QUASILINEAR SCHRÖDINGER EQUATIONS I: SMALL DATA AND QUADRATIC INTERACTIONS

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ABSTRACT. In this article we prove local well-posedness in low-regularity Sobolev spaces for general quasilinear Schrödinger equations. These results represent improvements of the pioneering works by Kenig-Ponce-Vega and Kenig-Ponce-Rolvung-Vega, where viscosity methods were used to prove existence of solutions in very high regularity spaces. Our arguments here are purely dispersive. The function spaces in which we show existence are constructed in ways motivated by the results of Mizohata, Ichinose, Doi, and others, including the authors.

1. INTRODUCTION

In this article we consider the local well-posedness for quasilinear Schrödinger equations

$$\begin{cases} iu_t + g^{jk}(u,\nabla u)\partial_j\partial_k u = F(u,\nabla u), \quad u:\mathbb{R}\times\mathbb{R}^d\to\mathbb{C}^m \\ u(0,x) = u_0(x) \end{cases} \tag{1.1}$$

with small initial data in a space with relatively low Sobolev regularity but with some extra decay assumptions. Here

$$g:\mathbb{C}^m\times(\mathbb{C}^m)^d\to\mathbb{R}^{d\times d}, \qquad F:\mathbb{C}^m\times(\mathbb{C}^m)^d\to\mathbb{C}^m$$

are smooth functions which satisfy

$$g(0) = I_d, \qquad F(y,z) = O(|y|^2+|z|^2) \text{ near } (y,z)=(0,0). \tag{1.2}$$

We also consider a second class of quasilinear Schrödinger equations

$$\begin{cases} iu_t + \partial_j g^{jk}(u)\partial_k u = F(u,\nabla u), \ u:\mathbb{R}^d\times\mathbb{R}\to\mathbb{C}^m \\ u(0,x) = u_0(x), \end{cases} \tag{1.3}$$

with g and F as in (1.2) but where the metric g depends on u but not on ∇u. Such an equation is obtained for instance by differentiating the first equation (1.1). Precisely, if u solves (1.1) then the vector $(u,\nabla u)$ solves an equation of the form (1.3), with a nonlinearity F which depends at most quadratically on ∇u.

We remark that the second order operator in (1.3) is written in divergence form. This is easily achieved by commuting the first derivative with g and moving the outcome to the right hand side. However, the second order operator in (1.1) cannot be written in divergence form without changing the type of the equation.

Naively one might at first consider the well-posedness of these problems in Sobolev spaces $H^s(\mathbb{R}^d)$ with large enough s. This is for instance what is done in the case of quasilinear wave equations, using energy estimates, Sobolev embeddings and Grönwall's inequality as in [6, 19]. However, this cannot work in general for the above Schrödinger equations.

The obstruction comes from the infinite speed of propagation phenomena. From [15, 16, 17, 11, 13], it is known that even in the case of linear problems of the form

$$(i\partial_t + \Delta_g)v = A_i(x)\partial_i v, \tag{1.4}$$

a necessary condition for L^2 well-posedness is an integrability condition for the magnetic potential A along the Hamilton flow of the leading order differential operator. In the case of (1.1), we would have to look instead at the corresponding linearized problem, which would exhibit a magnetic potential of the form $A = A(u, \nabla u)$. Such a potential in general does not satisfy Mizohata's integrability condition even if $A(u) = u$ or $A(u) = \nabla u$ with u solving the linear constant coefficient Schrödinger equation with H^s initial data and s arbitrarily large.

Given the above considerations, it is natural to add some decay to the H^s Sobolev spaces where the quasilinear problem (1.1) is considered. A traditional way to do that is to use weighted H^s spaces with polynomial weights. This avenue was pursued for instance in [7, 8, 9], where the first local well-posedness results for this problem were obtained for solutions in $H^s \cap L^2(\langle x\rangle^N)$, where $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$, for some unspecified sufficiently large s and N. sufficiently large depending upon complicated asymptotics.

One disadvantage of the above approach is that the results are not invariant with respect to translations. In this article we propose a different set-up, which is translation invariant. In the process we significantly lower the threshold s for local well-posedness.

To begin, for each u we denote by $\mathcal{F}u = \hat{u}$ the spatial Fourier transform of u. We say that the function u is localized at frequency 2^i if $\operatorname{supp} \hat{u}(t,\xi) \subset \mathbb{R} \times [2^{i-1}, 2^{i+1}]$. Next we introduce a standard

Littlewood-Paley decomposition with respect to spatial frequencies,

$$1 = \sum_{i=0}^{\infty} S_i.$$

Let $\phi_0 : [0,\infty) \to \mathbb{R}$ be a nonnegative, decreasing, smooth function such that $\phi_0(\xi) = 1$ on $[0,1]$ and $\phi_0(\xi) = 0$ if $\xi \geq 2$. Then, for each $i \geq 1$ we define $\phi_i : [0,\infty) \to \mathbb{R}$ by

$$\phi_i(\xi) = \phi_0(2^{-i}\xi) - \phi_0(2^{-i+1}\xi).$$

We define the operators S_i, which localize to frequency 2^i, by

$$\hat{f}_i(\xi) = \mathcal{F}(S_i f) = \phi_i(\xi)\hat{f}(\xi).$$

We also define the related operators

$$S_{\leq N} f = \sum_{i=0}^{N} f_i, \;\; S_{\geq N} f = \sum_{i=N}^{\infty} f_i.$$

For each nonnegative integer j we consider a partition $\mathcal{Q}_j$ of $\mathbb{R}^d$ into cubes of side length 2^j and an associated smooth partition of unity

$$1 = \sum_{Q \in \mathcal{Q}_j} \chi_Q.$$

Then we define the $l^1_j L^2$ norm by

$$\|u\|_{l^1_j L^2} = \sum_{Q \in \mathcal{Q}_j} \|\chi_Q u\|_{L^2}.$$

Our replacement for the H^s initial data space is the space $l^1 H^s$ with norm given by

$$\|u\|^2_{l^1 H^s} = \sum_{j \geq 0} 2^{2sj} \|S_j u\|^2_{l^1_j L^2}.$$

The motivation for this choice is as follows. Heuristically Schrödinger waves at frequency 2^j travel with speed 2^j. Hence on the unit time scale a partition on the 2^j spatial scale is exactly at the threshold where it does not interfere with the linear flow. In other words, the Schrödinger evolution in these spaces at frequency 2^j will be no different from the corresponding evolution in H^s. At the same time, the summability condition with respect to the 2^j spatial scale suffices in order to recover Mizohata's condition if s is sufficiently large.

As a point of reference, in [2] similar spaces are defined in the context of semilinear Schrödinger equations. There the trajectories of the Hamilton flow for the principal part are straight lines, and one sums $\|f\|_{L^2(Q)}$ over those $Q \in \mathcal{Q}_j$'s which intersect a line $L \subset \mathbb{R}^d$ and then take a supremum with respect to all lines L. However, such a definition relies heavily on the Hamilton flow associated with the Laplacian as the leading order differential operator. Here, as we are not guaranteed a nice Hamilton flow of the leading order operator, we simply sum over all cubes of scale 2^j.

