A CLASS OF STABLE PERTURBATIONS FOR A MINIMAL MASS SOLITON IN THREE DIMENSIONAL SATURATED NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. In this result, we develop the techniques of [11] and [3] in order to determine a class of stable perturbations for a minimal mass soliton solution of a saturated, focusing nonlinear Schrödinger equation in $\mathbb{R}^3$. Using dispersive estimates proved in [12], which are similar to those in [18] and [11], by projecting an initial perturbation ϕ onto a subspace of the continuous spectrum of the operator $\mathcal{H}$ resulting from linearization about a minimal mass soliton R_{min}, we are able to use a contraction mapping similar to that from [3] in order to show that there exist solutions of the form

$$e^{i\lambda_{\min}t}(R_{min} + e^{i\mathcal{H}t}\phi + w(x,t)),$$

where $e^{i\mathcal{H}t}\phi + w(x,t)$ either disperses as $t \to \infty$ or remains lower order on long time scales. Hence, we have long time persistance of a soliton of minimal mass despite the fact that these solutions are shown to be nonlinearly unstable in [5].

1. INTRODUCTION

In this result, we prove stability of solitons for a focusing, saturated nonlinear Schrödinger equation (NLS) in $\mathbb{R} \times \mathbb{R}^3$:

$$\begin{cases} iu_t + \Delta u + \beta(|u|^2)u = 0 \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $\beta : \mathbb{R} \to \mathbb{R}$, $\beta(s) \geq 0$ for all $s \in \mathbb{R}$, β has a specific structure outlined in one of the following definitions:

Definition 1.1. *Saturated nonlinearities of type* 1 *are of the form*

$$\beta(s) = s^{\frac{q}{2}} \frac{s^{\frac{p-q}{2}}}{1 + s^{\frac{p-q}{2}}}, \tag{1.2}$$

where $p > 2 + \frac{4}{3}$ *and* $\frac{4}{3} > q > 0$.

Definition 1.2. *Saturated nonlinearities of type* 2 *are of the form*

$$\beta(s) = \frac{s}{(1+s)^{\frac{2-q}{2}}}, \tag{1.3}$$

where $\frac{4}{3} > q > 0$.

Remark 1.1. *In both cases, for $|u|$ large, the behavior is L^2 subcritical and for $|u|$ small, the behavior is L^2 supercritical. For Definition 1.1, p is chosen much larger than the L^2 critical exponent, $\frac{4}{3}$, in order to allow sufficient regularity when linearizing the equation. In addition, there are clear extensions of these definitions for all dimensions, d, in the case of type 1 nonlinearities and dimensions $d \geq 3$ for type 2 nonlinearities.*

For a full statement of the stability theorems presented here for both types of nonlinearities, see Section 6 as a good deal of notation is required before the statements can be made rigorously. Saturated nonlinearities arise in various physical settings such as Bose superfluids, laser beam propagation and Bose-Einstein condenstates, see [22]. However, mathematically the author's interest was motivated by nonlinearities presented in the result on asymptotic stability in [17].

That such nonlinearities have minimal mass solitons can be observed numerically as seen in Figure 1 and discussed further below. In [5], the nonlinear instability of such a minimal mass soliton was proved, meaning that small generic perturbations of a minimal mass soliton can result in a large change in the profile of the solution on short time scales. It is precisely at this minimal mass that the celebrated variational requirements for stability/instability established in [23], [24] and generalized in [8], [19], [20], [21]. In this result, we follow the analysis presented in [3], who studies the existence of specific blow-up profiles, in order to build stable perturbations on both long time and global time scales depending on the structure of β.

The paper is structured as follows. In Sections 2 through 4, we recall some general properties of solutions to (1.1), introduce soliton solutions, discuss general soliton stability requirements and derive the matrix linear operator $\mathcal{H}$, which results from linearization of (1.1) about a soliton. In Section 4 we specifically discuss the existence of discrete and continuous spectrum for $\mathcal{H}$ and write down the necessary assumptions we require for our results. In Section 5, we recall the dispersive estimates and operator bounds on the solution operator generated by $\mathcal{H}$, which we refer to as $e^{it\mathcal{H}}$. Finally, in Section 6, we define our necessary function spaces and state the main theorems. In Section 7 we analyze the nonlinear terms resulting from our linearization for both types on nonlinearities, which are necessary to prove the global in time theorems for nonlinearities of type 1 in Sections 8-11 and the long time theorem for nonlinearities of type 2 in Section 12. In Section 13, we discuss the existence of manifold structure to our class of globally stable perturbations.

It should be noted that due to the nonlinear structure in the problem, though we initially perturb in a direction that is orthogonal to any instabilities predicted in the works [5] and similar to those expected from the works [23], [24], [8], [19], [20], [21], the resulting time dependent perturbation will not retain that orthogonality. Hence we are forced to instill some extra structure or "moment"-like conditions on our initial perturbation as in [3] to guarantee stability. Such conditions serve to provide stronger time dispersion of the solution provided one uses the proper weighted norm spaces.

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2. Conserved Quantities

In the sequel, we assume that $u_0 \in H^1$ and $|x|u_0 \in L^2$, or in other words, u_0 has finite variance. For this initial data, from the phase and time translation invariance of NLS, we have the following conserved quantities:

Conservation of Mass (or Charge):

$$Q(u) = \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 dx = \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2 dx,$$

and

Conservation of Energy:

$$E(u) = \int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} G(|u|^2) dx = \int_{\mathbb{R}^3} |\nabla u_0|^2 dx - \int_{\mathbb{R}^3} G(|u_0|^2) dx,$$

where

$$G(t) = \int_0^t \beta(s) ds.$$

We also have the pseudoconformal conservation law:

$$\|(x + 2it\nabla)u\|^2_{L^2(\mathbb{R}^3)} - 4t^2 \int_{\mathbb{R}^3} G(|u|^2) dx = \|x\phi\|^2_{L^2(\mathbb{R}^3)} - \int_0^t \theta(s) ds, \tag{2.1}$$

where

$$\theta(s) = \int_{\mathbb{R}^3} (4 \cdot (3+2) G(|u|^2) - 4 \cdot 3\beta(|u|^2)|u|^2) dx.$$

Note that $(x + 2it\nabla)$ is the invariant vector field given by the Hamilton flow of the linear Schrödinger equation, so the above identity relates how the solution to the nonlinear equation is effected by the linear flow.

Detailed proofs of these conservation laws can be arrived at easily using energy estimates or Noether's Theorem, which relates conservation laws to symmetries of an equation. Global well-posedness in $L^2(\mathbb{R}^3)$ of (NLS) with β of type 1 or 2 for finite variance initial data follows from standard theory for $L^2(\mathbb{R}^3)$ subcritical monomial nonlinearities. Proofs of the above results can be found in numerous excellent references for (NLS), including [4] and [22].

3. Soliton Solutions

A soliton solution is of the form

$$u(t,x) = e^{i\lambda t} R_\lambda(x)$$

where $\lambda > 0$ and $R_\lambda(x)$ is a positive, radially symmetric, exponentially decaying solution of the equation:

$$\Delta R_\lambda - \lambda R_\lambda + \beta(R_\lambda^2) R_\lambda = 0. \tag{3.1}$$

With this type of nonlinearity, soliton solutions exist and are known to be unique. Existence of solitary waves for nonlinearities of the type presented in Definitions 1.1 and 1.2 is proved by in [2] by minimizing the functional

$$T(u) = \int_{\mathbb{R}^3} |\nabla u|^2 dx$$

with respect to the functional

$$V(u) = \int_{\mathbb{R}^3} [G(|u|^2) - \frac{\lambda}{2}|u|^2] dx.$$

Then, using a minimizing sequence and Schwarz symmetrization, one sees the existence of the nonnegative, spherically symmetric, decreasing soliton solution. For uniqueness, see [15], where a shooting method is implemented to show that the desired soliton behavior only occurs for one particular initial value.

An important fact is that $Q_\lambda = Q(R_\lambda)$ and $E_\lambda = E(R_\lambda)$ are differentiable with respect to λ. This fact can be determined from the early works of Shatah, namely [19], [20]. By differentiating Equation (3.1), Q and E with respect to λ, we have

$$\partial_\lambda E_\lambda = -\lambda \partial_\lambda Q_\lambda.$$

Numerics show that if we plot Q_λ with respect to λ, we get a curve that goes to ∞ as $\lambda \to 0, \infty$ and has a global minimum at some $\lambda = \lambda_0 > 0$, see Figure 1. Variational techniques developed in [8] and [21] tell us that when $\delta(\lambda) = E_\lambda + \lambda Q_\lambda$ is convex, or $\delta''(\lambda) > 0$, we are guaranteed stability under small perturbations, while for $\delta''(\lambda) < 0$ we are guaranteed that the soliton is unstable under small perturbations. For brief reference on this subject, see [22], Chapter 4. For nonlinear instability at a minimum, see [5]. For notational purposes, we refer to a minimal mass soliton as R_{min}.

