d\mu = \int_{\partial\Omega}(Rh_2)h_1 d\mu$ for any $h_1, h_2 \in H^{1/2}(\partial\Omega)$, we obtain

$$\int_{\partial\Omega} [g_1 - \beta_0 R f_1](\gamma u_2) d\mu = \int_{\partial\Omega} [g_2 - \beta_0 R f_2](\gamma u_1) d\mu.$$

We next recall the relation $\omega(x, y) = \langle Jx, y \rangle_{\mathcal{H}}$ for all $x, y \in \mathcal{H}$, and compute using the above equality

$$\begin{aligned} \langle A(f_1, g_1), (f_2, g_2) \rangle_{\mathcal{H}} &- \langle A(f_2, g_2), (f_1, g_1) \rangle_{\mathcal{H}} \\ &= \omega\left(JA(f_2, g_2), (f_1, g_1)\right) - \omega\left(JA(f_1, g_1), (f_2, g_2)\right) \\ &= \int_{\partial\Omega} \left[f_1 g_2 - \beta_0 (Rf_1) f_2 - f_2 g_1 + \beta_0 (Rf_2) f_1\right] d\mu \\ &= \omega\left((f_1, g_1), (f_2, g_2)\right). \end{aligned}$$

The right-hand side vanishes because ρ is Lagrangian, and it follows that $A(\lambda, t)$ is selfadjoint.

In particular, this implies the graph $G_\rho(A(\lambda, t)) \subset \mathcal{H}$ is Lagrangian, and hence maximal. We also have from the definition of A and (24) that

$$(f, g) + JA(\lambda, t)(f, g) = \mathrm{Tr}_t(u_{\lambda,t})$$

for any $(f, g) \in \rho$, and so $G_\rho(A(\lambda, t)) \subset \mu(\lambda, t)$. Since $\mu(\lambda, t)$ is isotropic (by Lemma 3.4), the maximality of $G_\rho(A(\lambda, t))$ implies $G_\rho(A(\lambda, t)) = \mu(\lambda, t)$. Therefore $\mu(\lambda, t) \subset \mathcal{H}$ is Lagrangian, and the corresponding orthogonal projections in $B(\mathcal{H})$ vary smoothly with respect to λ and t. □

3.3. **The boundary space.** We next discuss the space ν defined in (9).

Lemma 3.6. *The boundary space ν is a Lagrangian subspace of $\mathcal{H}$.*

Proof. We first observe that ν can be written as

$$\nu = \gamma(\mathcal{X}) \oplus R\left[\gamma(\mathcal{X})^\perp\right], \tag{25}$$

where $\gamma(\mathcal{X})^\perp$ denotes the orthogonal complement of $\gamma(\mathcal{X})$ inside $H^{1/2}(\partial\Omega)$. By definition, $g \in R\left[\gamma(X)^\perp\right]$ if and only if $\left\langle R^{-1}g, \gamma u\right\rangle_{H^{1/2}(\partial\Omega)} = 0$ for all $u \in$

$\mathcal{X}$. Since $\left\langle R^{-1}g, \gamma u\right\rangle_{H^{1/2}(\partial\Omega)} = \langle g, R\gamma u\rangle_{H^{-1/2}(\partial\Omega)}$, this implies $R\left[\gamma(X)^\perp\right] = [R\gamma(\mathcal{X})]^\perp$.

The subspace $\nu \subset \mathcal{H}$ is closed because the Dirichlet trace $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ admits a bounded right inverse, and Definition (9) immediately gives that ν is isotropic.

It follows from a direct computation that

$$J\nu = \gamma(\mathcal{X})^\perp \oplus R\left[\gamma(\mathcal{X})\right] = \nu^\perp, \tag{26}$$

hence ν is Lagrangian. □

It should be observed that the boundary space ν in our construction is rather special within the class of Lagrangian subspaces. Specifically, ν decomposes as a direct sum of $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$ factors, as in (25), hence $(f,g) \in \nu$ precisely when $(f,0)$ and $(0,g)$ are both contained in ν. This fact, which is not true of generic Lagrangian subspaces, is crucial in establishing the following energy estimate, which will be used below in the proof of Lemma 3.8.

Lemma 3.7. *Let P_ν denote the $\mathcal{H}$-orthogonal projection onto ν, and $P_\nu^\perp = I - P_\nu$ the corresponding projection onto $\nu^\perp$. There is a constant $C = C(\lambda, t)$ such that*

$$\|u\|^2_{H^1(\Omega)} \leq C\left(\|u\|^2_{L^2(\Omega)} + \left\|P_\nu^\perp \operatorname{Tr}_t u\right\|^2\right)$$

for each $u \in K_{\lambda,t}$.

Proof. It follows immediately from (18) and the coercivity of D_t that there is a constant $C' > 0$ such that

$$\|u\|^2_{H^1(\Omega)} \leq C'\left(\|u\|^2_{L^2(\Omega)} + \int_{\partial\Omega}(B_t u)(\gamma u)d\mu\right)$$

for $u \in K_{\lambda,t}$. For the boundary term we let $(f,g) = \operatorname{Tr}_t u$ and define

$$(f_1, g_1) = P_\nu(f,g),$$
$$(f_2, g_2) = P_\nu^\perp(f,g),$$

so that $f = f_1 + f_2$ and $g = g_1 + g_2$. We then have

$$\begin{aligned}\int_{\partial\Omega}(B_t u)(\gamma u)d\mu &= \int_{\partial\Omega}(f_1+f_2)(g_1+g_2)d\mu \\ &= \int_{\partial\Omega} f_1 g_2 d\mu + \int_{\partial\Omega} g_2 f_1 d\mu,\end{aligned}$$

where we have used the fact that

$$\int_{\partial\Omega} f_1 g_1 d\mu = \left\langle f_1, R^{-1}g_1\right\rangle_{H^{1/2}(\partial\Omega)} = 0$$

by (25) because $(f_1, g_1) \in \nu$, and similarly for $(f_2, g_2) \in \nu^\perp$ using (26).

It follows that

$$\int_{\partial\Omega} (B_t u)(\gamma u) d\mu \leq \frac{1}{2}\left(\epsilon \|(f_1, g_1)\|^2_{\mathcal{H}} + \epsilon^{-1}\|(f_2, g_2)\|^2_{\mathcal{H}}\right)$$
$$\leq \epsilon C'' \|u\|^2_{H^1(\Omega)} + \epsilon^{-1}\left\| P_\nu^\perp \operatorname{Tr}_t u \right\|^2$$

for any $\epsilon > 0$. We choose ϵ small enough that $\epsilon C' C'' < 1$ and the result follows with $C = C'/\epsilon$. □

3.4. **The intersection.** We complete the section by proving the Fredholm property of $\mu(\lambda, t)$ and ν, and giving a proof of Proposition 2.2.