Our main result concerns the quasilinear problem (1.1) with small data $u_0(x) \in l^1 H^s$:

Theorem 1. *a) Let $s > \frac{d}{2} + 3$. Then there exists $\epsilon_0 > 0$ sufficiently small such that, for all initial data u_0 with $\|u_0\|_{l^1 H^s} \leq \epsilon_0$, the equation* (1.1) *is locally well-posed in $l^1 H^s(\mathbb{R}^d)$ on the time interval $I = [0, 1]$.*

b) The same result holds for the equation (1.3) *with $s > \frac{d}{2} + 2$.*

For comparison purposes we note that the scaling exponent for the principal part of (1.1) is $s = \frac{d}{2} + 1$, while for (1.3) with a quadratic nonlinearity in the gradient ∇u it is $s = \frac{d}{2}$. On the other hand, for the semilinear version of (1.3) the well-posedness result in $l^1 H^s$ in [1], [2] applies for $s > \frac{d}{2} + 1$; that result was shown to be sharp in [18].

We remark that our theorem also holds for the ultrahyperbolic operators studied in, e.g., [8, 9]. Indeed, if $g(0)$ is of different signature, we need only adjust the local smoothing estimates of Section 4. The wedge decomposition which is employed there allows this to be accomplished trivially.

The need to use the $l^1 H^s$ type spaces for the initial data is exclusively due to the bilinear interactions, both semilinear and quasilinear. However, we expect these spaces to be relaxed to H^s spaces if all the interactions which are present are cubic and higher. This problem is considered in a follow-up paper.

For simplicity the life span of the solutions in the above theorem has been taken to be $[0, 1]$. However, a simple rescaling argument shows that the life span can be made arbitrarily large by taking sufficiently small data. By contrast, the short time large data result cannot be obtained by scaling from the small data result. This is due to the fact that the spaces used are inhomogeneous Sobolev spaces, and spatial localization is not allowed due to the infinite speed of propagation. This problem will also be considered in subsequent work.

The definition of "well-posedness" in the above theorem is taken to include the following:

- Existence of a solution $u \in C([0, T_\epsilon); l^1 H^s)$ satisfying
$$\|u\|_{L^\infty l^1 H^s} \lesssim \epsilon.$$
- Uniqueness in the above class provided that s is large enough.

- Continuity of the solution map

$$l^1H^s \ni u_0 \to u \in C([0,T_\epsilon); l^1H^s)$$

for all s as in the theorem.

The above conditions allow one to interpret the rough solutions as the unique limits of smooth solutions. However, in the process of proving the theorem we introduce a stronger topology $l^1X^s \subset C([0,T_\epsilon); l^1H^s)$ and, for all s in the theorem, we show that the solutions belong to l^1X^s, are unique in l^1X^s and that the solution map $u_0 \to u$ is continuous from l^1H^s to l^1X^s.

We also remark that due to the quasilinear character of the problem the continuous dependence on the initial data is the best one can hope for. However, if we assume that the metric g does not depend on u, then the problem becomes semilinear and one obtains Lipschitz dependence on the initial data as in [2].

The paper is organized as follows. In Section 2, we describe the space-time function spaces in which we will solve (1.1) (1.3). In Section 3, we establish the necessary multilinear and nonlinear estimates in order to close the eventual bootstrap estimates. In Section 4, we prove the necessary Morawetz type estimate to establish local energy decay for a linear, inhomogeneous paradifferential version of the Schrödinger equation. Finally, in Section 5, we combine the above estimates with the proper paradifferential decomposition of the equation in order to prove Theorem 1.

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2. Function Spaces and Notations

2.1. **The l^p_jU spaces.** As a generalization of the $l^1_jL^2$ norm defined in the introduction, given any translation invariant Sobolev type space U we define the Banach spaces l^p_jU with norm

$$\|u\|^p_{l^p_jU} = \sum_{Q\in\mathcal{Q}_j} \|\chi_Q u\|^p_U$$

with the obvious changes when $p=\infty$. By a slight abuse we will employ the same notation whether U represents a space-time Sobolev space or a purely spatial Sobolev space. Note that in what follows we will work with inhomogeneous norms, so we take only cubes of size 1 or larger, i.e. $j \geq 0$. In particular we will use the dual space $l^\infty_j L^2$ to $l^1_j L^2$, with norm

$$\|u\|_{l^\infty_j L^2} = \sup_{Q \in \mathcal{Q}_j} \|\chi_Q u\|_{L^2}.$$

By replacing the sum over Q above with an integral, one can easily see that these spaces admit a translation invariant equivalent norm.

We also note that the smooth partition of compactly supported cutoffs in the $l^1_j U$ spaces can be replaced by cutoffs which are frequency localized. Indeed, we have that

$$\sum_{Q \in \mathcal{Q}_j} \|\chi_Q u\|_U \approx \sum_{Q \in \mathcal{Q}_j} \|(S_0 \chi_Q) u\|_U. \tag{2.1}$$

This follows simply from the fact that $S_0 \chi_Q$ decays rapidly away from Q. We will use frequency localized cutoffs whenever we need the components $\chi_Q u$ to retain the frequency localization of u.

2.2. **The X and Y spaces.** We next define a local energy type space X of functions on $[0,1] \times \mathbb{R}^d$ with norm

$$\|u\|_X = \sup_l \sup_{Q \in \mathcal{Q}_l} 2^{-\frac{l}{2}} \|u\|_{L^2_{t,x}([0,1] \times Q)}.$$

To measure the right hand side of the Schrödinger equation we use a dual local energy space $Y \subset L^2([0,1] \times \mathbb{R}^d)$, which as we will show satisfies the duality relation $X = Y^*$.

The space Y is an atomic space. A function a is an atom in Y if there exists some $j \geq 0$ and some cube $Q \in \mathcal{Q}_j$ so that a is supported in $[0,1] \times Q$ and

$$\|a\|_{L^2([0,1] \times Q)} \lesssim 2^{-\frac{j}{2}}.$$

The space Y is the Banach space of linear combinations of the form

$$f = \sum_k c_k a_k, \qquad \sum |c_k| < \infty, \qquad a_k \text{ atoms} \tag{2.2}$$

with respect to the norm

$$\|f\|_Y = \inf\{\sum |c_k| \,:\, f = \sum_k c_k a_k,\, a_k \text{ atoms}\}.$$

The core spaces X, Y are related via the following duality relation.