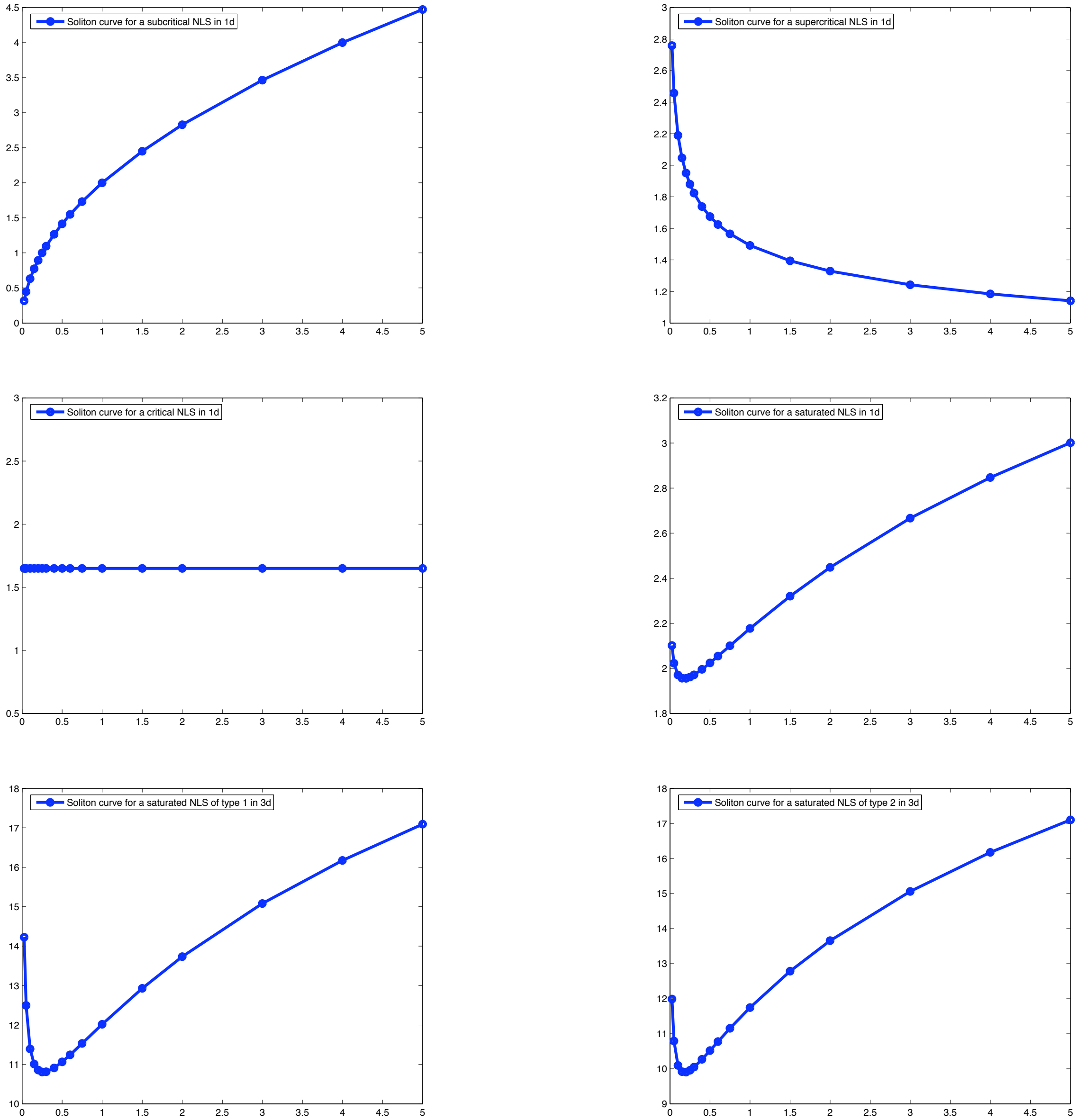

FIGURE 1. Plots of the soliton curves ($Q(\lambda)$ with respect to λ) for a subcritical nonlinearity ($d = 1$, $p = 3$), supercritical nonlinearity ($d = 3$, $p = 3$), critical nonlinearity ($d = 1$, $p = 5$), saturated nonlinearity of type 1 ($p = 7$, $q = 3$) in $\mathbb{R}$, saturated nonlinearity of type 1 in $\mathbb{R}^3$ ($p = 4$, $q = 2$), saturated nonlinearity of type 2 in $\mathbb{R}^3$ ($q = 2$). The curves for the monomial nonlinearities are found analytically, while the curves for the saturated nonlinearities are found numerically.

4. Linearization about a Soliton

Let us write down the form of NLS linearized about a soliton solution. First of all, we assume we have a solution $\psi = e^{i\lambda t}(R_\lambda + \phi(x,t))$. For simplicity, set $R = R_\lambda$. Inserting this into the equation we know that since ϕ is a soliton solution we have

$$i(\phi)_t + \Delta(\phi) \;=\; -\beta(R^2)\phi - 2\beta'(R^2)R^2\mathrm{Re}(\phi) + O(\phi^2), \tag{4.1}$$

by splitting ϕ up into its real and imaginary parts, then doing a Taylor Expansion. Hence, if $\phi = u + iv$, we get

$$i\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{H} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{4.2}$$

where

$$\mathcal{H} = \begin{pmatrix} 0 & iL_- \\ -iL_+ & 0 \end{pmatrix}, \tag{4.3}$$

where

$$L_- = -\Delta + \lambda - \beta(R_\lambda^2)$$

and

$$L_+ = -\Delta + \lambda - \beta(R_\lambda^2) - 2\beta'(R_\lambda^2)R_\lambda^2.$$

Let us fix the notation we will use in the sequel by defining the operator $e^{it\mathcal{H}}$ such that $\vec{u} = e^{it\mathcal{H}}\vec{f}$ is the solution to

$$\begin{cases} i\partial_t\vec{u} - \mathcal{H}\vec{u} = 0, \\ \vec{u}(0) = \vec{f}. \end{cases} \tag{4.4}$$

In order to study the behavior of solutions to (4.4), we must make the following assumptions:

Definition 4.1. *A Hamiltonian, $\mathcal{H}$ is called admissible if the following hold:*
1) There are no embedded eigenvalues in the essential spectrum, $(-\infty,\lambda]\cup[\lambda,\infty)$,
2) The only real eigenvalue in $[-\lambda,\lambda]$ *is* 0,
3) The values $\pm\lambda$ *are not resonances.*

Definition 4.2. *Let (NLS) be taken with nonlinearity β. We call β admissible if there exists a minimal mass soliton, R_{min}, for (NLS) and the Hamiltonian, $\mathcal{H}$, resulting from linearization about R_{min} is admissible in terms of Definition 4.1.*

For a minimal mass soliton, we have following the computations in [23] the following 10 dimensional set of generalized kernel functions

$$\left\{ \begin{bmatrix} 0 \\ R_{min} \end{bmatrix}, \begin{bmatrix} \partial_j R_{min} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x_j R_{min} \end{bmatrix}, \begin{bmatrix} (\partial_\lambda R)_{\lambda_{min}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \alpha \end{bmatrix}, \begin{bmatrix} \rho \\ 0 \end{bmatrix} \right\}.$$

From henceforward in this work, we assume that these functions span the entire discrete spectrum of $\mathcal{H}$. This assumption can be verified numerically away from the essential spectrum by adapting the techniques of [6]. Ideas for proving there are no eigenvalues embedded in the continuous spectrum will be explored in the forthcoming work [14].

The spectral properties we need for the linearized Hamiltonian equation in order to prove stability results are precisely those from Definition 4.1. Notationally, we refer to P_d and P_c as the projections onto the discrete spectrum of $\mathcal{H}$ and onto the continuous spectrum of $\mathcal{H}$ respectively.

Remark 4.1. *In this note we assume in the sequel that we work with admissible Hamiltonians satisfying the spectral assumptions in 4.1. However, analysis of these spectral conditions for a large class of $\mathcal{H}$ operators will be done both numerically and analytically in the forthcoming work* [14].

5. Review of Dispersive Estimates

We review here the dispersive estimates from [12]. Since [12] is a study of refined dispersive estimates related to the operator $e^{it\mathcal{H}}$, similar forms of many of the estimates stated here first appeared in several works, most notably [3], [7], [18]. When appropriate, the author will attribute credit for the theorems to the relevant works.

Let $\mathcal{S}$ be the Schwartz class of functions. Then, we have the following results:

Theorem 1 ([7],[12])**.** *Given an admissible Hamiltonian $\mathcal{H}$, P_c the projection on the continuous spectrum of $\mathcal{H}$, for initial data $\phi \in \mathcal{S}$, we have*

$$\|e^{it\mathcal{H}} P_c \phi\|_{L^\infty(\mathbb{R}^3)} \leq t^{-\frac{3}{2}} \|\phi\|_{L^1(\mathbb{R}^3)}.$$

Theorem 2 ([12])**.** *Let $\mathcal{H}$ be an admissible Hamiltonian as defined above. Let $\tilde{\phi}_\xi$ be an associated distorted Fourier basis as developed in* [12]*. Let*

$$\vec{\Psi}(\xi) = \int_y \tilde{\phi}_\xi(y) \vec{\psi}(y) dy.$$

Assume $\vec{\psi} \in L^{1,M}(\mathbb{R}^3)$ and

$$\partial_\xi^\alpha \partial_{|\xi|}^\beta \vec{\Psi}(0) = 0, \tag{5.1}$$

for multi-indices α, β such that $|\alpha| + |\beta| = 0, 1, 2, \dots, 2M$. Then,

$$\|e^{-c|x|} e^{it\mathcal{H}} P_c \vec{\psi}\|_{L^\infty(\mathbb{R}^3)} \leq C t^{-\frac{3}{2} - M} \|\vec{\psi}\|_{L^{1,M}(\mathbb{R}^3)}, \tag{5.2}$$

for any $c > 0$.