Lemma 3.8. *For each $(\lambda, t) \in \mathbb{R} \times [a, b]$ we have $\mu(\lambda, t) \in \mathcal{F}_\nu(\Lambda(\mathcal{H}))$.*

Proof. By Proposition 2.2, any element in $\mu(\lambda, t) \cap \nu$ is of the form $\operatorname{Tr}_t u$, where $u \in \ker(L_{\mathcal{X},t} - \lambda)$. Since $\ker(L_{\mathcal{X},t} - \lambda)$ is finite dimensional (see e.g. Theorem 7.21 of [8]), so is $\mu(\lambda, t) \cap \nu$.

To complete the proof we must show that $\mu(\lambda, t) + \nu$ is closed and has finite codimension in $\mathcal{H}$. For the remainder of the proof we fix (λ, t) and abbreviate $\mu = \mu(\lambda, t)$. To see that $\mu + \nu$ is closed, it suffices (by Theorem IV.4.2 of [14]) to show that the number

$$\kappa := \inf_{x \in \mu, x \notin \nu} \frac{\operatorname{dist}(x, \nu)}{\operatorname{dist}(x, \mu \cap \nu)} \tag{27}$$

is positive. Let P and $\widehat{P}$ denote the orthogonal projections onto ν and $\mu \cap \nu$, respectively, so that $\operatorname{dist}(x, \nu) = \|x - Px\|_{\mathcal{H}}$ and $\operatorname{dist}(x, \mu \cap \nu) = \|x - \widehat{P}x\|_{\mathcal{H}}$.

We first show that there is a positive constant K such that

$$\|x\|_{\mathcal{H}} \leq K\|x - Px\|_{\mathcal{H}} \tag{28}$$

for all $x \in \mu \cap (\mu \cap \nu)^\perp$. Suppose not, so there exists a sequence $\{u_i\}$ in $K_{\lambda,t} \subset H^1(\Omega)$ such that the traces $x_i = \operatorname{Tr}_t u_i$ are orthogonal to $\mu \cap \nu$ and satisfy

$$\|x_i\|_{\mathcal{H}} \geq i\|x - Px\|_{\mathcal{H}}.$$

Rescaling, we can assume that $\|u_i\|_{L^2(\Omega)} = 1$ for each i. It follows from Lemma 3.7 that

$$\|u_i\|^2_{H^1(\Omega)} \leq C\left(1 + i^{-1}\|u_i\|^2_{H^1(\Omega)}\right),$$

hence the sequence $\{u_i\}$ is bounded in $H^1(\Omega)$, and there exists an element $\bar{u} \in H^1(\Omega)$ and a subsequence $\{u_i\}$ such that $u_i \to \bar{u}$ in $L^2(\Omega)$ and $u_i \rightharpoonup \bar{u}$ in $H^1(\Omega)$. This implies $\|\bar{u}\|_{L^2(\Omega)} = 1$, and $D_t(u_i, v) \to D_t(\bar{u}, v)$ for any $v \in H^1(\Omega)$, hence $\bar{u} \in K_{\lambda,t}$ and $\operatorname{Tr}_t \bar{u} \in \mathcal{H}$ is well-defined.

Lemma 3.2 implies $\{x_i\}$ is bounded, so there is a weakly convergent subsequence $x_i \rightharpoonup \bar{x}$ in $\mathcal{H}$. We now show that $\bar{x} = \operatorname{Tr}_t \bar{u}$. For the Dirichlet trace we let X be a smooth vector field on Ω with $X|_{\partial\Omega} = N$, where N is the outward unit normal to $\partial\Omega$, and observe that

$$\int_{\partial\Omega} (\gamma u_i)(\gamma v) = \int_\Omega \operatorname{div}(u_i v X) \to \int_\Omega \operatorname{div}(\bar{u} v X) = \int_{\partial\Omega} (\gamma \bar{u})(\gamma v)$$

for any $v \in H^1(\Omega)$. Since $\{\gamma u_i\}$ converges weakly in $H^{1/2}(\partial\Omega)$ (hence strongly in $L^2(\partial\Omega)$), we conclude that its limit must be $\gamma\bar{u}$. It follows from (18) that $B_t u_i \rightharpoonup B_t\bar{u}$ in $H^{-1/2}(\partial\Omega)$, hence $\operatorname{Tr}_t u_i \rightharpoonup \operatorname{Tr}_t \bar{u}$ as claimed. In particular, this implies $\bar{x} \in \mu$. We also have $\|x_i - Px_i\|_{\mathcal{H}} \to 0$, hence $\bar{x} \in \nu$. Finally, since each $x_i \in (\mu \cap \nu)^\perp$, the weak convergence $x_i \rightharpoonup \bar{x}$ implies $\bar{x} \in (\mu \cap \nu)^\perp$, and we conclude that $\bar{x} = 0$. This implies $\bar{u} = 0$ (by Lemma 3.3), a contradiction. This completes the proof of (28).

Recalling that P and $\widehat{P}$ are the orthogonal projections onto ν and $\mu \cap \nu$, and letting $x \in \mu$, we thus have

$$\begin{aligned} \operatorname{dist}(x, \mu \cap \nu) &= \|x - \widehat{P}x\|_{\mathcal{H}} \\ &\leq K\|(x - \widehat{P}x) - P(x - \widehat{P}x)\|_{\mathcal{H}} \\ &= K\|x - Px\|_{\mathcal{H}} \end{aligned}$$

where in the last equality we have used the fact that $P\widehat{P} = \widehat{P}$ because $\mu \cap \nu \subset \nu$. Referring to (27), we have shown that $\kappa \geq K^{-1} > 0$, hence $\mu + \nu$ is closed.

Finally, using the fact that μ and ν are Lagrangian, we find that the codimension of $\mu + \nu$ is equal to the dimension of

$$(\mu + \nu)^\perp = \mu^\perp \cap \nu^\perp = J\mu \cap J\nu = J(\mu \cap \nu),$$

which is finite because $\dim(\mu \cap \nu)$ is finite and J is an isomorphism. □

We conclude with the proof of Proposition 2.2, first proving a simple lemma about the Dirichlet trace restricted to a subspace of $H^1(\Omega)$.

Lemma 3.9. *Let $\mathcal{X} \subset H^1(\Omega)$ be a closed subspace that contains $H^1_0(\Omega)$, and let $u \in H^1(\Omega)$. Then $\gamma u \in \gamma(\mathcal{X})$ if and only if $u \in \mathcal{X}$.*

Proof. Suppose $\gamma u \in \gamma(\mathcal{X})$. Then there exists $w \in \mathcal{X}$ such that $\gamma u = \gamma w$, hence $\gamma(u - w) = 0$. This implies $u - w \in H^1_0(\Omega) \subset \mathcal{X}$, so $u = (u - w) + w \in \mathcal{X}$. □

Proof of Proposition 2.2. First suppose there exists a function $u \in \mathcal{D}(L_{\mathcal{X},t})$, not identically zero, such that $L_{\mathcal{X},t}u = \lambda u$. Then $D_t(u, v) = \lambda \langle u, v \rangle_{L^2(\Omega)}$ for all $v \in \mathcal{X}$, hence for all $v \in H^1_0(\Omega)$, so $u \in K_{\lambda,t}$. Comparing with (18), we find that

$$\int_{\partial\Omega} (B_t u)(\gamma v) d\mu = 0$$

for all $v \in \mathcal{X}$. This implies $\operatorname{Tr}_t u = (\gamma u, B_t u) \in \mu(\lambda, t) \cap \nu$. It follows from Lemma 3.3 that $\mu(\lambda, t) \cap \nu \neq \{0\}$.