Proposition 2.1. *The following duality relation holds with respect to the standard L^2 duality: $Y^* = X$.*

Proof. It is clear by construction that

$$(u, v)_{t,x} \lesssim \|u\|_X \|v\|_Y.$$

Hence, we need to show for any $L \in Y^*$, there exists $u \in X$ such that

$$(u, v)_{t,x} = L(v), \quad \|u\|_X \leq \|L\|_{Y^*}.$$

Applying L to all atoms associated to a cube $Q \in \mathcal{Q}_j$, we obtain

$$|Lv| \lesssim 2^{\frac{j}{2}} \|L\|_{Y^*} \|v\|_{L^2}$$

for all functions $v \in L^2$ with support in Q. Hence by Riesz's theorem there exists a function u_Q in Q so that

$$Lv = \langle u_Q, v \rangle, \qquad \|u_Q\|_{L^2} \lesssim 2^{\frac{j}{2}} \|L\|_{Y^*}.$$

A priori the functions u_Q depend on Q. However, given two intersecting cubes Q_1 and Q_2, the actions of u_{Q_1} and u_{Q_2} must coincide as L^2 functions in $Q_1 \cap Q_2$. Hence we must have $u_{Q_1} = u_{Q_2}$ on $Q_1 \cap Q_2$. Thus there is a single global function u so that, for every cube Q, u_Q is the restriction of u to Q. Then the last estimate shows that

$$\|\chi_Q u\|_{L^2} \lesssim 2^{\frac{j}{2}} \|L\|_{Y^*}, \qquad Q \in \mathcal{Q}_j$$

or equivalently

$$\|u\|_X \lesssim \|L\|_{Y^*}.$$

□

2.3. **The $l^1 X^s$ and $l^1 Y^s$ spaces.** We first remark that the X norm corresponds exactly to the local energy decay estimates for $H^{-\frac{1}{2}}$ solutions to the Schrödinger equation. Precisely, in the constant coefficient case we have the following dyadic bound

$$\|e^{it\Delta} S_j f\|_X \lesssim 2^{-\frac{j}{2}} \|f\|_{L^2}.$$

Thus for L^2 solutions to the linear Schrödinger equations which are localized at frequency 2^j it is natural to use the space

$$X_j = 2^{-\frac{j}{2}} X \cap L^\infty L^2$$

with norm

$$\|u\|_{X_j} = 2^{\frac{j}{2}} \|u\|_X + \|u\|_{L^\infty L^2}.$$

Adding the l^1 spatial summation on the 2^j scale we obtain the space $l^1_j X_j$ with norm

$$\|u\|_{l^1_j X_j} = \sum_{Q \in \mathcal{Q}_j} \|\chi_Q u\|_{X_j}.$$

Then we define the space $l^1 X^s$ where we seek solutions to the nonlinear Schrödinger equations (1.1), (1.3) with $l^1 H^s$ data by

$$\|u\|_{l^1 X^s}^2 = \sum_j 2^{2js} \|S_j u\|_{X_j}^2.$$

The appropriate space for the inhomogeneous term for L^2 solutions to the Schrödinger equation at frequency 2^j is

$$Y_j = 2^{\frac{j}{2}} Y + L^1 L^2$$

with norm

$$\|f\|_{Y_j} = \inf_{f = 2^{\frac{j}{2}} f_1 + f_2} \|f_1\|_Y + \|f_2\|_{L^1 L^2}.$$

To fit it to the context in the present paper we add the l^1_j summation and work with the space $l^1_j Y_j$. Finally, we define the space $l^1 Y^s$ with norm

$$\|f\|_{l^1 Y^s}^2 = \sum_j 2^{2js} \|S_j f\|_{l^1_j Y_j}^2.$$

2.4. **Frequency envelopes.** For both technical and expository reasons it is convenient to present our bilinear and nonlinear estimates using the method of frequency envelopes. Given a Sobolev type space U so that

$$\|u\|_U^2 \sim \sum_{k=0}^{\infty} \|S_k u\|_U^2$$

a frequency envelope for u in U is a positive sequence a_j so that

$$\|S_j u\|_U \le a_j \|u\|_U, \qquad \sum a_j^2 \approx 1. \tag{2.3}$$

We say that a frequency envelope is admissible if $a_0 \approx 1$ and it is slowly varying,

$$a_j \le 2^{\delta|j-k|} a_k, \qquad j, k \ge 0, \qquad 0 < \delta \ll 1.$$

An admissible frequency envelope always exists, say by

$$a_j = 2^{-\delta j} + \|u\|_U^{-1} \max_k 2^{-\delta|j-k|} \|S_k u\|_U. \tag{2.4}$$

In the sequel we will use frequency envelopes for the spaces $l^1 H^s$, $l^1 X^s$ and $l^1 Y^s$. The parameter δ is a sufficiently small parameter, which will only depend on the value of s in our main theorem. For instance in the case of part (b) of the theorem, we will choose δ so that

$$0 < \delta < s - \frac{d}{2} - 2.$$

3. Multilinear and nonlinear estimates

In this section we prove the main bilinear and nonlinear estimates in the paper. We begin with a shorter proposition containing our bilinear and Moser estimates in terms of the l^1X^s and l^1Y^s spaces.

Proposition 3.1. *We have the following:*
a) Let $s > \frac{d}{2}$. Then the l^1X^s spaces satisfy the algebra property

$$\|uv\|_{l^1X^s} \lesssim \|u\|_{l^1X^s}\|v\|_{l^1X^s}, \tag{3.1}$$

as well as the Moser estimate

$$\|F(u)\|_{l^1X^s} \lesssim \|u\|_{l^1X^s}(1 + \|u\|_{l^1X^s})c(\|u\|_{L^\infty}). \tag{3.2}$$

for all smooth F with $F(0) = 0$.
b) Bilinear $X \cdot X \to Y$ bounds. Let $s > \frac{d}{2} + 2$. Then

$$\|uv\|_{l^1Y^\sigma} \lesssim \|u\|_{l^1X^{s-1}}\|v\|_{l^1X^{\sigma-1}}, \qquad 0 \le \sigma \le s, \tag{3.3}$$

$$\|uv\|_{l^1Y^\sigma} \lesssim \|u\|_{l^1X^{s-2}}\|v\|_{l^1X^{\sigma}}, \qquad 0 \le \sigma \le s-1. \tag{3.4}$$

The estimates in the above proposition suffice for most of our purposes, but not all. Instead we need a sharper version of it, which is phrased in terms of frequency envelopes. Thus Proposition 3.1 is a direct consequence of the next proposition:

Proposition 3.2. *We have the following:*
a) Let $s > \frac{d}{2}$, and $u, v \in l^1X^s$ with admissible frequency envelopes a_k, respectively b_k. Then the l^1X^s spaces satisfy the algebra type property

$$\|S_k(uv)\|_{l^1X^s} \lesssim (a_k + b_k)\|u\|_{l^1X^s}\|v\|_{l^1X^s}, \tag{3.5}$$

as well as the Moser type estimate

$$\|S_kF(u)\|_{l^1X^s} \lesssim a_k\|u\|_{l^1X^s}(1 + \|u\|_{l^1X^s})c(\|u\|_{L^\infty}). \tag{3.6}$$

for all smooth F with $F(0) = 0$.
b) Bilinear $X \cdot X \to Y$ bounds. Let $s > \frac{d}{2} + 2$, $\sigma \le s$ and $u \in l^1X^s$, $v \in l^1X^\sigma$ with admissible frequency envelopes a_k, respectively b_k. Then

$$\|S_k(uv)\|_{l^1Y^\sigma} \lesssim (a_k + b_k)\|u\|_{l^1X^{s-1}}\|v\|_{l^1X^{\sigma-1}}, \qquad 0 \le \sigma \le s, \tag{3.7}$$

$$\|S_k(uv)\|_{l^1Y^\sigma} \lesssim (a_k + b_k)\|u\|_{l^1X^{s-2}}\|v\|_{l^1X^{\sigma}}, \qquad 0 \le \sigma \le s-1, \tag{3.8}$$

$$\|S_k(uS_{\ge k-4}v)\|_{l^1Y^\sigma} \lesssim (a_k + b_k)\|u\|_{l^1X^{s-2}}\|v\|_{l^1X^{\sigma}}, \qquad 0 \le \sigma \le s. \tag{3.9}$$

c) Commutator bound. For $s > \frac{d}{2} + 2$ and any multiplier $A \in S^0$ we have

$$\|\nabla[S_{<k-4}g, A(D)]\nabla S_ku\|_{l^1Y^0} \lesssim \|g - I\|_{l^1X^s}\|S_ku\|_{l^1X^0}. \tag{3.10}$$