Overview of Proof Techniques. The results in Theorems 1 and 2 as presented in [12] follow from using standard scattering theory and microlocal analytic techniques presented in either [1], [9] and [10] to construct a distorted Fourier basis and hence write the non-self-adjoint operator $\mathcal{H}$ as an oscillatory integral. From such a construction, one is able to prove the necessary time decay using stationary/non-stationary phase techniques.

The estimate in Theorem 1 was first proved in [7] using resolvent estimates and a Birman-Schwinger approximation, which are clearly very closely related to the scattering theory techniques of the author. However the contributions of [12] is to fully represent the distorted Fourier basis in order to allow null or "moment" conditions to be defined with respect to such a basis. These are then applied as in the work [3] to get refined dispersive estimates as appear in Theorem 2.

It should be noted similar estimates for a problem in $\mathbb{R}$ also appear in [11] in order to improve the time integrability associated with the linear operator in 1 dimension. However, there the distorted Fourier basis is constructed using ODE/Wronskian methods. □

From Theorem 1, we also have the following results:

Theorem 3 ([3])**.** *Let P_c and P_d be projections onto the continuous and discrete spectrum of $\mathcal{H}$ respectively. Then,*

$$\begin{aligned}
(i)\ \|e^{it\mathcal{H}}(P_c\phi)\|_{H^1(\mathbb{R}^3)} &\leq C\|\phi\|_{H^1(\mathbb{R}^3)} \\
(ii)\ \|e^{it\mathcal{H}}(P_c\phi)\|_{H^s(\mathbb{R}^3)} &\leq C\|\phi\|_{H^s(\mathbb{R}^3)} \\
(iii)\ \|e^{it\mathcal{H}}(P_d\phi)\|_{H^s(\mathbb{R}^3)} &\leq C(1+|t|^3)\int e^{-c|x|}|\phi(x)|dx \\
(iv)\ \||x|^\alpha e^{it\mathcal{H}}(P_c\phi)\|_{L^2(\mathbb{R}^3)} &\leq C(\||x|^\alpha\phi\|_{L^2}+(1+|t|^\alpha)\|\phi\|_{H^\alpha}(\mathbb{R}^3)) \\
(v)\ \||x|^\alpha e^{it\mathcal{H}}(P_d\phi)\|_{L^2(\mathbb{R}^3)} &\leq C(1+|t|^3)\int |\phi|e^{-c|x|}dx,
\end{aligned}$$

for $s, \alpha \geq 0$.

Theorem 4 ([18])**.** *For p and p' such that $\frac{1}{p}+\frac{1}{p'}=1$, with $2 \leq p \leq \infty$, and $t \neq 0$, the transformation $e^{i\mathcal{H}t}$ maps continuously $L^{p'}(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$ and*

$$\|e^{i\mathcal{H}t}\phi\|_{L^p(\mathbb{R}^3)} \lesssim \frac{1}{|t|^{3(\frac{1}{2}-\frac{1}{p})}}\|\phi\|_{L^{p'}(\mathbb{R}^3)}. \tag{5.3}$$

Theorem 5 ([18])**.** *For every $\phi \in L^2$ and every admissible pair (q,r), the function $t \to e^{i\mathcal{H}t}\phi$ belongs to $L^q(\mathbb{R}, L^r(\mathbb{R}^3)) \cap C(\mathbb{R}, L^2(\mathbb{R}^3))$, and there exists a constant C depending only on q such that*

$$\|e^{i\mathcal{H}t}\phi\|_{L^q(\mathbb{R},L^r(\mathbb{R}^3))} \leq C\|\phi\|_{L^2(\mathbb{R}^3)}. \tag{5.4}$$

The proof of Theorem 3 follows from the proof of Proposition 2.8 in [3] and relies on the spectral assumptions presented in Definitions 4.1, 4.2 in addition to the decay properties of R. As shown in [18], the proof of Theorem 4 follows in the standard way by interpolation between the decay in Theorem 1 and the L^2 boundedness of $e^{it\mathcal{H}}$. Since $\mathcal{H}$ is not self-adjoint, Theorem 5 does not follow immediately from Theorem 1 from the usual duality arguments.

However, in Corollary 48 from Section 8 of [18], the result is established once the dispersive estimate in Theorem 1 is known to hold by applying the Christ-Kiselev lemma.

6. Main Results

To begin, we define the function space

$$\begin{aligned}\mathcal{P}_1^A(\mathbb{R}^3) &= \Big\{\phi \in L^2(\mathbb{R}^3) \mid \|\phi\|_{H^A(\mathbb{R}^3)} < \infty,\ \||x|^A\phi\|_{L^2(\mathbb{R}^3)} < \infty, \\ &\qquad \int_{\mathbb{R}^3} x^\alpha \phi(x)dx = 0 \text{ for } |\alpha| \leq 2A\Big\},\end{aligned}$$

with norm given by

$$\|\phi\|_{\mathcal{P}_1^A(\mathbb{R}^3)} = \left(\|\phi\|^2_{H^A(\mathbb{R}^3)} + \||x|^A\phi\|^2_{L^2(\mathbb{R}^3)}\right)^{\frac{1}{2}}.$$

We similarly define the function space

$$\begin{aligned}\mathcal{P}_2^A(\mathbb{R}^3) &= \big\{\phi : \mathbb{R}^3 \to \mathbb{C},\ \phi = P_c\phi \mid \|\phi\|_{H^A(\mathbb{R}^3)} < \infty,\ \||x|^A\phi\|_{L^2(\mathbb{R}^3)} < \infty, \\ &\qquad (5.1) \text{ is satisfied for } |\alpha| + |\beta| \leq A\big\},\end{aligned}$$

with norm given by

$$\|\phi\|_{\mathcal{P}_2^A(\mathbb{R}^3)} = \left(\|\phi\|^2_{H^A(\mathbb{R}^3)} + \||x|^A\phi\|^2_{L^2(\mathbb{R}^3)}\right)^{\frac{1}{2}}.$$

In this result, we prove that minimal mass solitons for nonlinear Schrödinger equations in three dimensions have stable perturbations in some sense. These minimal solitons are generically unstable as discussed in [5]. The main goal is to prove the following three theorems:

Theorem 6. *Let $\delta > 0$ be small and define $t_0 = \delta^{-1}$. Take the equation in $\mathbb{R} \times \mathbb{R}^3$*

$$\begin{cases} iu_t + \Delta u + \beta(|u|^2)u = 0 \\ u(t_0, x) = u_0(x), \end{cases} \tag{6.1}$$

where β is an admissible saturated nonlinearity of type 1. For any $\phi \in \mathcal{P}_1^A(\mathbb{R}^3)$, Equation (6.1) has a solution u for $t \in [t_0, \infty)$ of the form

$$u(x,t) = R_{min} + v(t) = R_{min} + e^{i\Delta t}\phi + w(x,t),$$

where R_{min} is the minimal mass soliton in Definition 4.1 and $\|w(\cdot,t)\|_{H^2(\mathbb{R}^3)} \to 0$ as $t \to \infty$. For Equation (6.1) in $\mathbb{R}^3$ of type 1, we have $A > \frac{13}{2}$.

Theorem 7. *Let $\delta > 0$ be small and define $t_0 = \delta^{-1}$. Take Equation (6.1), where β is an admissible saturated nonlinearity of type 1. For any $\phi \in \mathcal{P}_2^A(\mathbb{R}^3)$, Equation (6.1) has a solution u for $t \in [\frac{1}{\delta}, \infty)$ of the form*

$$u(x,t) = R_{min} + v(t) = R_{min} + e^{i\mathcal{H}t}\phi + w(x,t),$$

where R_{min} is the minimal mass soliton in Definition 4.1 and $\|w(\cdot,t)\|_{L^2(\mathbb{R}^3)} \to 0$ as $t \to \infty$. In this theorem, for Equation (6.1) in $\mathbb{R}^3$ of type 1, we have $A > \frac{5}{2}$.

Remark 6.1. *We note here that though we take t_0 large in order to use δ as a small parameter for the simplicity of the analysis. However, by time translation our results are valid for any $t_0 \in \mathbb{R}$.*

Theorem 8. *Let $\delta > 0$ be small. Given Equation (6.1), where β is an admissible saturated nonlinearity of type 2, for any $\phi = P_c\phi \in W^{2,1}(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$ with $\|\phi\|_{W^{2,1}(\mathbb{R}^3)\cap H^2(\mathbb{R}^3)} < \delta < 1$, Equation (6.1) has a solution for $t \in [0, \left(\frac{1}{2\delta}\right)^{\frac{1}{4}})$ of the form*

$$u(x,t) = R_{min} + v(t) = R_{min} + e^{i\mathcal{H}t}\phi + w(x,t),$$

where R_{min} is the minimal mass soliton in Definition 4.1 and

$$u(x,0) = R_{min} + \phi.$$

Remark 6.2. *In Theorems 6 and 7, the stable perturbations can be shown to live on a finite codimension manifold for powers p large enough in Definition 1.1.*

The class of functions $\mathcal{P}_i^A(\mathbb{R}^3)$ for $i = 1, 2$ will be developed throughout the course of this work. They will result from projecting onto a distorted Fourier basis for the linearized problem. For a further discussion these topics and the notion of distorted Fourier basis, see [12] and the refences contained within.