Now suppose that $\mu(\lambda, t) \cap \nu \neq \{0\}$. By definition, there exists $u \in K_{\lambda,t}$ with nonvanishing trace $\operatorname{Tr}_t u \in \mu(\lambda, t) \cap \nu$. Since $\operatorname{Tr}_t u \in \nu$ we have $\gamma u \in \gamma(\mathcal{X})$, hence $u \in \mathcal{X}$ by Lemma 3.9. We also have from the definition of ν that

$$\int_{\partial\Omega} (B_t u)(\gamma v) d\mu = 0$$

for any $v \in \mathcal{X}$, hence

$$D_t(u,v) = \langle L_t u, v\rangle_{L^2(\Omega)}$$

for any $v \in \mathcal{X}$. It follows that $u \in \mathcal{D}(L_{\mathcal{X},t})$ and $L_{\mathcal{X},t}u = L_t u = \lambda u$. □

4. Proof of Theorem 1

We now prove the main theorem of the paper. As in [7], this will follow from the homotopy invariance of the Maslov index, together with a straightforward monotonicity computation (with respect to λ) and a uniform (in t) lower bound on the eigenvalues of $L_{\mathcal{X},t}$.

For any $\lambda_0 < 0$, $\mu(\lambda,t)$ defines a homotopy $[\lambda_0,0]\times[a,b] \to \mathcal{F}_\nu(\Lambda(\mathcal{H}))$, hence

(29)

$$\mathbf{Mas}(\mu(\lambda,a);\nu) + \mathbf{Mas}(\mu(0,t);\nu) = \mathbf{Mas}(\mu(\lambda_0,t);\nu) + \mathbf{Mas}(\mu(\lambda,b);\nu).$$

To prove Theorem 1 we simply analyze each of the terms in the above equation.

Lemma 4.1. *There exists a constant $\lambda_0 < 0$ such that $\mu(\lambda,t)\cap\nu = \{0\}$ for all $t \in [a,b]$ and $\lambda \leq \lambda_0$.*

In other words, the operators $L_{\mathcal{X},t}$ have eigenvalues bounded uniformly below for $t \in [a,b]$. We can thus choose λ_0 to ensure $\mathbf{Mas}(\mu(\lambda_0,t);\nu) = 0$.

Proof. By Proposition 2.2 it suffices to show that $D_t(u,u) \geq C\|u\|_{L^2(\Omega)}$ for all $u \in H^1(\Omega)$ and $t \in [a,b]$, where $C \in \mathbb{R}$ is independent of t. This follows from the continuity of the coefficients of D_t with respect to t and the compactness of the interval $[a,b]$ (cf. the proof of Proposition C.1 in Appendix A). □

The following lemma, along with (29), completes the proof of Theorem 1.

Lemma 4.2. *Fix $t_0 \in [a,b]$. Then $\boldsymbol{Mas}(\mu(\lambda,t_0);\nu) = -M(L_{\mathcal{X},t_0})$.*

Proof. Since the path $\lambda \mapsto \mu(\lambda,t_0)$ is smooth, we can determine its Maslov index from a crossing form computation. We claim that the path is negative definite (as defined in Appendix B) hence

$$\begin{aligned}\mathbf{Mas}(\mu(\lambda,t_0);\nu) &= -\sum_{\lambda_0\leq\lambda<0} \dim\left[\mu(\lambda,t_0)\cap\nu\right] \\ &= -\sum_{\lambda<0}\dim\left[\mu(\lambda,t_0)\cap\nu\right] \\ &= -M(L_{\mathcal{X},t_0}),\end{aligned}$$

where in the last two equalities we have used Lemma 4.1 and Proposition 2.2, respectively.

To prove the claimed monotonicity result, we assume there is a crossing at λ_*, so there exists a path $\{x_\lambda\}$ in $\mathcal{H}$ with $x_\lambda \in \mu(\lambda,t_0)$ for $|\lambda-\lambda_*| \ll 1$, and $x_{\lambda_*} \in \nu$. By Lemma 3.3 there is a smooth path $\{u_\lambda\}$ in $H^1(\Omega)$ such that $\mathrm{Tr}_{t_0} u_\lambda = x_\lambda$. Differentiating the equation $D_{t_0}(u_\lambda, v) = \lambda \langle u_\lambda, v\rangle_{L^2(\Omega)}$ with respect to λ and letting $' = \frac{d}{d\lambda}$, we find

$$D_{t_0}(u'_\lambda, v) = \left\langle \lambda u'_\lambda + u_\lambda, v \right\rangle_{L^2(\Omega)}$$

for all $v \in H^1_0(\Omega)$, so (18) implies

$$D_{t_0}(u'_\lambda, u_\lambda) = \left\langle \lambda u'_\lambda + u_\lambda, u_\lambda \right\rangle_{L^2(\Omega)} + \int_{\partial\Omega} \left(B_{t_0} u'_\lambda\right) u_\lambda d\mu,$$
$$D_{t_0}(u_\lambda, u'_\lambda) = \left\langle \lambda u_\lambda, u'_\lambda \right\rangle_{L^2(\Omega)} + \int_{\partial\Omega} \left(B_{t_0} u_\lambda\right) u'_\lambda d\mu.$$

Since D_{t_0} is symmetric, we obtain

$$\begin{aligned} Q(x_{\lambda_*}, x_{\lambda_*}) &= \omega\left(\mathrm{Tr}_{t_0} u_\lambda, \mathrm{Tr}_{t_0} u'_\lambda\right)\big|_{\lambda=\lambda_*} \\ &= -\|u_{\lambda_*}\|^2_{L^2(\Omega)}, \end{aligned}$$

which completes the proof. □

5. The crossing form

Having completed the proof of Theorem 1, we study the Maslov index in (11) in greater detail. This gives a signed count of the number of conjugate times in $[a, b]$, with the sign depending on the direction in which the subspace $\mu(0, t)$ passes though ν. This is intimately related to the monotonicity (or lack thereof) of the eigenvalues of $L_{\mathcal{X},t}$ with respect to t, which depends nontrivially on the boundary conditions. We elucidate this dependence by computing crossing forms for the Dirichlet and Robin problems introduced in Section 2.5, corresponding to the spaces $\mathcal{X}^0 = H^1_0(\Omega)$ and $\mathcal{X}^1 = H^1(\Omega)$.