Proof. A preliminary step in the proof is to observe that we have a Bernstein type inequality,

$$\|S_k u\|_{l^1_k L^\infty} \lesssim 2^{\frac{dk}{2}} \|S_k u\|_{l^1_k L^\infty L^2} \lesssim 2^{\frac{dk}{2}} \|S_k u\|_{l^1_k X_k}.$$

This is easily proved using the classical Bernstein inequality, with frequency localized cube cutoffs. After dyadic summation this gives

$$\|u\|_{L^\infty} \lesssim \|u\|_{l^1 X^s}, \qquad s > d/2, \tag{3.11}$$

respectively

$$\|S_{<j} u\|_{l^1_j L^\infty} \lesssim \|u\|_{l^1 X^s}, \qquad s > d/2. \tag{3.12}$$

To prove the X algebra property we consider the usual Littlewood-Paley dichotomy. In a dyadic expression $S_k(S_i u S_j v)$ we need to consider two cases:

High-low interactions: $j < i - 4$ and $|i - k| < 4$ (or the symmetric alternative). Then the $l^1_k X_k$ and $l^1_i X_i$ norms are comparable therefore we have

$$\|S_i u S_j v\|_{l^1_k X_k} \lesssim \|S_i u\|_{l^1_i X_i} \|S_j v\|_{L^\infty} \lesssim 2^{\frac{dj}{2}} \|S_i u\|_{l^1_i X_i} \|S_j v\|_{L^\infty L^2}.$$

The multiplier S_k is bounded in $l^1_k X_k$, therefore we obtain

$$\|S_k(S_i u S_j v)\|_{l^1 X^s} \lesssim 2^{(\frac{d}{2}-s)j} a_i b_j \|u\|_{l^1 X^s} \|v\|_{l^1 X^s}.$$

Upon summation over i, j, we get the desired bound for the high-low interactions.

High-high interactions: $|i - j| \leq 4$ and $i, j \geq k - 4$. For $j > k$ we use Bernstein's inequality at frequency 2^k to obtain

$$\|S_k(S_i u S_j v)\|_{l^1_k X_k} \lesssim 2^{\frac{kd}{2}} \|S_i u\|_{l^1_k X_k} \|S_j v\|_{L^\infty L^2}.$$

Each $\mathcal{Q}_i$ cube contains about $2^{d(i-k)}$ $\mathcal{Q}_k$ cubes and $X_i \subset X_k$; therefore we obtain

$$\|S_k(S_i u S_j v)\|_{l^1_k X_k} \lesssim 2^{d(i-k)} 2^{\frac{kd}{2}} \|S_i u\|_{l^1_i X_i} \|S_j v\|_{L^\infty L^2},$$

i.e.

$$\|S_k(S_i u S_j v)\|_{l^1 X^s} \lesssim 2^{(\frac{d}{2}-s)(2i-k)} a_i b_j \|u\|_{l^1 X^s} \|v\|_{l^1 X^s}. \tag{3.13}$$

The corresponding part of the bound (3.5) follows after summation over i, j.

Next we turn our attention to the Moser estimate (3.6). Following an idea in [22] we consider a multilinear paradifferential expansion, which follows from the Fundamental Theorem of Calculus. For the purpose of this proof we replace the discrete Littlewood-Paley decomposition by a continuous one

$$Id = S_0 + \int_0^\infty S_k \, dk,$$

denote $u_k = S_k u$, and, by a slight abuse of notation, $u_0 = S_0 u$. Then we can write

$$S_k F(u) = S_k F(u_0) + \int_0^\infty S_k(u_{k_1} F'(u_{<k_1}))\, dk_1. \tag{3.14}$$

To estimate the first term, we begin with

$$\|\partial^\alpha u_0\|_{L^\infty} \lesssim \|u_0\|_{L^\infty}, \quad \|\partial^\alpha u_0\|_{l_0^1 X} \lesssim \|u_0\|_{l_0^1 X}.$$

Then, repeated applications of the chain rule lead to

$$\|\partial^\alpha F(u_0)\|_{L^\infty} \lesssim \|u_0\|_{L^\infty} c(\|u_0\|_{L^\infty}),$$
$$\|\partial^\alpha F(u_0)\|_{l_0^1 X_0} \lesssim \|u_0\|_{l_0^1 X_0} c(\|u_0\|_{L^\infty}).$$

Hence

$$\|S_k F(u_0)\|_{l_k^1 X_k} \lesssim 2^{\frac{k}{2}} \|S_k F(u_0)\|_{l_0^1 X_0} \lesssim 2^{-Nk} \|u_0\|_{l_0^1 X_0} c(\|u_0\|_{L^\infty})$$

for any N. The $l^1 X^s$ bound for the first term of (3.14) then follows trivially.

For the second term in (3.14), we consider three cases.

Case I: $k-4 \leq k_1 \leq k+4$. This is the easiest case as

$$\|S_k(u_{k_1} F'(u_{<k_1}))\|_{l_k^1 X_k} \lesssim \|u_{k_1}\|_{l_{k_1}^1 X_{k_1}} c(\|u_{<k_1}\|_{L^\infty}),$$

therefore

$$\|S_k(u_{k_1} F'(u_{<k_1}))\|_{l^1 X^s} \lesssim a_{k_1} \|u\|_{l^1 X^s} c(\|u_{<k_1}\|_{L^\infty}).$$

For $|k - k_1| \leq 4$ we have $a_{k_1} \sim a_k$, and the k_1 integration is trivial.