Remark 6.3. *It should be noted that similar results should hold in all dimensions provided one has the corresponding dispersive estimates. Particularly, for $d > 3$, one should be able to generalize the dispersive estimates proved in [12] using the same techniques. For $d = 1$, the estimates likely follow from careful analysis of the distorted Fourier basis constructed in [11].*

We also mention that using modulation techniques introduced in [23] and used extensive in [18] and [11], the codimension in the theorems above be capable of being reduced to at most 2 unstable direction as the others will be generated by symmetries of the soliton. Numerical evidence recently reported in [13] and suggested in [16] suggest in fact there should be a codimension 1 manifold of stable perturbations.

7. Preliminaries

We wish to construct a contraction argument similar to that presented in [3] in the case where we have a more general nonlinearity. In particular, we have the equation

$$\begin{cases} iu_t + \Delta u + F(|u|^2)u = 0, \\ u(0,x) = u_0(x), \end{cases}$$

where F is chosen to be of type 1 or type 2.

Many estimates that hold for the L^2 critical equation hold at that soliton because they share the property that $\partial_\lambda Q(u_\lambda) = 0$ where λ is the soliton parameter and Q is the L^2 mass. As this is a minimal mass soliton, there are many possible perturbations. One could perturb onto the manifold of stable solitons, onto the manifold of unstable solitons, or in fact, reduce the L^2 energy so that solitons no longer formed. Unfortunately, due to a lack of scaling and general difficulties, very little is known about stable perturbations to such a soliton. Also, it is a major question whether or not we have dispersion and scattering for initial data with L^2 mass below the minimal soliton mass. We hope to address this in future work, but for now we wish to prove the existence of stable solutions to the minimal mass soliton. We may assume that the minimal mass soliton occurs at $\lambda_0 = 1$. In other words, if R is the desired soliton, we seek a solution of the form

$$u = Re^{it} + z_\phi(x,t)e^{it} + we^{it}, \tag{7.1}$$

where $w \in C([\frac{1}{\delta}, \infty]; X)$ and $\|w\|_X \leq \frac{1}{t^N}$ for some normed space X and some large N to be determined. The goal is to solve this problem for both $z_\phi(x,t)$ in 7.1 defined as the solution to

$$\begin{aligned} iz_t - \Delta z &= 0, \\ z(0,x) &= \phi(x) \end{aligned}$$

and

$$\begin{aligned} iz_t - \mathcal{H} z &= 0, \\ z(0,x) &= \phi(x), \end{aligned}$$

where $\mathcal{H}$ the matrix Hamiltonian that results from linearizing about the minimal mass soliton. In other words, we take

$$z_\phi = e^{it\Delta}\phi \text{ or } z_\phi = e^{it\mathcal{H}}\phi.$$

To begin, we run through the contraction argument assuming that we are using the linear Schrodinger operator, $e^{i\Delta t}$, and the space $X = X_A$ defined by

$$X_A = \{\phi | \|\phi\|_{H^A} + \|(1+|x|)^A \phi\|_{L^2} < \infty\}.$$

Let $v_0 = z_\phi e^{-it}$ and let $u(x,t) = e^{it}(R+v)$ for $v = w + v_0$. Then, v must satisfy

$$iv_t + \Delta v - v + [F(|R+v|^2)(R+v) - F(R^2)R] = 0,$$

or

$$iv_t + \Delta v - v + (F(R^2) + F'(R^2)R^2)v + (F'(R^2)R^2)\bar{v} + O(|v|^2) = 0.$$

Since $i(v_0)_t + \Delta(v_0) - v_0 = 0$, we have

$$iw_t + \Delta w - w + [F(|R+v_0+w|^2)(R+v_0+w) - F(R^2)R] = 0.$$

Let

$$\begin{aligned}
f_0 &= F(|R+v_0|^2)(R+v_0) - F(R^2)R, \\
a &= [F(|R+v_0|^2) + F'(|R+v_0|^2)|R+v_0|^2] - [F(R^2) + F'(R^2)R^2], \\
b &= F'(|R+v_0|^2)(R+v_0)^2 - F'(R^2)R^2, \\
G(w) &= F(|R+v_0+w|^2)(R+v_0+w) - F(|R+v_0|^2)(R+v_0) \\
&- [F(|R+v_0|^2) + F'(|R+v_0|^2)|R+v_0|^2]w - F'(|R+v_0|^2)(R+v_0)^2\bar{w}.
\end{aligned}$$

Then, we have

$$iw_t + \Delta w - w + (F(R^2) + F'(R^2)R^2)w + F'(R^2)R^2\bar{w} + f_0 + aw + b\bar{w} + G(w) = 0.$$

In other words, we have

$$iw_t + \Delta w - w + (F(R^2) + F'(R^2)R^2)w + F'(R^2)R^2\bar{w} + aw + b\bar{w} + f_0 + G(w) = 0,$$

where G is at least quadratic in w and f_0 is linear in v_0.

To see this, note that for nonlinearities of type 1, we have

$$\begin{aligned}
F(x) &= \frac{x^{\frac{p}{2}}}{1 + x^{\frac{p-q}{2}}} \\
F'(x) &= \frac{x^{\frac{p}{2}-1}(\frac{p}{2} + \frac{q}{2}x^{\frac{p-q}{2}})}{(1 + x^{\frac{p-q}{2}})^2} \\
F''(x) &= \frac{x^{\frac{p}{2}-2}(\frac{p}{2}(\frac{p}{2}-1) + (pq - \frac{q^2}{4} - \frac{q}{2} - \frac{p^2}{2})x^{\frac{p-q}{2}} + (\frac{q^2}{4} - \frac{q}{2})x^{p-q})}{(1 + x^{\frac{p-q}{2}})^3},
\end{aligned}$$

and for type 2,

$$\begin{aligned}
F(x) &= \frac{x}{(1+x)^{\frac{2-q}{2}}} \\
F'(x) &= \frac{1 + \frac{q}{2}x}{(1+x)^{2-\frac{q}{2}}} \\
F''(x) &= \frac{(q-2) + \left(\frac{q^2}{4} - \frac{q}{2}\right)x}{(1+x)^{3-\frac{q}{2}}}.
\end{aligned}$$

Note that in both cases, $F \in C^1$ and in the second case, $F \in C^\infty$. However, we can define $G(z, \bar{z}) = F(|R+z|^2)(R+z)$. This is C^2 at $z = 0$ in both cases. To see this, note

$$\begin{aligned}
\partial_z G &= F'(|R+z|^2)(R+z)(R+\bar{z}) + F(|R+z|^2) \\
\partial_{\bar{z}} G &= F'(|R+z|^2)(R+z)^2 \\
\partial_{zz} G &= 2F'(|R+z|^2)(R+\bar{z}) + F''(|R+z|^2)(R+z)(R+\bar{z})^2 \\
\partial_{\bar{z}\bar{z}} G &= F''(|R+z|^2)(R+z)^3 \\
\partial_{z\bar{z}} G &= 2F'(|R+z|^2)(R+z) + F''(|R+z|^2)(R+z)^2(R+\bar{z}),
\end{aligned}$$

hence at $z = 0$, the terms resulting in exponential growth from F'' are controlled. In the resulting Taylor expansion, we see

$$G(z, z') = F(R^2)R + F'(R^2)R^2\bar{z} + (F'(R^2)R^2 + F(R^2))z + O(|R+z|^{p-1}|z|^2).$$

Let us make the assumption that we are working with type 1 nonlinearities with $\frac{10}{3} < p < \infty$. The author believes that similar results should hold even if p is not restricted to allow more regularity of the nonlinearity, however for the expansions in the sequel to be accurate, we must restrict the nonlinearities to have sufficient regularity, as well as to provide sufficient decay in t.