5.1. **The general framework.**

We start with some computations that are valid for any boundary conditions, letting D'_t denote the derivative of the form D_t with respect to t, so that

$$\frac{d}{dt} D_t(u_t, v_t) = D'_t(u_t, v_t) + D_t(u'_t, v_t) + D_t(u_t, v'_t)$$

when u_t, v_t are differentiable paths in $H^1(\Omega)$.

Lemma 5.1. *Suppose $U \subset [a, b]$ is open and $u_t \in C^1(U, H^1(\Omega))$, with $u_t \in K_{0,t}$ for each t. Letting $' = d/dt$, we have*

$$\omega\left(\mathrm{Tr}_t u_t, (\mathrm{Tr}_t u_t)'\right) = D'_t(u_t, u_t). \tag{30}$$

Proof. From the definition of ω we have

$$\omega\left(\mathrm{Tr}_t u_t, (\mathrm{Tr}_t u_t)'\right) = \int_{\partial\Omega} \left[(B_t u_t)' \gamma u_t - (B_t u_t) \gamma u'_t\right] d\mu.$$

Recalling that $D_t(u_t, v) = \int_{\partial\Omega} (B_t u_t)(\gamma v) d\mu$ for all $v \in H^1(\Omega)$, we differentiate with respect to t and then evaluate at $v = u_t$ to find

$$D'_t(u_t, u_t) + D_t(u'_t, u_t) = \int_{\partial\Omega} (B_t u_t)'(\gamma u_t) d\mu.$$

We also have

$$D_t(u_t, u_t') = \int_{\partial\Omega} (B_t u_t)(\gamma u_t') d\mu$$

and the result follows from the symmetry of D_t.

□

It thus remains to compute $D_t'(u_t, u_t)$ when t is a conjugate time. We start by writing the Dirichlet form D abstractly as

$$D(u,u) = \int_\Omega F(x, u, \nabla u) dx \tag{31}$$

for $u \in H^1(\Omega)$.

Proposition 5.2. *Suppose $t_* \in [a,b]$ is a conjugate time and $u_{t_*} \in \ker L_{\mathcal{X},t_*}$. Let $\widehat{u} = u_{t_*} \circ \varphi_{t_*}^{-1}$ and $x_* = \operatorname{Tr}_{t_*} u_{t_*}$. Then the crossing form satisfies*

$$Q(x_*, x_*) = \int_{\partial\Omega_t} \left[F\left(y, \widehat{u}, \nabla\widehat{u}\right)(X \cdot N_t) - 2(\widehat{B}_t \widehat{u})(X\widehat{u}) \right] d\mu_t(y) \tag{32}$$

where $X = \varphi_t'$, N_t is the outward unit normal to $\partial\Omega_t$, $d\mu_t$ is the induced volume form on $\partial\Omega_t$, F is defined in (31)*, and $\widehat{B}_t$ is the boundary operator defined in* (10)*.*

Proof. From (31) and the definition of D_t we have

$$\begin{aligned} D_t(u,u) &= D|_{\Omega_t}\left(u \circ \varphi_t^{-1}, u \circ \varphi_t^{-1}\right) \\ &= \int_{\Omega_t} F\left(x, u \circ \varphi_t^{-1}, \nabla(u \circ \varphi_t^{-1})\right) dx. \end{aligned}$$

Differentiating and using Theorem 1.11 from [12] we obtain

$$\begin{aligned} D_t'(u,u) = &-2\, D|_{\Omega_t}\left(X(u \circ \varphi_t^{-1}), u \circ \varphi_t^{-1}\right) \\ &+ \int_{\partial\Omega_t} F\left(y, u \circ \varphi_t^{-1}, \nabla(u \circ \varphi_t^{-1})\right)(X \cdot N_t) d\mu_t(y). \end{aligned}$$

Setting $t = t_*$ and $u = u_{t_*}$ in the above formula, the result follows. □

We now consider some specific examples for the operator $L = -\Delta + V(x)$.

5.2. **The Dirichlet ($\mathcal{X}^0$) problem.** We use the Dirichlet form

$$D(u,v) = \int_\Omega \left[\nabla u \cdot \nabla v + V uv\right] dx, \tag{33}$$

which has boundary operator $\widehat{B}_t = \frac{\partial \widehat{u}}{\partial N_t}$. Suppose that t_* is a crossing time. With x_*, u_{t_*} and $\widehat{u}$ defined as in Proposition 5.2 we have

$$Q(x_*, x_*) = \int_{\partial\Omega_t} \left[\left(|\nabla\widehat{u}|^2 + V(y)\widehat{u}^2\right)(X \cdot N_t) - 2X\widehat{u}\frac{\partial\widehat{u}}{\partial N_t} \right] d\mu_t(y). \tag{34}$$

Since $\widehat{u}$ vanishes on $\partial\Omega_t$, this reduces to

$$Q(x_*, x_*) = \int_{\partial\Omega_t} \frac{\partial\widehat{u}}{\partial N_t} \left[(X \cdot N_t)\frac{\partial\widehat{u}}{\partial N_t} - 2X\widehat{u} \right] d\mu_t(y).$$

To simplify further, we decompose the velocity field X into normal and tangential components, $X = X^\top + (X \cdot N_t)N_t$, and observe that

$$X\widehat{u} = (X \cdot N_t)\frac{\partial \widehat{u}}{\partial N_t}$$

because $X^\top \widehat{u} = 0$. It follows that

$$Q(x_*, x_*) = -\int_{\partial\Omega_t} \left(\frac{\partial \widehat{u}}{\partial N_t}\right)^2 (X \cdot N_t) d\mu_t. \tag{35}$$

More generally, for the operator $L = -\partial_i(a^{ij}\partial_j) + c$, the same computation yields

$$Q(x_*, x_*) = -\int_{\partial\Omega_t} a^{ij}(\partial_i \widehat{u})(\partial_j \widehat{u})(X \cdot N_t) d\mu_t.$$

In either case, we see that crossings for the Dirichlet problem are isolated and negative definite as long as $X \cdot N_t > 0$; the proof of Corollary 2.3 follows. Geometrically the condition $X \cdot N_t > 0$ means that Ω_t is flowing outward as t increases. If the family of diffeomorphisms is such that $X \cdot N_t$ changes sign on $\partial\Omega_t$, then the signature of the crossing form is more difficult to determine, as it depends on the structure of $\partial\widehat{u}/\partial N_t$ on the boundary.

The expression for Q given in (34) is in fact valid for any boundary value problem corresponding to the Dirichlet form (33) (i.e. for any choice of $\mathcal{X}$). In particular, we can use this to compute crossing forms for the Neumann problem, as well as the $\mathcal{X}^2$ and $\mathcal{X}^3$ problems defined above. The Robin boundary value problem requires a modification to the form (33), and is considered in detail below.