Case II: $k_1 < k-4$. In this case,

$$S_k(u_{k_1} F'(u_{<k_1})) = S_k(u_{k_1} \tilde{S}_k F'(u_{<k_1})),$$

for a multiplier $\tilde{S}_k$ which similarly localizes to frequency 2^k and

$$S_k \tilde{S}_k = S_k.$$

Applying the chain rule as above, it follows that

$$\|\tilde{S}_k F'(u_{<k_1})\|_{L^\infty} \lesssim 2^{-N(k-k_1)} c(\|u_{<k_1}\|_{L^\infty}), \qquad k_1 \leq k \tag{3.15}$$

and thus,

$$\begin{aligned}\|S_k(u_{k_1} F'(u_{<k_1}))\|_{l_k^1 X_k} &\lesssim 2^{\frac{k-k_1}{2}} \|u_{k_1}\|_{l_{k_1}^1 X_{k_1}} \|\tilde{S}_k F'(u_{<k_1})\|_{L^\infty} \\ &\lesssim 2^{-N(k-k_1)} \|u_{k_1}\|_{l_{k_1}^1 X_{k_1}} c(\|u_{<k_1}\|_{L^\infty}),\end{aligned}$$

which leads to

$$\|S_k(u_{k_1}F'(u_{<k_1}))\|_{l^1X^s} \lesssim 2^{-N(k-k_1)}a_{k_1}\|u\|_{l^1X^s}c(\|u\|_{L^\infty}).$$

The k_1 integration is now straightforward.

Case III: $k_1 > k+4$. In this case, we apply the Fundamental Theorem of Calculus again to see that

$$\int_{k+4}^{\infty} S_k(u_{k_1}F'(u_{<k_1}))\,dk_1 = \int_{k+4}^{\infty} S_k(u_{k_1}F'(u_0))\,dk_1 + \int_{k+4}^{\infty}\int_0^{k_1} S_k(u_{k_1}u_{k_2}F''(u_{<k_2}))\;dk_2\;dk_1. \tag{3.16}$$

For the first term in the right of (3.16), we have that

$$S_k(u_{k_1}F'(u_0)) = S_k(u_{k_1}\tilde{S}_{k_1}F'(u_0)).$$

Therefore, as there are $2^{d(k_1-k)}$ cubes of sidelength 2^k contained in a cube with sidelength 2^{k_1}, it follows that

$$\begin{aligned}\|S_k(u_{k_1}F'(u_0))\|_{l^1_kX_k} &\lesssim 2^{d(k_1-k)}\|u_{k_1}\|_{l^1_{k_1}X_{k_1}}\|\tilde{S}_{k_1}F'(u_0)\|_{L^\infty}\\ &\lesssim 2^{d(k_1-k)-Nk_1}\|u_{k_1}\|_{l^1_{k_1}X_{k_1}}\|u_0\|_{L^\infty}c(\|u_0\|_{L^\infty}).\end{aligned}$$

This yields

$$\|S_k(u_{k_1}F'(u_0))\|_{l^1X^s} \lesssim 2^{(d-s)(k_1-k)}2^{-Nk_1}a_{k_1}\|u\|_{l^1X^s}c(\|u_0\|_{L^\infty}).$$

The desired estimate follows easily after a k_1 integration.

We now examine the second term in the right of (3.16). Here we have two subcases to examine separately.

Case III(a): $k_1 - 4 \le k_2 \le k_1$. Then we can argue as in (3.13) to obtain

$$\|S_k(u_{k_1}u_{k_2}F''(u_{<k_2}))\|_{l^1_kX_k} \lesssim 2^{dk_1}2^{-\frac{dk}{2}}\|u_{k_1}\|_{l^1_{k_1}X_{k_1}}\|u_{k_2}\|_{L^\infty L^2}c(\|u\|_{L^\infty}).$$

Case III(b): $0 < k_2 \le k_1 - 4$. Then

$$S_k(u_{k_1}u_{k_2}F''(u_{<k_2})) = S_k(u_{k_1}u_{k_2}\tilde{S}_{k_1}F''(u_{<k_2})).$$

Therefore using (3.15) for $\tilde{S}_{k_1}F''(u_{<k_2})$ and Bernstein's inequality at frequency 2^k we have

$$\begin{aligned}&\|S_k(u_{k_1}u_{k_2}F''(u_{<k_2}))\|_{l^1_kX_k}\\ &\qquad\lesssim\; 2^{d(k_1-k)}2^{\frac{dk}{2}}2^{-N(k_1-k_2)}\|u_{k_1}\|_{l^1_{k_1}X_{k_1}}\|u_{k_2}\|_{L^\infty L^2}c(\|u\|_{L^\infty}).\end{aligned}$$

Combining the two cases and adding in the Sobolev weights this leads to

$$\|S_k(u_{k_1}u_{k_2}F''(u_{<k_2}))\|_{l^1X^s} \lesssim 2^{(2k_1-k)(\frac{d}{2}-s)-N(k_1-k_2)}a_{k_1}a_{k_2}\|u\|^2_{l^1X^s}c(\|u\|_{L^\infty})$$

which can be integrated with respect to k_1, k_2.

b) As a general rule, here we always estimate the bilinear expressions in Y, and never in L^1L^2. By the definition of the Y space, for each $l \leq j$ we have

$$\|f\|_{l_j^1 Y} \lesssim 2^{\frac{l}{2}} \|f\|_{l_l^1 L^2}. \tag{3.17}$$

We use the standard Littlewood-Paley dichotomy, and consider expressions of the form $S_k(S_i u S_j v)$. There are two cases to examine.

High-low interactions: $|i-k| \leq 4$ and $j < i-4$. Applying (3.17) with $l = j$ we obtain

$$\|S_i u S_j v\|_{l_k^1 Y_k} \lesssim 2^{\frac{j-k}{2}} \|S_i u S_j v\|_{l_j^1 L^2} \lesssim 2^{\frac{j-k}{2}} \|S_i u\|_{l_j^\infty L^2} \|S_j v\|_{l_j^1 L^\infty}.$$

For the first factor we use the X norm and for the second we use Bernstein's inequality. This yields

$$\|S_i u S_j v\|_{l_k^1 Y_k} \lesssim 2^{\frac{d+2}{2} j - k} \|S_i u\|_{X_i} \|S_j v\|_{l_j^1 L^\infty L^2},$$

and further

$$\|S_k(S_i u S_j v)\|_{l_k^1 Y_k} \lesssim 2^{\frac{d+2}{2} j - k} \|S_i u\|_{l_i^1 X_i} \|S_j v\|_{l_j^1 X_j}. \tag{3.18}$$

The alternative low-high interactions can be handled by similar arguments.