Let us explore the behaviors of the the above functions. For simplicity, let $v_0 = v_1 + iv_2$. To begin,

$$\begin{aligned}
|f_0| &= |F(R^2 + 2Rv_1 + v_1^2 + v_2^2)(R+v_0) - F(R^2)R| \\
&\lesssim |[F(R^2)R + O(|R+v_0|^p|v_0|) - F(R^2)R| \\
&\lesssim O(R^p|v_0|) + O(|v_0|^{p+1})|.
\end{aligned}$$

A similar calculation gives that

$$|a| \lesssim O(R^{p-1}|v_0|) + O(|v_0|^p),$$

and similarly for b, we have

$$|b| \lesssim O(R^{p-1}|v_0|) + O(|v_0|^p).$$

Finally, we have

$$|G| \lesssim O(|w|^2).$$

Hence, we solve the integral equation for w

$$w(t) = -i\int_t^\infty e^{i(\tau-t)\mathcal{H}}[f_0 + aw + b\bar{w} + G(w)]d\tau.$$

Though it is admittedly a slight abuse of notation to remain in the complex, scalar function domain, we mention here that the linear evolution of the operator

$$iw_t + \Delta w - w + (F(R^2) + F'(R^2)R^2)w + F'(R^2)R^2\bar{w},$$

which can be expressed as

$$i \begin{pmatrix} w \\ \bar{w} \end{pmatrix}_t - \tilde{\mathcal{H}} \begin{pmatrix} w \\ \bar{w} \end{pmatrix}$$

with

$$\tilde{\mathcal{H}} = \begin{bmatrix} -\Delta + 1 - F(R^2) - F'(R^2)R^2 & -F'(R^2)R^2 \\ F'(R^2)R^2 & \Delta - 1 + F(R^2) + F'(R^2)R^2 \end{bmatrix},$$

is equivalent to that of

$$i\vec{u} + \mathcal{H}\vec{u}$$

by a simple linear transform. Indeed, as seen in [18] we have

$$\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}^{-1} \tilde{\mathcal{H}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} = \mathcal{H},$$

the operator for which the appropriate linear estimates have been proved in [12, 7]. In order to follow more closely the analogous argument in [3] and hence simplify the exposition, we will consider the inhomogeneous terms in w but refer to the associated linear operator as $\mathcal{H}$ as well.

We would like to see that $\|w(t)\|_{X_A} \leq \frac{1}{t^M}$ for some M to be determined in order to show that we have a stable manifold of perturbations on the function space X^A. However, we are actually only able to prove $\|w(t)\|_{X^0} \leq \frac{1}{t^M}$, where $X^A \subset X^0$. The resulting effects of this will appear later in Section 13.

8. Contraction Argument

Making the assumption that

$$\int x^\beta \phi(x) dx = 0,$$

for the multi-index $|\beta| = 0, 1, 2, ..., 2M$ for M to be determined, we have by Taylor expanding the exponential in the fundamental solution that

$$z_\phi(x,t) = O\left(\frac{1}{t^{M+\frac{3}{2}}}\right)$$

where $|x| \lesssim 1$.

We have that

$$\begin{aligned} \|D^\alpha v_0\|_{L^\infty(\mathbb{R}^3)} &\leq \frac{C}{t^{\frac{3}{2}}}, \\ |e^{-c|x|} D^\alpha v_0(x,t)| &< \frac{C}{t^N}, \end{aligned}$$

for t and N large and the range of α's to be determined. In general, we explore low regularity perturbations but are in need of L^∞ bounds, so α will be small, but positive.

Then, we have for s in some range to be determined

$$\|(Rv_0)(t)\|_{H^s(\mathbb{R}^3)} \leq \frac{C}{t^{M+\frac{3}{2}}},$$

and hence, we have to check that for our space X^A,

$$\|f_0\|_{X^A(\mathbb{R}^3)} \leq \frac{C}{t^{\frac{3}{2}(p+1)}}.$$

where p is determined by the supercritical power in the nonlinearity and we have assumed the moments condition above for all $|\beta| \leq 2M$. In particular, note that $\gamma > \frac{4}{3}$, hence these terms have at least quadratic decay in t of the L^∞ norm.

We also have

$$|a| = O(R^{p-1}|v_0| + O(|v_0|^p)),$$

hence

$$\begin{aligned} |D^\alpha a(x,t)| &\leq \frac{C}{t^{\frac{3}{2}p}}, \\ |e^{-c|x|}D^\alpha a(x,t)| &\leq \frac{C}{t^{M+\frac{3}{2}}}. \end{aligned}$$

Similarly,

$$\begin{aligned} |D^\alpha b(x,t)| &\leq \frac{C}{t^{\frac{3}{2}p}}, \\ |e^{-c|x|}D^\alpha b(x,t)| &\leq \frac{C}{t^{M+\frac{3}{2}}}. \end{aligned}$$

Now, we look at the integral formulation of the equation for w

$$\begin{aligned} w(t) &= -i\int_t^\infty e^{i(\tau-t)\mathcal{H}} P_d[f_0 + aw + b\bar{w} + G(w)](\tau)d\tau \\ &+ -i\int_t^\infty e^{i(\tau-t)\mathcal{H}} P_c[f_0 + aw + b\bar{w} + G(w)](\tau)d\tau \\ &= w_d^\Delta + w_c^\Delta \end{aligned}$$

where P_S projects onto the singular part of the spectrum and P_M projects onto the discrete part of the spectrum.

Since we are interested in minimal regularity perturbations, we first want to see that

$$\|w(t)\|_{H^2(\mathbb{R}^3)} \leq \frac{1}{t^{N_1}},$$

for $t \geq \frac{1}{\delta}$ and N_1 to be determined.

Remark 8.1. *Though we begin with a perturbation that lives in the codimension* 10 *manifold orthogonal to the generalized null space, note that we do not expect the error terms to remain on such a manifold. Hence, we must control the projection of the error term onto the discrete spectrum as well, or the term* w_d.

To do this, we will discuss w_d^Δ and w_c^Δ separately.

From the following Corollary in [12],

$$\|P_c e^{it\mathcal{H}} f\|_{L^2(\mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R}^3)},$$

we have

$$\begin{aligned} \|w_d^\Delta\|_{H^2(\mathbb{R}^3)} &\leq \int_t^\infty [1+(\tau-t)^3] \left\{ \int |f_0 + aw + b\bar{w} + G(w)](x,\tau)| e^{-c|x|} dx \right\} dt \\ &\leq \int_t^\infty [1+(\tau-t)^3] [\frac{C}{\tau^{M+\frac{3}{2}}} + \frac{C}{\tau^{M+\frac{3}{2}}} \|w(\tau)\|_{H^2(\mathbb{R}^3)} + C\|w(\tau)\|^2_{H^2(\mathbb{R}^3)}] d\tau. \end{aligned}$$

Hence, by assuming M, N_1 large enough and using a bootstrapping argument

$$\|w_d^\Delta\|_{H^2(\mathbb{R}^3)} \leq \int_t^\infty [1+(\tau-t)^3] \left[\frac{C}{\tau^{2N_1}} \right] d\tau < \frac{C}{t^{2N_1-4}} < \frac{C\delta}{t^{N_1}}.$$

For the second part of this argument, we see

$$\begin{aligned} \|w_c^\Delta\|_{H^2(\mathbb{R}^3)} &\leq \int_t^\infty \|f_0 + aw + b\bar{w} + G(w)](\tau)\|_{H^2(\mathbb{R}^3)} d\tau \\ &\leq \int_t^\infty \left\{ \frac{C}{\tau^{\frac{3}{2}p}} + \frac{C}{\tau^{\frac{3}{2}p}} \|w(\tau)\|_{H^2(\mathbb{R}^3)} + C\|w(\tau)\|^2_{H^2(\mathbb{R}^3)} \right\} d\tau \\ &\leq \int_t^\infty \left\{ \frac{C}{\tau^{\frac{3}{2}p}} + \frac{C}{\tau^{N_1+\frac{3}{2}p}} + \frac{C}{\tau^{2N_1}} \right\} d\tau \\ &\leq \frac{C}{t^{N_1+1}} \leq \frac{C\delta}{t^{N_1}}, \end{aligned}$$

provided $\frac{3}{2}p$ is large enough.

We are also be able to show

$$\|w\|_{L^2(|x|^A dx)} \leq \frac{C}{t^{N_2}}$$

for $t > \frac{1}{\delta}$ and N_2 to be determined. Then, we will have the desired contraction argument for the linear perturbation.

From the necessary dispersive estimate given by

$$\||x|^\alpha e^{it\mathcal{H}}(P_S \phi)\|_{L^2} \leq C(1+|t|^3) \int |\phi| e^{-c|x|} dx,$$

the estimate for w_d^Δ follows immediately from the H^s argument.

For w_c^Δ, we need the following estimate

$$\||x|^\alpha e^{it\mathcal{H}}(P_M\phi)\|_{L^2(\mathbb{R}^3)} \leq C\||x|^\alpha \phi\|_{L^2(\mathbb{R}^3)} + C(1+|t|^\alpha)\|\phi\|_{H^\alpha(\mathbb{R}^3)}.$$

Then,

$$\begin{aligned}
\|w_c^\Delta\|_{L^2(|x|^A dx)} &\leq C\int_t^\infty \|[f_0 + aw + b\bar{w} + G(w)](\tau)\|_{L^2(|x|^A dx)} d\tau \\
&+ C\int_t^\infty (1+|t-\tau|^A)\|[f_0 + aw + b\bar{w} + G(w)](\tau)\|_{H^A(\mathbb{R}^3)} d\tau \\
&= I_{c,\Delta} + II_{c,\Delta}.
\end{aligned}$$

Again, we look at each integral separately. For $I_{c,\Delta}$,

$$\begin{aligned}
I_{c,\Delta} &\leq \int_t^\infty \left\{ \frac{C}{\tau^{2N}} + \frac{C}{\tau^2}\|w(\tau)\|_{L^2(|x|^A dx)} + C\|w(\tau)\|_{L^\infty}\|w(\tau)\|_{L^2(|x|^A dx)} \right\} d\tau \\
&< \left\{ \frac{C}{\tau^{2N}} + \frac{C}{\tau^{2+N1}} + \frac{C}{\tau^{N+N1}} \right\} d\tau \\
&< \frac{C}{t^{N1+1}} < \frac{C\delta}{t^{N1}}.
\end{aligned}$$

For $II_{c,\Delta}$,

$$\begin{aligned}
II_{c,\Delta} &\leq C\int_t^\infty [(1+(\tau-t)^A]\left\{ \frac{C}{\tau^{\frac{3}{2}p+A}} + C\|w(\tau)\|^2_{H^A(\mathbb{R}^3)} \right\} d\tau \\
&\leq C\int_t^\infty \tau^A \left\{ \frac{C}{\tau^{2N_2}} + \frac{C}{\tau^{\frac{3}{2}p}} \right\} d\tau \\
&\leq \frac{C}{t^{2N_2-A-1}} < \frac{C\delta}{t^{N_2}},
\end{aligned}$$

for N_2, p sufficiently large.