5.3. **The Robin ($\mathcal{X}^1$) problem.** We now consider the Dirichlet form

$$D(u, v) = \int_\Omega \left[\nabla u \cdot \nabla v + \nabla \cdot (uvY) + Vuv\right] dx$$

where Y is a smooth vector field on Ω. This corresponds to $L = -\Delta + V$, with the boundary operator

$$\widehat{B}_t u = \frac{\partial u}{\partial N_t} + \beta u$$

on Ω_t, where we have defined $\beta := (Y|_{\partial\Omega_t}) \cdot N_t$. Without loss of generality we may assume that Y has no component tangential to $\partial\Omega_{t_*}$, hence $Y|_{\partial\Omega_{t_*}} = \beta N_{t_*}$.

Since $\mathcal{X}^1 = H^1(\Omega)$ we have $\widehat{B}_t \widehat{u} = 0$ at a crossing time, so Proposition 5.2 yields

$$Q(x_*, x_*) = \int_{\partial\Omega_t} \left[|\nabla \widehat{u}|^2 + \nabla \cdot (\widehat{u}^2 Y) + V(y)\widehat{u}^2\right] (X \cdot N_t) d\mu_t(y).$$

Using the fact that $Y|_{\partial\Omega_t} = \beta N_t$ to compute the second term explicitly, we obtain

$$\begin{aligned}\nabla \cdot (\widehat{u}^2 Y) &= \nabla \cdot \big(\beta \widehat{u}^2 N_t\big) \\ &= \frac{\partial \beta}{\partial N_t}\widehat{u}^2 + 2\beta\widehat{u}\frac{\partial \widehat{u}}{\partial N_t} + \beta\widehat{u}^2(\nabla\cdot N_t) \\ &= \left(\frac{\partial \beta}{\partial N_t} - 2\beta^2 + \beta H_{\partial\Omega_t}\right)\widehat{u}^2\end{aligned}$$

where we have used the fact that $\widehat{B}_t\widehat{u} = 0$, and the mean curvature is defined to be $H_{\partial\Omega_t} = \nabla \cdot N_t$. Decomposing $\nabla\widehat{u} = \nabla^\top\widehat{u} + \frac{\partial\widehat{u}}{\partial N_t}$ into tangential and normal components, then applying the boundary conditions, we have

$$|\nabla\widehat{u}|^2 = |\nabla^\top\widehat{u}|^2 + \beta^2\widehat{u}^2,$$

and so the crossing form is

$$Q(x_*, x_*) =$$

$$\int_{\partial\Omega_t}\left[|\nabla^\top\widehat{u}|^2 + \left(V(y) - \beta^2 + \beta H_{\partial\Omega_t} + \frac{\partial\beta}{\partial N_t}\right)\widehat{u}^2\right](X\cdot N_t)d\mu_t(y). \tag{36}$$

We observe that the above crossing form precisely coincides with the formula for the first variation of a Robin eigenvalue derived by other means in [4] and [10]. One advantage of our symplectic formulation is that it describes the change in Morse index with respect to the domain parameter t, rather than the velocity of a single eigenvalue curve—as a result, one does not need to worry about multiplicities of eigenvalues, continuous vs. analytic eigenvalue curves, and so on.

5.4. **The star-shaped case.** We finally revisit the star-shaped case for the Dirichlet and Neumann problems. With

$$D(u,v) = \int_\Omega [\nabla u \cdot \nabla v + V(x)uv]\,dx$$

as above, $\varphi_t(x) = tx$ and $\Omega_t = \{tx : x \in \Omega\}$, a simple computation shows that

$$D_t(u,v) = t^{n-2}\int_\Omega \big[(\nabla u\cdot\nabla v) + t^2V(tx)uv\big]\,dx$$

and so

$$\begin{aligned}D_t'(u,v) =& (n-2)t^{n-3}\int_\Omega\big[(\nabla u\cdot\nabla v) + t^2V(tx)uv\big]\,dx \\ &+ t^{n-2}\int_\Omega \frac{d}{dt}\big[t^2V(tx)\big]\,uv\;dx.\end{aligned}$$

Now evaluating at a solution u_t to the equation $-\Delta u_t + t^2V(tx)u_t = 0$ (i.e. $L_tu_t = 0$) we find that

$$D_t'(u_t,u_t) = (n-2)t^{n-3}\int_{\partial\Omega} u_t\frac{\partial u_t}{\partial N}d\mu + t^{n-2}\int_\Omega u_t^2\frac{d}{dt}\big[t^2V(tx)\big]\,dx. \tag{37}$$

In particular, for either Dirichlet or Neumann boundary conditions, we have

$$D_t'(u_t, u_t) = t^{n-2} \int_\Omega \frac{d}{dt} u_t^2 \left[t^2 V(tx)\right] dx. \tag{38}$$

Replacing $V(x)$ by $V(x) - \lambda$, this simplifies to

$$D_t'(u_t, u_t) = t^{n-2} \int_\Omega u_t^2 \frac{d}{dt} \left[t^2 V(tx) - t^2 \lambda\right] dx, \tag{39}$$

and we can conclude that all crossings are negative definite provided

$$\frac{d}{dt}\left[t^2 V(tx) - t^2\lambda\right] < 0$$

for all $x \in \Omega$. This is equivalent to (13), so the proof of Corollary 2.4 follows immediately.

Appendix A. Selfadjoint operators and bilinear forms

In this appendix we review the correspondence between symmetric bilinear forms and selfadjoint operators, as described in Proposition 2.1. While the result is standard, it is worth reviewing the construction in some detail, as some of the constructions in the main body of the paper (in particular of the boundary space) require the explicit identification of the domain of the unbounded operator corresponding to a given form.

Our starting point is a Dirichlet form

$$D(u,v) = \int_\Omega \left[a^{ij}(\partial_i u)(\partial_j v) + b^i(\partial_i u)v + b^i u(\partial_i v) + cuv\right]$$

with real coefficients $a^{ij}, b^i, c \in C^\infty(\bar\Omega)$ satisfying $a^{ij} = a^{ji}$. We assume that D is strongly elliptic, so there exists a constant $\lambda_0 > 0$ such that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda_0 |\xi|^2$$

for all $x \in \bar\Omega$ and $\xi \in \mathbb{R}^n$. By construction D is symmetric, in the sense $D(u,v) = D(v,u)$ for all $u, v \in H^1(\Omega)$. Integrating by parts, we find that

$$D(u,v) = \int_\Omega \left[-\partial_i(a^{ij}\partial_j u) + (c - \partial_i b^i)u\right] v + \int_{\partial\Omega} N_i \left(b^{ij}\partial_j u + b^i u\right) v \tag{40}$$

provided $u, v \in H^1(\Omega)$ and $\partial_i(a^{ij}\partial_j u) \in L^2(\Omega)$, where $\{N_j\}$ are the components of the outward-pointing unit normal to $\partial\Omega$.