High-High interactions. $|i-j| \leq 4$ and $i, j \geq k-4$. Applying (3.17) with $l = k$, Cauchy-Schwarz to transition from 2^k sized cubes to 2^j sized cubes and then Bernstein's inequality, we have

$$\begin{aligned}
\|S_k(S_i u S_j v)\|_{l_k^1 Y_k} &\lesssim \|S_k(S_i u S_j v)\|_{l_k^1 L^2} \\
&\lesssim 2^{\frac{d}{2}(j-k)} \|S_k(S_i u S_j v)\|_{l_j^1 L^2} \\
&\lesssim 2^{\frac{jd}{2}} \|S_k(S_i u S_j v)\|_{l_j^1 L_t^2 L_x^1} \\
&\lesssim 2^{\frac{jd}{2}} \|S_i u\|_{l_i^1 L^2} \|S_j v\|_{L^\infty L^2}.
\end{aligned}$$

Thus we obtain

$$\|S_k(S_i u S_j v)\|_{l_k^1 Y_k} \lesssim 2^{\frac{jd}{2}} \|S_i v\|_{l_i^1 X_i} \|S_j u\|_{l_j^1 X_j}. \tag{3.19}$$

The desired bounds (3.8), (3.7) and (3.9) follow easily from the dyadic bounds (3.18) and (3.19) after summation.

c) For the commutator we claim the representation

$$\nabla [S_{<k-4} g, A(D)] \nabla S_k u = L(\nabla S_{<k-4} g, \nabla S_k u) \tag{3.20}$$

where L is a disposable operator, i.e. a translation invariant operator of the form

$$L(f,g)(x) = \int f(x+y) g(x+z) w(y,z) \, dy dz, \qquad \|w\|_{L^1} \lesssim 1.$$

Assume this representation holds. Then, since the l^1X^s spaces are translation invariant (i.e. they admit translation invariant equivalent norms), the commutator bound (3.10) becomes a direct consequence of (3.7).

To prove (3.20) we first observe that we can harmlessly replace the multiplier $A(D)$ by $\tilde{S}_kA(D)$ and $S_{<k-4}g$ by $\tilde{S}_{<k-4}S_{<k-4}g$. Denoting $g_1 = S_{<k-4}g$ and $u_1 = \nabla S_k u$, the above commutator is written in the form

$$C(g_1, u_1) = \nabla[\tilde{S}_{<k-4}g_1, \tilde{S}_kA(D)]u_1.$$

The operators $\tilde{S}_kA(D)$ and $\tilde{S}_{<k-4}$ have kernels $K(y)$, $H(y)$ which satisfies bounds of the form

$$|\partial^\alpha K(y)|, |\partial^\alpha H(y)| \lesssim_\alpha 2^{(d+|\alpha|)k}(1 + 2^k|y|)^{-N}.$$

Then we can write

$$\begin{aligned} C(g_1, u_1)(x) =& \nabla_x \int (g_1(x-z) - g_1(x-y-z))H(z)K(y)u_1(x-y)dydz \\ =& \nabla_x \int_0^1 \int y\nabla g_1(x-z-hy)H(z)K(y)u_1(x-y)dydzdh \\ =& \nabla_x \int_0^1 \int y\nabla g_1(x-z)H(z+hy)K(y)u_1(x-y)dydzdh \end{aligned}$$

Distributing the x derivative in front and integrating by parts with respect to either y or z this leads to the representation (3.20) where the kernel w of L is given by

$$w(y, z) = (\nabla_y + \nabla_z)\int_0^1 yH(z+hy)K(y)dh.$$

The L^1 bound on w follows from the above bounds on H, K and their derivatives. $\square$

4. Local Energy Decay

In this section we consider a frequency localized linear Schrödinger equation

$$(4.1)\qquad (i\partial_t + \partial_k g^{kl}_{<j-4}\partial_l)u_j = f_j, \qquad u_j(0) = u_{0j}.$$

The main result of this section is as follows:

Proposition 4.1. *Assume that the coefficients g^{kl} in (4.1) satisfy*

$$(4.2)\qquad \|g^{kl} - \delta^{kl}\|_{l^1X^s} \ll 1$$

for some $s > \frac{d}{2} + 2$. *Let* u_j *be a solution to* (4.1) *which is localized at frequency* 2^j. *Then the following estimate holds:*

$$\|u_j\|_{l^1_j X_j} \lesssim \|u_{0j}\|_{l^1_j L^2} + \|f_j\|_{l^1_j Y_j}. \tag{4.3}$$

Proof. Dropping the l^1_j summation, our main task will be to prove the simpler bound

$$\|u_j\|_{X_j} \lesssim \|u_{0j}\|_{L^2} + \|f_j\|_{Y_j}. \tag{4.4}$$

Then (4.3) will follow easily via $\mathcal{Q}_j$ localizations. We rewrite the equation (4.1) in the form

$$(i\partial_t - A)u_j = f_{j1} + f_{j2}, \qquad u_j(0) = u_{0j},$$

where $A = -\partial_k g^{kl}_{<j-4}\partial_l$ is self-adjoint and $f_{j1} \in L^1L^2$, $f_{j2} \in Y$.

The estimate (4.4) has two components, an energy bound and local energy decay. We have the trivial inequality $\|u\|_X \lesssim \|u\|_{L^\infty L^2}$; therefore the energy estimate suffices for small j.

The energy-type estimate is standard if the right hand side is in $L^1_t L^2_x$, but we would like to allow the right hand side to be in the dual smoothing space as well. Using the common notation $D_t = \frac{1}{i}\partial_t$, we frame it in an abstract framework as follows:

Lemma 4.2. *Let* A *be a self-adjoint operator. Let* u *solve the equation*

$$(D_t + A)u = f \qquad u(0) = u_0 \tag{4.5}$$

in the time interval $[0,T]$. *Then we have*

$$\|u\|^2_{L^\infty_t L^2_x} \lesssim \|u_0\|^2_{L^2} + \|u\|_{X_j}\|f\|_{Y_j}. \tag{4.6}$$

Proof. We need only compute

$$\frac{d}{dt}\frac{1}{2}\|u(t)\|^2_{L^2} = \mathrm{Im}\langle u, f\rangle, \tag{4.7}$$

and notice that for each $t \in [0,T]$ we have by duality

$$\|u(t)\|^2_{L^2} \lesssim \|u(0)\|^2_{L^2} + \|u\|_{X_j}\|f\|_{Y_j}.$$

We take the supremum over t on the left and the conclusion follows. □

Next we consider the local energy decay estimate. We will prove that the following holds for $Q \in \mathcal{Q}_l$ and $0 \le l \le j$:

$$2^{j-l}\|u_j\|^2_{L^2(Q)} \lesssim \|u_j\|^2_{L^\infty L^2} + \|u_j\|_{X_j}\|f_j\|_{Y_j} + (2^{-j} + \|g - I\|_{l^1X^s})\|u_j\|^2_{X_j}. \tag{4.8}$$

Suppose this is true. Taking the supremum over $Q \in \mathcal{Q}_l$ and over l, we obtain

$$2^j\|u_j\|^2_X \lesssim \|u_j\|^2_{L^\infty L^2} + \|u_j\|_{X_j}\|f_j\|_{Y_j} + (2^{-j} + \|g - I\|_{l^1X^s})\|u_j\|^2_{X_j}.$$

The last term on the right can be discarded for large enough j since $\|g - I\|_{l^1 X^s} \ll 1$. Then we obtain

$$2^j \|u_j\|_X^2 \lesssim \|u_j\|_{L^\infty L^2}^2 + \|u_j\|_{X_j} \|f_j\|_{Y_j}.$$

Combined with (4.6) this gives (4.4) by the Cauchy-Schwarz inequality.