Hence, the contraction argument goes through and we have the desired bound on $\|w\|_X$.

9. Optimization for LS

We seek optimal values for the spaces and decay in the case where the perturbation solves the linear Schrodinger equation.

To begin, allow A and M to be arbitrary for now and we will select them later. Assume that $\phi \in X_A$ and that the first $2M$ moments of ϕ vanish. By writing the linear solution in integral form, we see that

$$\begin{aligned} \|D^\alpha v_0\|_{L^\infty(\mathbb{R}^3)} &\leq \frac{C}{t^{\frac{3}{2}}}, \\ |e^{-c|x|} D^\alpha v_0(x,t)| &\leq \frac{C}{t^{\frac{3}{2}+M}}, \end{aligned}$$

for $\alpha < M$.

To gain in time decay for the linear Schrödinger equation, we make the assumption that

$$\int x^\alpha \phi(x) dx = 0, \tag{9.1}$$

where α is a multi-index where $|\alpha| \leq M$ for M to be determined below. Note that the function space $\mathcal{P}_1^A$ in Section 6 is determined by functions $\phi \in X^A$ coupled with taking moments conditions for $|\alpha| \leq A$.

Lemma 9.1. *Since $R \in \mathcal{S}$,*

$$\|(Rv_0)(t)\|_{H^s} \leq \frac{C}{t^M}.$$

Proof. We have

$$\|(Rv_0)(t)\|_{L^2} \leq \|\langle x \rangle^{-N} v_0\|_{L^\infty} \|\langle x \rangle^N R\|_{L^\infty}.$$

Hence, using the principle of nonstationary phase away from the origin and the moments condition near the origin in the fundamental solution for linear Schrödinger, we gain in time decay. Note that in order to gain in time decay away from the origin, it is essential that we have the weight in order to control all of the terms resulting from integrating by parts. The higher derivative terms follow similarly.

□

Lemma 9.2. *For f_0 described above for $v_0 = e^{i\Delta t}\phi$, we have*

$$\|f_0\|_{H^2} \leq \frac{C}{t^{\frac{3}{2}p}}$$

and

$$\|e^{-c|x|} f_0\|_{L^\infty} \leq \frac{C}{t^{(\frac{3}{2}+M)p}}.$$

Proof. The $O(Rv_0)$ term is controlled by similar analysis to that in 9.1. Hence, we concern ourselves with the $O(|v_0|^{p+1})$ term. To that end, we have

$$\|v_0^{p+1}\|_{L^2} \leq \|v_0\|^p_{L^\infty} \|v_0\|_{L^2}.$$

Consequently, we have

$$\|v_0^{p+1}\|_{L^2} \lesssim \langle t \rangle^{-\frac{3}{2}p}.$$

□

Once again, since the decay rate is determined by the number of moments for the linear equation,

$$
\begin{aligned}
|e^{-c|x|} D^\alpha a(x,t)| &\leq \frac{C}{t^{M+\frac{3}{2}}}, \\
|e^{-c|x|} D^\alpha b(x,t)| &\leq \frac{C}{t^{M+\frac{3}{2}}}.
\end{aligned}
$$

As we desire to work with low regularity perturbations, let us simply assume that $\|w\|_{H^2}^2 \lesssim t^{-N}$. Now, we must choose A and M optimally for the contraction argument to work. For the analysis on w_d^Δ, we require that

$$
\begin{aligned}
2N - 4 &\geq N, \\
\frac{3}{2} + M - 4 &\geq N,
\end{aligned}
$$

where the moments condition is determined by the $O(v_0 R^p)$ term. So, we gather that $N > 4$ and $M > 8 - \frac{3}{2}$. The number of moments necessary will depend upon the the dimension d. In $\mathbb{R}^3$, we have $M > \frac{13}{2}$.

For the analysis on w_c^Δ, we have only one more requirement

$$
\frac{3}{2} p - 1 > N.
$$

At this stage, we see that given $N > 4$, we need $p > \frac{10}{3}$. Clearly, the restrictions on p lessen as d gets large. In particular, we cannot show the existence of stable perturbations for minimal mass solitons of NLS equations with nonlinearities of type 2 in $\mathbb{R}^3$. A variation of this argument will be explored later to show long time stability under restricted perturbations.

10. Linearization Scheme for $\mathcal{H}$-LS perturbations

Again, let

$$
u = Re^{it} + z_\phi e^{it} + we^{it},
$$

except now we have

$$
\begin{aligned}
iz_t + \mathcal{H} z &= 0, && (10.1) \\
z(0,x) &= \phi(x), && (10.2)
\end{aligned}
$$

where $\mathcal{H}$ is linear operator resulting from linearizing about the minimal mass soliton. We refer to Equation (10.1) as the $\mathcal{H}$linear Schrödinger equation ($\mathcal{H}$-LS).

Now, let $v_0 = z_\phi e^{-it}$. Again, we have the same equation,

$$
iv_t + \Delta v - v + F(|R+v|^2)(R+v) - F(R^2)R = 0,
$$

where $u(x,t) = e^{it}(R+v)$.

However, since

$$i(v_0)_t + \Delta v_0 - v_0 + [F(R^2) + F'(R^2)R^2]v_0 + F'(R^2)R^2\overline{v_0} = 0,$$

we have

$$\begin{aligned} iw_t &+ \Delta w - w + [F(|R+v_0+w|^2)(R+v_0+w) - F(R^2)R - (F(R^2)+F'(R^2)R^2)v_0 \\ &- F'(R^2)R^2\overline{v_0}] = 0. \end{aligned}$$

Hence, let

$$\begin{aligned} f_0 &= F(|R+v_0|^2)(R+v_0) - F(R^2)R - (F(R^2)+F'(R^2)R^2)v_0 - F'(R^2)R^2\overline{v_0}, \\ a &= [F(|R+v_0|^2) + F'(|R+v_0|^2)|R+v_0|^2] - [F(R^2)+F'(R^2)R^2], \\ b &= F'(|R+v_0|^2)(R+v_0)^2 - F'(R^2)R^2, \\ G(w) &= F(|R+v_0+w|^2)(R+v_0+w) - F(|R+v_0|^2)(R+v_0) \\ &- [F(|R+v_0|^2) + F'(|R+v_0|^2)|R+v_0|^2]w - F'(|R+v_0|^2)(R+v_0)^2\bar{w}. \end{aligned}$$

Hence, we now have

$$\begin{aligned} |f_0| &\lesssim O(R^p|v_0|^2) + O(|v_0|^{p+1})|, \\ |a| &\lesssim O(R^{p-1}|v_0|) + O(|v_0|^p), \\ |b| &\lesssim O(R^{p-1}|v_0|) + O(|v_0|^p) \\ |G| &\lesssim O(|w|^2). \end{aligned}$$

Notice that since the linear terms in v_0 have been removed from f_0, we expect to require fewer moments conditions for the contraction argument to hold.

11. Optimization for $\mathcal{H}$-LS

In the scheme where we solve the linear perturbation using $\mathcal{H}$, we have now introduced gain in the f_0 term. Specifically, we now have

$$|f_0| \lesssim O(R|v_0|^2) + O(|v_0|^{p+1}).$$

For the $\mathcal{H}$-LS case, we use the moment conditions derived in [12] in order to gain decay in time locally in space. Note that the function space $\mathcal{P}_2^A$ in Theorem 7 is determined by functions $\phi \in X^A$ coupled with taking moments conditions for $|\beta| \leq A$.