Lemma A.1. *Let $L = -\partial_i a^{ij}\partial_j + (c - \partial_i b^i)$ and $B = N_i(b^{ij}\partial_j + b^i)$. If $u, v \in H^1(\Omega)$ and $Lu \in L^2(\Omega)$, then*

$$D(u,v) = \langle Lu, v\rangle_{L^2(\Omega)} + \int_{\partial\Omega} (Bu)(\gamma v) d\mu.$$

It is easily shown that such a D is coercive and bounded over $H^1(\Omega)$, i.e. there exist constants $C_1, C_2 > 0$ and $\lambda \geq 0$ such that

$$D(u,u) \geq C_1 \|u\|^2_{H^1(\Omega)} - \lambda \|u\|^2_{L^2(\Omega)} \tag{41}$$

and

$$|D(u,v)| \leq C_2 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

for all $u, v \in H^1(\Omega)$. We now let $\mathcal{X}$ be a closed subspace of $H^1(\Omega)$ that contains $H^1_0(\Omega)$, and view D as an unbounded form on $L^2(\Omega)$ with domain $\mathcal{X}$. The coercivity estimate implies that

$$\|u\|_{+1} := \sqrt{D(u,u) + \lambda \|u\|^2_{L^2(\Omega)}}$$

is equivalent to the $H^1(\Omega)$ norm, hence D is closed and semibounded. Theorem XIII.15 of [19] (cf. Theorem VI.2.1 of [14]) implies there is a unique selfadjoint operator $L_{\mathcal{X}}$, with domain $D(L_{\mathcal{X}}) \subset \mathcal{X}$, such that

$$D(u,v) = \langle L_{\mathcal{X}} u, v \rangle_{L^2(\Omega)}$$

for all $u \in D(L_{\mathcal{X}})$ and $v \in \mathcal{X}$. Moreover, if there exist $u \in \mathcal{X}$ and $w \in L^2(\Omega)$ such that $D(u,v) = \langle w, v \rangle_{L^2(\Omega)}$ for all v contained in a core of D, then $u \in D(L_{\mathcal{X}})$ and $L_{\mathcal{X}} u = w$. (Recall that a *core* for D is an $H^1(\Omega)$-dense subspace $V \subset \mathcal{X}$.)

One expects $L_{\mathcal{X}}$ to formally agree with the differential operator L defined above. To verify this intuition, and explicitly identify the domain of $L_{\mathcal{X}}$, we now review the explicit construction given in [19]. Choosing λ as in (41), there is a well-defined operator ψ with domain

$$D(\psi) = \left\{ u \in \mathcal{X} : \exists w \in L^2(\Omega) \text{ with } \langle w, v \rangle_{L^2(\Omega)} = D(u,v) + \lambda \langle u, v \rangle_{L^2(\Omega)} \right\}$$

that is dense in $L^2(\Omega)$. If such a w exists (hence $u \in D(\psi)$), we set $\psi u = w$.

We define $L_{\mathcal{X}} = \psi - \lambda I$, with $D(L_{\mathcal{X}}) = D(\psi)$, and the following result is immediate.

Lemma A.2. *A function $u \in \mathcal{X}$ is an element of $D(L_{\mathcal{X}})$ if and only there exists $w \in L^2(\Omega)$ such that $\langle w, v \rangle_{L^2(\Omega)} = D(u,v)$ for all $v \in \mathcal{X}$; if this is the case, then $L_{\mathcal{X}} u = w$.*

We mention in passing that, if it exists, such a w is necessarily unique. If two such elements existed they would by definition satisfy

$$\langle w_1 - w_2, v \rangle_{L^2(\Omega)} = 0$$

for each $v \in \mathcal{X}$. Because $\mathcal{X}$ contains $H^1_0(\Omega)$ (and hence is dense in $L^2(\Omega)$) this implies $w_1 = w_2$.

We can relate this construction to the formal differential operators L and B defined above.

Lemma A.3. *Let $u \in \mathcal{X}$. Then $u \in D(L_{\mathcal{X}})$ if and only if $Lu \in L^2(\Omega)$ and*

$$\int_{\partial\Omega} (Bu)(\gamma v) d\mu = 0 \tag{42}$$

for every $v \in \mathcal{X}$.

Appendix B. The Maslov index in symplectic Hilbert spaces

We now review the definitions and basic properties of symplectic Hilbert spaces, the Fredholm–Lagrangian–Grassmannian, and the Maslov index. These will be our main tools in the proof of Theorem 1. Unless stated otherwise, technical details can be found in [9].

Let H be a real, infinite-dimensional, separable Hilbert space, and $\omega : H \times H \to \mathbb{R}$ a bilinear, skew-symmetric form. If the induced map $\omega^\sharp(x) := \omega(x, \cdot)$ is an isomorphism $H \to H^*$ we say that ω is *nondegenerate*, and call the pair (H, ω) a *symplectic Hilbert space*. For example, if E is a Hilbert space, we can set $H = E \oplus E^*$ and define

$$\omega((x, \phi), (y, \psi)) := \psi(x) - \phi(y),$$

which is easily seen to be nondegenerate.

A subspace μ of a symplectic Hilbert space H is said to be *isotropic* if $\omega(x, y) = 0$ for all $x, y \in \mu$. A *Lagrangian subspace* is then defined to be a maximal closed isotropic subspace of H. The set of all Lagrangian subspaces in H is called the Lagrangian–*Grassmannian*, and denoted by $\Lambda(H)$. Given the gap topology (where the distance between subspaces μ and ν, with respective orthogonal projections P_μ and P_ν, is the operator norm $\|P_\mu - P_\nu\|$), the Lagrangian–Grassmannian becomes a smooth, contractible Banach manifold, locally equivalently to the space of bounded, self-adjoint operators on H.

If follows that any homotopy invariant $C^0(\mathbb{S}^1; \Lambda(H)) \to \mathbb{Z}$ is necessarily trivial. This contrasts sharply with the finite-dimensional case, where we have $\pi_1(\Lambda(\mathbb{R}^{2n})) = \mathbb{Z}$. It is for this reason that we later introduce the Fredholm–Lagrangian–Grassmannian, which will have nontrivial topology and hence admit a nontrival homotopy invariant on its path space.

Before doing so, we give a useful criteria for finding Lagrangian subspaces.

Lemma B.1. *Suppose $\mu \subset H$ is a closed isotropic subspace. If there exists an isotropic subspace $V \subset H$ such that $H = \mu + V$, then μ is Lagrangian.*

Proof. Since μ is closed and isotropic, it suffices to prove that it is maximal. Suppose this is not the case, so there exists an isotropic subspace $\nu \supset \mu$ and a nonzero element $z \in \nu \cap V$. Now for $x + y \in \mu + V$ we have

$$\omega(x + y, z) = \omega(x, z) + \omega(y, z).$$

The first term vanishes since $x, z \in \nu$, and the second term vanishes since $y, z \in V$. But $x+y \in H$ was arbitrary so this contradicts the nondegeneracy of ω on H. $\square$

We next introduce the notion of a *Fredholm pair* in the Lagrangian–Grassmannian. This is precisely a pair of closed subspaces $\mu, \nu \in \Lambda(H)$ such that

(1) $\dim(\mu \cap \nu) < \infty$, and
(2) $\mu + \nu$ is closed and of finite codimension in H.