We now turn our attention to the proof of (4.8). For a self-adjoint multiplier $\mathcal{M}$, we have

$$\frac{d}{dt}\langle u, \mathcal{M}u\rangle = -2\,\mathrm{Im}\langle (D_t + A)u, \mathcal{M}u\rangle + \langle i[A,\mathcal{M}]u, u\rangle. \tag{4.9}$$

We then wish to construct $\mathcal{M}$ so that

(1) $\|\mathcal{M}u\|_{L^2_x} \lesssim \|u\|_{L^2_x}$,
(2) $\|\mathcal{M}u\|_X \lesssim \|u\|_X$,
(3) $i\langle [A,\mathcal{M}]u, u\rangle \gtrsim 2^{j-\ell}\|u\|^2_{L^2_{t,x}([0,1]\times Q)} - O(2^{-j} + \|g - I\|_{l^1 X^s})\|u\|^2_{X_j}$.

If these three properties hold for $u = u_j$ and $(D_t + A)u_j = f_j$, then the bound (4.8) follows.

As a general rule, we will choose $\mathcal{M}$ to be a first order differential operator with smooth coefficients localized at frequency $\lesssim 1$,

$$i2^j \mathcal{M} = a^k(x)\partial_k + \partial_k a^k(x) \tag{4.10}$$

A key step in our analysis is to dispense with the contribution of the difference $g - I$ in the commutator $[A,\mathcal{M}]$. Precisely, we have

Lemma 4.3. *Let $A = \partial_k g^{kl}\partial_l$ with $g = g_{<j-4}$ and $\mathcal{M}$ be as above. Suppose that $s > \frac{d}{2} + 2$. Then we have*

$$|\langle [A,\mathcal{M}]u_j, u_j\rangle| \lesssim \|g\|_{l^1 X^s}\|u_j\|^2_{X_j}. \tag{4.11}$$

Proof. The commutator $[A,\mathcal{M}]$ can be written in the form

$$i[A,\mathcal{M}] = 2^{-j}(\nabla(g\nabla a + a\nabla g)\nabla + \nabla g \nabla^2 a + g\nabla^3 a).$$

All the a factors are bounded and low frequency, and can therefore trivially be discarded. Hence the worst term we need to estimate is

$$2^{-j}\langle a\nabla g \nabla u_j, \nabla u_j\rangle.$$

Due to the frequency localization of u_j we have

$$\|\nabla u_j\|_{X_j} \lesssim 2^j \|u_j\|_{X_j}.$$

Hence by the $Y_j^* = X_j$ duality it remains to show that

$$\|(\nabla g_{<j-4})v_j\|_{Y_j} \lesssim 2^{-j}\|g\|_{l^1 X^s}\|v_j\|_{X_j}$$

for $v_j = \nabla u_j$. But this is a consequence of the bilinear bound (3.3). $\square$

The next step is to prove (4.8) under the additional assumption that u_j is frequency localized in an angle

$$\operatorname{supp} \hat{u}_j \subset \{|\xi| \lesssim \xi_1\}. \tag{4.12}$$

Here, we take a small angle about the first coordinate axis, and the argument can be repeated similarly near the other axes. By translation invariance we can assume that $Q = \{|x_j| \leq 2^l : j = 1, \dots, d\}$. Then we consider a multiplier $\mathcal{M}$ of the form

$$i2^j \mathcal{M} = m_l(x_1)\partial_1 + \partial_1 m_l(x_1)$$

where $m_l(s) = m(2^{-l}s)$ with m a smooth bounded increasing function with $m'(s) = \psi^2(s)$ for some Schwartz function ψ localized at frequency $\lesssim 1$ with $\psi \sim 1$ for $|s| \leq 1$.

The properties (1) and (2) clearly hold for $\mathcal{M}$ and $u = u_j$ due to the frequency localizations of u_j and m_l. It remains to verify (3). By the previous lemma applied for $g - I$, we can set $A = -\Delta$. Then

$$-i2^j[A, \mathcal{M}] = 2^{-l+2}\partial_1 \psi^2(2^{-l}x_1)\partial_1 + O(1).$$

The last term is bounded, therefore

$$i2^j\langle [A, \mathcal{M}]u_j, u_j\rangle = 2^{-l+2}\|\psi(2^{-l}x_1)\partial_1 u_j\|_{L^2}^2 + O(\|u_j\|_{L^2}^2).$$

Given the frequency and angular localization of u_j, we obtain

$$2^{-l}2^{2j}\|\psi(2^{-l}x_1)u_j\|_{L^2}^2 \lesssim i2^j\langle [A, \mathcal{M}]u_j, u_j\rangle + O(\|u_j\|_{L^2}^2).$$

Hence (3) follows. Thus we have proved (4.8) under the additional frequency localization condition (4.12).

To prove (4.8) in general we use a wedge decomposition in the frequency variables. To this end, we consider a partition of unity $\{\theta_k(\omega)\}_{k=1}^d$,

$$1 = \sum_k \theta_k(\omega) \qquad \text{in } \mathbb{S}^{d-1},$$

where, for each k, $\theta_k(\omega)$ is supported in a small angle. We then define the localized functions $u_{j,k} = \Theta_{j,k}u_j$ via

$$\mathcal{F}\Theta_{j,k}u = \theta_k\Big(\frac{\xi}{|\xi|}\Big) \sum_{j-1\leq l\leq j+1} \phi_l(\xi)\hat{u}(t,\xi).$$

These solve the equations

$$(i\partial_t - A)u_{j,k} = \Theta_{j,k}f_j - [A, \Theta_{j,k}]u_j.$$

By Plancherel's theorem, it is trivial to see that $\Theta_{j,k}$ is L^2 bounded. We note further that the kernel of the operator $\Theta_{j,k}$ has Schwartz class decay outside a ball of radius 2^{-j}. Thus, it is easy to show that $\Theta_{j,k}$ is bounded on X, and by duality on Y.

To prove (4.8) for u_j we apply the appropriate multipliers to each of the $u_{j,k}$ and sum up. We obtain

$$\begin{aligned} 2^{j-l}\|u_j\|^2_{L^2(Q)} \lesssim \|u_j\|^2_{L^\infty L^2} + \|u_j\|_{X_j}(\|f_j\|_{Y_j} + \sum_k \|[A,\Theta_{j,k}]u_j\|_{Y_j}) \\ + (2^{-j} + \|g-I\|_{l^1X^s})\|u_j\|^2_{X_j} \end{aligned} \tag{4.13}$$

It remains to estimate the commutator, which is done via (3.10). Then (4.8) follows.