Lemma 11.1. *Since $R \in \mathcal{S}$, let $v_0 = e^{it\mathcal{H}}\vec{\psi}$ be such that conditions*

$$\partial_\xi^\alpha \partial_{|\xi|}^\beta \vec{\Psi}(0) = 0, \tag{11.1}$$

for multi-indices α, β such that $|\alpha| + |\beta| = 0, 1, 2, \ldots, 2M$, where

$$\vec{\Psi}(\xi) = \int_y \tilde{\phi}_\xi(y)\vec{\psi}(y)dy$$

and $\tilde{\phi}_\xi$ is a distorted Fourier basis for $\mathcal{H}$, defined in [12]*, Sections* 8 *and* 9*. Then, we have*

$$\|(Rv_0)(t)\|_{H^s} \leq \frac{C}{t^{\frac{3}{2}+M}}.$$

Proof. We have

$$\|(Rv_0)(t)\|_{L^2} \leq \|\langle x\rangle^{-N} v_0\|_{L^\infty} \|\langle x\rangle^N R\|_{L^\infty}.$$

Hence, , we gain in time decay using the principal of nonstationary phase away from the origin and the moments condition near the origin on

$$e^{i\mathcal{H}t} P_c \phi = Q^{-1} e^{itW} Q P_c \phi, \tag{11.2}$$

where

$$W = \begin{bmatrix} (\lambda^2+\xi^2) & 0 \\ 0 & -(\lambda^2+\xi^2) \end{bmatrix}$$

is a self-adjoint operator and

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} i(\lambda^2+\xi^2)^{\frac{1}{2}} \mathcal{F} L_-^{-\frac{1}{2}} & (\lambda^2+\xi^2)^{-\frac{1}{2}} \mathcal{F} L_-^{\frac{1}{2}} \\ -i(\lambda^2+\xi^2)^{\frac{1}{2}} \mathcal{F} L_-^{-\frac{1}{2}} & (\lambda^2+\xi^2)^{-\frac{1}{2}} \mathcal{F} L_-^{\frac{1}{2}} \end{bmatrix}.$$

We briefly note that $L_-^{-\frac{1}{2}}$ is not a well-defined operator on L^2 since L_- has nontrivial kernel. However, as seen in [12], the operator QP_c is well-defined on L^2 since we project away from the generalized null space of $\mathcal{H}$. Note that in order to gain in time decay away from the origin, it is essential that we have the weight in order to control all of the terms resulting from integrating by parts. The higher derivative terms follow similarly.

□

Lemma 11.2. *For f_0 described above for $v_0 = e^{i\mathcal{H}t}\phi$, we have*

$$\|f_0\|_{X_A} \leq \frac{C}{t^{\frac{3}{2}p}},$$

for $s < M$.

Proof. The $O(Rv_0)$ term is controlled by similar analysis to that in 11.1. Hence, we concern ourselves with the $O(|v_0|^{p+1})$ term. To that end, we have

$$\|v_0^{p+1}\|_{L^2} \leq \|v_0\|_{L^\infty}^p \|v_0\|_{L^2}.$$

Using a similar analysis from Lemma 9.2 on Equation (11.2), we have

$$\|v_0^{p+1}\|_{L^2} \lesssim \langle t\rangle^{\frac{3}{2}p}$$

using the L^2 boundedness results for $e^{i\mathcal{H}t}$.

□

Now, since we are dealing with nonlinearities with minimal smoothness, we wish to run the contraction argument with minimal assumptions on $w(x,t)$. Then, we assume

$$\|w(x,t)\|_{L^2} < \frac{1}{t^N},$$

for some N to be determined. Then,

$$\begin{aligned} w(t) &= -i\int_t^\infty e^{i(\tau-t)H} P_d[f_0 + aw + b\bar{w} + G(w)](\tau)d\tau \\ &+ -i\int_t^\infty e^{i(\tau-t)H} P_c[f_0 + aw + b\bar{w} + G(w)](\tau)d\tau \\ &= w_d^{\mathcal{H}} + w_c^{\mathcal{H}}, \end{aligned}$$

where P_d projects onto the discrete part of the spectrum and P_c projects onto the continuous part of the spectrum. So, using the dispersive estimates, we have

$$\begin{aligned} \|w_d^{\mathcal{H}}\|_{H^2} &\leq \| \int_t^\infty e^{i(\tau-t)H} P_S[f_0 + aw + b\bar{w} + G(w)](\tau)d\tau\|_{H^2} \\ &\leq \int_t^\infty [1+(\tau-t)^3] \left\{ \int |f_0 + aw + b\bar{w}G(w)](x,t)e^{-c|x|}dx \right\} d\tau \\ &\leq \int_t^\infty [1+(\tau-t)^3] \left\{ \int |f_0 + aw + b\bar{w}G(w)](x,t)e^{-c|x|}dx \right\} d\tau \\ &\lesssim \int_t^\infty [1+(\tau-t)^3] \left\{ \frac{1}{\tau^{2N}} + \frac{1}{\tau^N}\|w(\tau)\|_{L^2} + \|w(\tau)\|_{L^2}^2 \right\} d\tau \\ &\lesssim \frac{1}{t^{(\frac{3}{2}+M)2-4}} + \frac{1}{t^{(\frac{3}{2}+M)p+N-4}} + \frac{1}{t^{2N-4}}, \\ \|w_c^{\mathcal{H}}\|_{L^2} &\leq \| \int_t^\infty e^{i(\tau-t)H} P_M[f_0 + aw + b\bar{w} + G(w)](\tau)d\tau\|_{L^2} \\ &\leq \int_t^\infty \| \int |f_0 + aw + b\bar{w}G(w)](x,t)e^{-c|x|}dx\|_{L^2} d\tau \\ &\leq \left\{ \frac{1}{\tau^{\frac{3}{2}p}} + \frac{1}{\tau^N}\|w(\tau)\|_{H^2} + \|w(\tau)\|_{H^2}^2 \right\} \\ &\lesssim \frac{1}{t^{2N-1}} + \frac{1}{t^{M+N-1}} + \frac{1}{t^{2M-1}}. \end{aligned}$$

Hence, we require once again that that $2N-4 \geq N$, but the moments condition is determined now by the $O(v_0^2 R^{p-1})$ term, so we have

$$3 + 2M - 4 > 4.$$

In $\mathbb{R}^3$ that $M > \frac{5}{2}$, or $M > 2$. The condition on p however does not change whatsoever, therefore are again only considering nonlinearities of type 1 with $p > \frac{10}{3}$.

12. Long Time Analysis for Type 2 Nonlinearities

For NLS with saturated nonlinearities of type 2, we can no longer do the global scattering analysis from above. Instead, we have Theorem 8:

Theorem 9. *Given Equation* (6.1)*, where β is an admissible saturated nonlinearity of type 2, for any $\phi = P_c\phi \in W^{2,1} \cap H^2$ with $\|\phi\|_{W^{2,1}\cap H^2} < \delta < 1$, Equation* (6.1) *has a solution for $t \in [0, (2\delta)^{-\frac{1}{4}})$ of the form*

$$u(x,t) = R_{min} + v(t) = R_{min} + e^{i\mathcal{H}t}\phi + w(x,t),$$

where

$$u(x,0) = R_{min} + \phi.$$

Proof. Instead of the scattering point of view, we look at solving for the perturbation forward in time. Namely, we have

$$\begin{aligned} w(t) &= i\int_0^t e^{i\mathcal{H}(t-\tau)}[f_0 + aw + b\bar{w} + G(w)](\tau)d\tau \\ &= i\int_0^t e^{i\mathcal{H}(t-\tau)}P_d[f_0 + aw + b\bar{w} + G(w)](\tau)d\tau \\ &+ i\int_0^t e^{i\mathcal{H}(t-\tau)}P_c[f_0 + aw + b\bar{w} + G(w)](\tau)d\tau. \end{aligned}$$

Let us assume that

$$\|w\|_{L^\infty[0,T]H^2(\mathbb{R}^3)} < \delta. \tag{12.1}$$

Then

$$\begin{aligned}
\|w\|_{L^\infty[0,T]H^2(\mathbb{R}^3)} &\lesssim \Bigg\| \int_0^t [1+(t-\tau)^3] \left(\int f_0 e^{-c|x|} dx + \int a w e^{-c|x|} dx \right. \\
&+ \left. \int b \bar{w} e^{-c|x|} dx + \int G(w) e^{-c|x|} dx \right) d\tau \\
&+ \int_0^t [\|f_0\|_{H^2(\mathbb{R}^3)} + \|aw\|_{H^2(\mathbb{R}^3)} + \|b\bar{w}\|_{H^2} + \|G(w)\|_{H^2(\mathbb{R}^3)}] d\tau \Bigg\|_{L_t^\infty[0,T]} \\
&\lesssim \Bigg\| \int_0^t [1+(t-\tau)^3] \left(\langle \tau \rangle^{-3\left(\frac{3}{2}\right)} \|\phi\|_{L^1}^3 \right. \\
&+ \left. \langle \tau \rangle^{-\frac{3}{2}} \|w\|_{H^2(\mathbb{R}^3)} \|\phi\|_{L^1(\mathbb{R}^3)} + \|w\|_{H^2(\mathbb{R}^3)}^2 \right) d\tau \\
&+ \int_0^t [\langle \tau \rangle^{-3} \|\phi\|_{L^1}^2 \|\phi\|_{H^2} + \langle \tau \rangle^{-\frac{3}{2}} \|\phi\|_{L^1} \|w\|_{H^2(\mathbb{R}^3)} + \|w\|_{H^2(\mathbb{R}^3)}^2] d\tau \Bigg\|_{L_t^\infty[0,T]} \\
&\lesssim T^4 \|w\|_{L^\infty[0,T]H^2(\mathbb{R}^3)}^2 + T^{\frac{5}{2}} \|\phi\|_{L^1}^2 \|w\|_{L^\infty[0,T]H^2(\mathbb{R}^3)} + T^{-\frac{1}{2}} \|\phi\|_{L^1}^3 \\
&\lesssim \frac{\delta}{2} \|w\|_{L^\infty[0,T]H^2(\mathbb{R}^3)} + \delta^{3+\frac{1}{8}}.
\end{aligned}$$