Then the *Fredholm–Lagrangian–Grassmannian* of (H, ω), with respect to fixed $\mu \in \Lambda(H)$, is the set

$$\mathcal{F}\Lambda_\mu(H) := \{\nu \in \Lambda(H) : (\mu, \nu) \text{ is a Fredholm pair}\}.$$

It is shown in [9] that this is an open subset of $\Lambda(H)$, and hence a smooth Banach manifold, with $\pi_1(\mathcal{F}\Lambda_\mu(H)) \cong \mathbb{Z}$.

We conclude our review by defining the *Maslov index* of a continuous path $\mu\colon [a,b] \to \mathcal{F}\Lambda_\nu(H)$. We first define a continuous path of operators $S = S_\nu(\mu)$ by the formula

$$S(t) := (2P_{\mu(t)} - I)(2P_\nu - I),$$

where P denotes orthogonal projection onto the designated subspace. This operator precisely consists of reflection across the subspace ν followed by reflection across $\mu(t)$. It can be shown that there exist times $a = t_0 < t_1 < \cdots < t_N = b$ and positive constants $\epsilon_j \in (0, \pi)$ for $1 \leq j \leq N$, such that

(1) $e^{\pm i\epsilon_j} \notin \sigma(S(t))$, and
(2) $\sum_{|\theta| \leq \epsilon_j} \dim\ker\left(S(t) - e^{i\theta}\right) < \infty$

for all $t \in [t_{j-1}, t_j]$. These conditions mean that as t increases from t_{j-1} to t_j, the number of eigenvalues of $S(t)$ on the arc $\{|\theta| \leq \epsilon_j\} \subset \mathbb{S}^1$ is always finite, and that no eigenvalues enter or exit this arc.

Following [18] and [6], we define the Maslov index by the formula

$$\mathbf{Mas}(\mu(t);\nu) := \sum_{j=1}^{N} \sum_{0 \leq \theta \leq \epsilon_j} \left[\dim\ker\left(S(t_j) - e^{i\theta}\right) - \dim\ker\left(S(t_{j-1}) - e^{i\theta}\right)\right].$$

This should be interpreted as a count (with appropriate sign and multiplicity) of the number of eigenvalues of $S(t)$ that cross through the point $1 \in \mathbb{S}^1$ in a counterclockwise direction as t increases from a to b.

To compute the Maslov index in practice, we will make frequent use of crossing forms (see [21, 22]). Suppose $\mu\colon [0,1] \to \mathcal{F}_\nu(\mathcal{H})$ is a C^1 path, and $t_* \in [a,b]$ is a *crossing time*, i.e. $\mu(t_*) \cap \nu \neq \{0\}$. Then for each t close to t_* there exists a bounded operator $A_t\colon \mu(t_*) \to \mu(t_*)$ such that $\mu(t)$ is equal to the graph

$$G_{\mu(t_*)}(A_t) = \{x + JA_t(x) : x \in \mu(t_*)\}.$$

The *crossing form* is the symmetric, bilinear form defined by

$$Q(x,y) = \frac{d}{dt}\omega\left(x, JA_t(y)\right)\bigg|_{t=t_*} \tag{43}$$

for all x, y in the finite-dimensional space $\mu(t_*) \cap \nu$. This is useful for the following reason.

Proposition B.2. *Let $\mu(t)$ by a C^1 path in $\mathcal{F}_\nu(\mathcal{H})$, and suppose $t_* \in [a,b]$ is a crossing time. Assume Q is nondegenerate, with p positive and q negative eigenvalues. If $t_* \in (a,b)$ and $\delta > 0$ is sufficiently small, then*

$$\textbf{\textit{Mas}}\left(\mu(t)|_{[t_*-\delta,t_*+\delta]};\nu\right) = p - q.$$

If $t_* = a$, then

$$\textbf{\textit{Mas}}\left(\mu(t)|_{[a,a+\delta]};\nu\right) = -q$$

and if $t_* = b$, then

$$\textbf{\textit{Mas}}\left(\mu(t)|_{[b-\delta,b]};\nu\right) = p.$$

In other words, the net contribution to $\textbf{Mas}(\mu(t);\nu)$ as t passes through t_* is determined by the signature of the associated crossing form. In the case where the crossing occurs at an endpoint of the curve, we see that an initial crossing ($t_* = a$) can only contribute negatively to the Maslov index, while a terminal crossing ($t_* = b$) can only contribute positively to the Maslov index. For instance, if $\mu(t)$ is a negative path (in the sense that at any crossing time the form Q is strictly negative), then

$$\textbf{Mas}(\mu(t);\nu) = -\sum_{t\in[a,b)} \dim\left(\mu(t)\cap\nu\right),$$

where $t = a$ is included in the above sum but $t = b$ is not. Similarly, for a positive curve one has

$$\textbf{Mas}(\mu(t);\nu) = \sum_{t\in(a,b]} \dim\left(\mu(t)\cap\nu\right),$$

where the sum now include $t = b$ but omits $t = a$.

Appendix C. Smooth families of Dirichlet forms

We say a Dirichlet form D on Ω is *invertible* if, for any nonzero $u \in H^1(\Omega)$, there exists $v \in H^1(\Omega)$ such that $D(u,v) \neq 0$.

Proposition C.1. *Let*

$$D_t(u,v) = \int_\Omega \left[a_t^{ij}(\partial_i u)(\partial_j v) + b_t^i(\partial_i u)v + c_t^i u(\partial_i v) + d_t uv\right]$$

be a one-parameter family of invertible, strongly elliptic Dirichlet forms, defined for t in some compact interval I. Assume that the coefficients $a_t^{ij}, b_t^i, c_t^i, d_t$ are in $C^\infty(\bar{\Omega})$ for each t, and are contained in $C^k\left(I; L^\infty(\Omega)\right)$ for some $k \geq 0$.

Let $\{F_t\}$ be a one-parameter family of bounded linear functionals on $H^1(\Omega)$, contained in $C^k(I; H^1(\Omega)^*)$. Then for each $t \in I$, there exists a unique $u_t \in H^1(\Omega)$ such that $D_t(u_t, v) = F_t(v)$ for every $v \in H^1(\Omega)$. Moreover, the path $t \mapsto u_t$ is contained in $C^k(I; H^1(\Omega))$.

Therefore, when as the boundary-value problem corresponding to D_t is uniquely solvable for each t (in the weak sense given by the Dirichlet form), the corresponding path of solutions u_t will be at least as smooth as the coefficients of D_t and the source term F_t.