Hence, by a continuity argument, w exists on $[0,T]$ with $\|w\|_{L^\infty[0,T]H^2(\mathbb{R}^3)} \leq \delta^{3+\frac{1}{8}}$. □

13. Manifolds of Perturbations for Type 1 Nonlinearities

From Theorems 6 and 7, we would like to know that our perturbative solution actually lives on a finite codimension submanifold. Specifically, given spaces Z_1, Z_2 with $Z_1 \subset Z_2$ and norms $\|\cdot\|_{Z_1}$, $\|\cdot\|_{Z_2}$ respectively, we require a finite codimension subset $\mathcal{S} \subset Z_1$ and a map $\Psi : \mathcal{B} \cap \mathcal{S} \to Z_2$ where

$$\mathcal{B} = \{\phi \in Z_1 | \|\phi\|_{Z_1} < \delta\}.$$

For Theorems 6 and 7 above, we have

$$\Psi(\phi) = w(t_0). \tag{13.1}$$

For our analysis, let us define

$$Z_1 = L^2(|x|^{3+} dx) \cap H^2, \tag{13.2}$$

$$Z_2 = H^2, \tag{13.3}$$

and

$$\mathcal{S} = \{\phi \in H^2 | \phi = P_c \phi,\ \phi \in \mathcal{P}_2^A\} \tag{13.4}$$

for some $A > 2$.

Lemma 13.1. *For the map* Ψ *defined by* (13.1)*, we have*

$$\|\Psi(\phi)\|_{Z_2} \lesssim \|\phi\|^2_{Z_1},\ \phi \in \mathcal{B} \cap \mathcal{S}, \tag{13.5}$$

$$\|\Psi(\phi_1) - \Psi(\phi_2)\|_{Z_2} \lesssim \delta \|\phi_1 - \phi_2\|_{Z_1},\ \phi_1, \phi_2 \in \mathcal{B} \cap \mathcal{S}. \tag{13.6}$$

Remark 13.1. *In Lemma 13.1, Equation* (13.5) *shows that the tangent space at* 0 *of the stable submanifold,* $\mathcal{M}$*, is the space* $\mathcal{S}$*, while Equation* (13.6) *shows that* $\mathcal{M}$ *is given by a Lipschitz parametrization.*

Remark 13.2. *The codimension of* $\mathcal{S}$ *will be at most* $2d+4$ *since* $H^1 \times H^1 = N_g(\mathcal{H}) \oplus \{N_g(\mathcal{H}^*)\}^\perp$ *and* $N_g(\mathcal{H}) = 2d+4$*. It is possible that the size of* $\mathcal{S}$ *can be improved beyond this codimension, which the author will explore in future work.*

Proof of Lemma 13.1. Let us first prove (13.5). Assume that $\|w\|_{Z_2} \leq \|\phi\|^2_{Z_1}$. Then,

$$\begin{aligned} \|w\|_{Z_2} &\leq \int_{t_0}^\infty (1 + (t_0 - \tau)^3) \left[\int v_0^2 e^{-c|x|} dx + \int v_0^2 e^{-c|x|} dx + \int v_0^2 e^{-c|x|} dx \right] d\tau \\ &+ \int_{t_0}^\infty [\|f_0\|_{X_2} + \|aw\|_{X_2} + \|b\bar{w}\|_{X_2} + \|G(w)\|_{X_2}]\, d\tau \\ &\lesssim t_0^{-\epsilon} \|\phi\|^2_{Z_1} \end{aligned}$$

using our assumptions as well as the decay of $\|w\|_{H^2}$ from the proof of Theorem 7. Hence, taking t_0 to be large, the result follows.

Now, for (13.6), we have

$$w_1 - w_2 = \int_{t_0}^\infty e^{i\mathcal{H}(\tau - t_0)} [(f_0^1 - f_0^2) + a(w_1 - w_2) + b(\bar{w}_1 - \bar{w}_2) + (G(w_1) - G(w_2))] d\tau.$$

Since

$$|G(w_1) - G(w_2)| \sim |w_1 + w_2||w_1 - w_2|$$

and

$$|f_0^1 - f_0^2| \sim R|\phi_1 + \phi_2||\phi_1 - \phi_2| + (|\phi_1|^p + |\phi_2|^p)|\phi_1 - \phi_2|,$$

the result follows from a similar continuity argument to that above using (13.5). $\square$

Remark 13.3. *Note that for p large enough in type* 1 *nonlinearities, using the dispersive estimates (iv), (v) from Theorem 3 and the fact that*

$$\|\phi\|_{L^1} \leq C(\epsilon) \|\phi \langle x \rangle^{3+\epsilon}\|_{L^\infty(\mathbb{R}^3)},$$

we can take $Z_1 = Z_2 = X_{3+}$ *to have a true manifold of perturbations.*

References

[1] S. Agmon. Spectral properties for Schrödinger operators and scattering theory, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **2**, 151-218 (1975).

[2] H. Berestycki and P. L. Lion. Nonlinear scalar field equations, I: Existence of a ground state, *Arch. Rational Mech. Anal.*, **82**, no. 4, 313-345 (1983).

[3] J. Bourgain, W. Wang. Construction of Blowup Solutions for the Nonlinear Schrodinger Equation with Critical Nonlinearity, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **25**, 197-215 (1998).

[4] T. Cazenave. *Semilinear Schrodinger Equations,* Courant Lecture Notes in Mathematics, **10**. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI (2003).

[5] A. Comech and D. Pelinovsky. Purely nonlinear instability of standing waves with minimal energy, *Comm. Pure Appl. Math.*, **56**, no. 11, 1565-1607 (2003).

[6] L. Demanet and W. Schlag. Numerical verification of a gap condition for a linearized nonlinear Schrödinger equation, *Nonlinearity*, **19**, No. 4, 829-852 (2006).

[7] B. Erdogan and W. Schlag. Dispersive estimates for Schrödinger operators in the presence of a resonance and/or eigenvalue at zero energy in dimension three. II, *J. Anal. Math*, **99**, 199-248 (2006).

[8] M. Grillakis, J. Shatah and W. Strauss. Stability theory of solitary waves in the presence of symmetry. II, *J. Funct. Anal.*, **94**, no. 2, 308-348 (1990).

[9] L. Hörmander. *The Analysis of Linear Partial Differential Operators III*, Grundlehren der Mathematischen Wissenschaften, **274** . Springer-Verlag, Berlin (1994).

[10] L. Hörmander. *The Analysis of Linear Partial Differential Operators IV*, Grundlehren der Mathematischen Wissenschaften, **275** . Springer-Verlag, Berlin (1994).

[11] J. Krieger and W. Schlag. Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension, *J. Amer. Math. Soc.*, **19**, no. 4, 815-920 (2006).

[12] J.L. Marzuola. Dispersive estimates using scattering theory for matrix Hamiltonian equations, submitted. *Arxiv preprint 0906.0351.*

[13] J.L. Marzuola, S. Raynor, G. Simpson. A system of ODE's for a perturbation of a minimal mass soliton, submitted. *Arxiv preprint 0905.0513.*

[14] J.L. Marzuola, G. Simpson. Spectral Analysis for Matrix Hamiltonian Operators, in preparation.

[15] K. McLeod. Uniqueness of Positive Radial Solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$, II, *Transactions of the American Mathematical Society*, **339**, no. 2, 495-505 (1993).

[16] D. Pelinovsky, V. Afanasjev and Y. Kivshar. Nonlinear theory of oscillating, decaying, and collapsing solitons in the generalized nonlinear Schrödinger equation, *Physical Review E*, **53**, no. 2, 1940-1953 (1996).

[17] I. Rodnianski, W. Schlag, and A. Soffer. Asymptotic stability of N-soliton states of NLS, preprint (2003).

[18] W. Schlag. *Stable manifolds for an orbitally unstable NLS*, to appear in *Annals of Math*, preprint (2004).

[19] J. Shatah. Stable Standing Waves of Nonlinear Klein-Gordon Equations, *Communications in Mathematical Physics*, **91**, 313-327 (1983).

[20] J. Shatah. Unstable Ground State of Nonlinear Klein-Gordon Equations, *Transactions of the American Mathematical Society*, **290**, no. 2, 701-710 (1985).

[21] J. Shatah and W. Strauss. Instability of Nonlinear Bound States, *Communications in Mathematical Physics*, **100**, 173-190 (1985).

[22] C. Sulem and P. Sulem. *The Nonlinear Schrodinger Equation. Self-focusing and wave-collapse*, Applied Mathematical Sciences, **39**. Springer-Verlag, New York (1999).

[23] M. Weinstein. Modulation Stability of Ground States of Nonlinear Schrodinger Equations. *SIAM Journal of Mathematical Analysis*, **16**, no. 3, 472-491 (1985).

[24] M. Weinstein. Lyapunov Stability of Ground States of Nonlinear Dispersive Evolution Equations. *Communications on Pure and Applied Mathematics*, **39**, 51-68 (1986).

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