Proof. From Theorem 7.13 of [8] we have that each D_t is coercive, i.e. there exist positive constants C_t and λ_t such that

$$|D_t(u,v)| \geq C_t \|u\|^2_{H^1(\Omega)} - \lambda_t \|u\|^2_{L^2(\Omega)}$$

for all $u, v \in H^1(\Omega)$. Moreover, it is immediate from the proof that C_t and λ_t only depend on the ellipticity constant of the form D_t and the norms $\|b^i_t\|_{L^\infty(\Omega)}$, $\|c^i_t\|_{L^\infty(\Omega)}$, and $\|d_t\|_{L^\infty(\Omega)}$. By compactness there exist constants C and λ such that

$$|D_t(u,v)| \geq C \|u\|^2_{H^1(\Omega)} - \lambda \|u\|^2_{L^2(\Omega)} \tag{44}$$

for all $u, v \in H^1(\Omega)$ and $t \in I$. The existence of u_t follows from the coercivity estimate and the invertibility of D_t (see Theorem 7.21 of [8]).

We next claim that there exists $A > 0$, independent of t (and $\{F_t\}$), such that

$$\|u_t\|_{H^1(\Omega)} \leq A \|F_t\|_{H^1(\Omega)^*}. \tag{45}$$

Assume this is not the case, so there exists a sequence of times $\{t_i\}$ such that $\|u_{t_i}\|_{H^1(\Omega)} \geq i \|F_{t_i}\|_{H^1(\Omega)^*}$. By linearity, we can rescale so that $\|u_{t_i}\|_{L^2(\Omega)} = 1$. Then the uniform coercivity bound implies $\{u_{t_i}\}$ is bounded in $H^1(\Omega)$, so there exists a subsequence with

$$\begin{aligned} u_{t_i} &\rightharpoonup \bar{u} \text{ in } H^1(\Omega), \\ u_{t_i} &\to \bar{u} \text{ in } L^2(\Omega), \\ t_i &\to \bar{t} \end{aligned}$$

for some $\bar{u} \in H^1(\Omega)$ and $\bar{t} \in I$. We also have that $F_{t_i} \to 0$ in $H^1(\Omega)^*$, so we obtain $D_{\bar{t}}(\bar{u}, v) = 0$ for all v, hence $\bar{u} = 0$. This contradicts the fact that $\|u_{t_i}\|_{L^2(\Omega)} = 1$ for each i and u_{t_i} converges strongly to $\bar{u}$ in $L^2(\Omega)$, so the proof of (45) is complete.

With this uniform estimate in place, we are ready to prove continuity of the path $t \mapsto u_t$. It suffices to prove continuity at a single point, say $t = 0$ (which we can assume is contained in I by performing a suitable translation, in necessary). From the definition of u_t we obtain

$$D_0(u_t - u_0, v) = (F_t - F_0)(v) - (D_t - D_0)(u_t, v).$$

Defining a functional $G_t \in H^1(\Omega)^*$ by $G_t(v) := (F_t - F_0)(v) - (D_t - D_0)(u_t, v)$, we have from (45) that

$$\|u_t - u_0\|_{H^1(\Omega)} \leq A \|G_t\|_{H^1(\Omega)^*}.$$

Since $\|u_t\|_{H^1(\Omega)}$ is uniformly bounded for t close to zero (again using (45)), the continuity of D_t and F_t implies $\|G_t\|_{H^1(\Omega)^*} \to 0$ as $t \to 0$, as desired.

This completes the proof for $k = 0$. Now assume the result holds for some $k \geq 0$, and assume that the coefficients of D_t and F_t are of class C^{k+1}. Differentiating the equation $D_t(u_t, v) = F_t(v)$ with respect to t, we obtain

$$D_t(u_t^{(k)}, v) = F_t^{(k)} - \sum_{j=0}^{k-1} \binom{k}{j} D_t^{(k-j)} \left(u_t^{(j)}, v\right) \tag{46}$$

For convenience we let $u_0^{(j)}$ denote the jth derivative of u_t evaluated at $t = 0$, and similarly for D_t. We claim that $u_0^{(k+1)}$ exists, and is equal to the unique function $w \in H^1(\Omega)$ that satisfies

$$\begin{aligned} D_0(w, v) =& F_0^{(k+1)}(v) - D^{(1)}(u_0^{(k)}, v) \\ & - \sum_{j=0}^{k-1} \binom{k}{j} \left[D_0^{(k+1-j)} \left(u_0^{(j)}, v\right) + D_0^{(k-j)} \left(u_0^{(j+1)}, v\right)\right] \end{aligned}$$

for all $v \in H^1(\Omega)$. Using (46), we find

$$\begin{aligned} D_0\left(w - \frac{u_t^{(k)} - u_0^{(k)}}{t}, v\right) &= \left(F_0^{(k+1)} - \frac{F_t^{(k)} - F_0^{(k)}}{t}\right)(v) \\ &+ \sum_{j=0}^{k} \left[\frac{D_t^{(k-j)}(u_t^{(j)}, v) - D_0^{(k-j)}(u_0^{(j)}, v)}{t} \right. \\ & \left. - D_0^{(k+1-j)}(u_0^{(j)}, v) - D_0^{(k-j)}(u_0^{(j+1)}, v) \right]. \end{aligned}$$

Since D_t and F_t are of class C^{k+1}, the right-hand side defines a linear functional $H_t(v)$ with $\|H_t\|_{H^1(\Omega)^*} \to 0$ as $t \to 0$, hence the uniform estimate (45) yields

$$\lim_{t=0} \left\| w - \frac{u_t^{(k)} - u_0^{(k)}}{t} \right\|_{H^1(\Omega)} = 0$$

as was to be shown. The continuity of $u_t^{(k+1)}$ is proved in a similar fashion. □

We finally observe that the assumption of invertibility on the entire interval I is not unreasonable.

Lemma C.2. *Let $\{D_t\}$ satisfy the hypothesis of Proposition C.1 with $k = 0$. If D_{t_0} is invertible for some $t_0 \in I$, then there exists $\epsilon > 0$ such that D_t is invertible when $|t - t_0| < \epsilon$.*

Proof. It suffices to consider $t_0 = 0$. Suppose the claimed result is false; then there exist real numbers $t_i \to 0$, and nonzero functions $u_i \in H^1(\Omega)$, such that $D_{t_i}(u_i, v) = 0$ for each i. Rescaling, we can assume that $\|u_i\|_{L^2(\Omega)} = 1$, hence the uniform coercivity estimate (44) implies that the sequence $\{u_i\}$ is bounded in $H^1(\Omega)$. We can thus find a subsequence (again denoted $\{u_i\}$)

and a function $\bar{u} \in H^1(\Omega)$ such that $u_i \rightharpoonup \bar{u}$ in $H^1(\Omega)$ and $u_i \to \bar{u}$ in $L^2(\Omega)$. It follows that $\|\bar{u}\|_{L^2(\Omega)} = 1$. However, since the coefficients of D_t are continuous, and weakly convergent subsequences are bounded, we find that

$$D_0(\bar{u}, v) = \lim_{i \to \infty} D_{t_i}(u_i, v) = 0$$

for each $v \in H^1(\Omega)$. The invertibility of D_0 yields $\bar{u} = 0$, which is not possible. $\square$

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