We now show how (4.3) follows from (4.4). We consider a partition of unity χ_Q corresponding to cubes Q of scale $M2^j$. Allowing rapidly decreasing tails, we can assume that the functions χ_Q are localized at frequencies $\lesssim 1$. We can also assume that χ_Q are smooth on the $M2^j$ scale, in particular

$$|\nabla \chi_Q| \lesssim (2^jM)^{-1}, \qquad |\nabla \chi_Q| \lesssim (2^jM)^{-2}.$$

The functions $\chi_Q u_j$ solve

$$(i\partial_t - A)(\chi_Q u_j) = \chi_Q f_j - [A,\chi_Q]u_j.$$

We apply (4.4) to each of the functions $\chi_Q u_j$ and add them up. This gives

$$\sum_Q \|\chi_Q u_j\|_{X_j} \lesssim \sum_Q \|\chi_Q u_{0j}\|_{L^2} + \|\chi_Q f_j\|_{Y_j} + \|[A,\chi_Q]u_j\|_{L^1L^2}. \tag{4.14}$$

It remains to estimate the commutators. Using the bounds on the derivatives of χ_Q we obtain

$$\sum_Q \|[A,\chi_Q]u_j\|_{L^1L^2} \lesssim M^{-1} \sum_Q \|\chi_Q u_j\|_{L^\infty L^2}.$$

Hence if M is large enough (independently of j) then the last term on the right in (4.14) can be discarded, and we are left with

$$\sum_Q \|\chi_Q u_j\|_{X_j} \lesssim \sum_Q \|\chi_Q u_{0j}\|_{L^2} + \|\chi_Q f_j\|_{Y_j}. \tag{4.15}$$

The transition from cubes of size $M2^j$ to cubes of size 2^j is straightforward, and (4.3) follows.

□

5. Proof of Theorem 1

We recall that the equation (1.1) turns into an equation of the form (1.3) by differentiation. Hence it suffices to prove part (b) of the theorem. We recast the equation (1.3) in a paradifferential form, given by

$$\begin{cases} L_j u_j = f_j, \\ u_j(0) = (u_0)_j, \end{cases} \tag{5.1}$$

where

$$L_j = (i\partial_t + \partial_k g^{kl}_{<j-4}\partial_l)$$

and

$$f_j = S_j F(u, \nabla u) - S_j \partial_k g^{kl}_{>j-4}\partial_l u - [S_j, \partial_k g^{kl}_{<j-4}\partial_l]u. \tag{5.2}$$

5.1. **A formal bootstrap.** Using the bounds in Proposition 3.1 one can estimate the f_j's by the following

Lemma 5.1. *Let $s > \frac{d}{2} + 1$, and $u \in l^1 X^s$ with frequency envelope $\{a_j\}$. Then the functions f_j in (5.2) satisfy*

$$\|f_j\|_{l^1 Y^s} \lesssim a_j \|u\|^2_{l^1 X^s} c(\|u\|_{l^1 X^s}). \tag{5.3}$$

Proof. The first term is estimated using (3.6) followed by (3.8) with $\sigma = s$, taking advantage of the fact that F is at least quadratic at zero. The second term is estimated using (3.6) and (3.9) with $\sigma = s$. For the third term we use (3.10). $\square$

As a corollary of the above lemma it follows that

$$\sum_j \|f_j\|^2_{l^1 Y^s} \lesssim \|u\|^4_{l^1 X^s} c(\|u\|_{l^1 X^s}).$$

For each of the equations in (5.1) we can apply Proposition 4.1. Square summing we obtain

$$\|u\|^2_{l^1 X^s} \lesssim \|u_0\|^2_{l^1 H^s} + \|u\|^4_{l^1 X^s} c(\|u\|_{l^1 X^s}).$$

From here a continuity argument *formally* leads to

$$\|u\|_{l^1 X^s} \lesssim \|u_0\|_{l^1 H^s}$$

assuming that the initial data u_0 is small enough.

5.2. **The linear problem.** Here we consider the linear equation

$$\begin{cases} (i\partial_t + \partial_k g^{kl}\partial_l)u + V\nabla u + Wu = h, \\ u(0) = u_0, \end{cases} \tag{5.4}$$

and we prove the following:

Proposition 5.2. *a) Assume that the metric g and the potentials V and W satisfy*

$$\|g - I\|_{l^1 X^s} \ll 1, \quad \|V\|_{l^1 X^{s-1}} \ll 1, \quad \|W\|_{l^1 X^{s-2}} \ll 1 \qquad s > \frac{d}{2} + 2.$$

Then the equation (5.4) *is well-posed for initial data $u_0 \in l^1 H^\sigma$ with $0 \leq \sigma \leq s - 1$. and we have the estimate*

$$\|u\|_{l^1 X^\sigma} \lesssim \|u_0\|_{l^1 H^\sigma} + \|h\|_{l^1 Y^\sigma}. \tag{5.5}$$

b) Assume in addition that $W = 0$. Then the equation (5.4) *is well-posed for initial data $u_0 \in l^1 H^\sigma$ with $0 \leq \sigma \leq s$, and the estimate* (5.5) *holds.*

Proof. We rewrite the equation as a family of equations for the dyadic parts of u,

$$\begin{cases} (i\partial_t + \partial_k g^{kl}_{<j-4} \partial_l) u_j = g_j + h_j, \\ u_j(0) = u_{0j}, \end{cases}$$

where

$$g_j = -S_j \partial_k g^{kl}_{>j-4} \partial_l u_j - [S_j, \partial_k g^{kl}_{<j-4} \partial_l] u_j - S_j V \nabla u - S_j W u.$$

As in Lemma 5.1, we apply Proposition 3.2 for each of the terms in g_j to obtain

$$\sum_j \|g_j\|^2_{l^1 Y^\sigma} \lesssim \|u\|^2_{l^1 X^\sigma} (\|g - I\|^2_{l^1 X^s} + \|V\|^2_{l^1 X^{s-1}} + \|W\|^2_{l^1 X^{s-2}}).$$

The estimate (5.5) follows by applying Proposition 4.1 to each of these equations and summing in j. The more restrictive range of σ in part (a) arises due to the similar range in (3.7). □

5.3. **The iteration scheme: uniform bounds.** Here we seek to construct solutions to (1.3) iteratively, based on the scheme

$$\begin{cases} (i\partial_t + \partial_j g^{jk}(u^{(n)}) \partial_k) u^{(n+1)} = F(u^{(n)}, \nabla u^{(n)}), \\ u^{(n+1)}(0, x) = u_0(x) \end{cases} \tag{5.6}$$

with the trivial initialization

$$u^{(0)} = 0.$$

Applying at each step Proposition 5.2 and assuming that u_0 is small in $l^1 H^s$ we inductively obtain the uniform bound

$$\|u^{(n)}\|_{l^1 X^s} \lesssim \|u_0\|_{l^1 H^s}. \tag{5.7}$$

Our next goal is to consider the convergence of this scheme.

5.4. **The iteration scheme: weak convergence.** Here we prove that our iteration scheme converges in the weaker l^1H^{s-1} topology. For this we write an equation for the difference $v^{(n+1)} = u^{(n+1)} - u^{(n)}$:

$$\begin{cases} (i\partial_t + \partial_j g^{jk}(u^{(n)})\partial_k)v^{(n+1)} = V_n\nabla v^{(n)} + W_n v^{(n)}, \\ v^{(n+1)}(0,x) = 0, \end{cases} \tag{5.8}$$

where

$$\begin{aligned} V_n &= V_n(u^{(n)}, \nabla u^{(n)}, u^{(n-1)}, \nabla u^{(n-1)}), \\ W_n &= h_1(u^{(n)}, u^{(n-1)}) + h_2(u^{(n)}, u^{(n-1)})\nabla^2 u^{(n)}. \end{aligned}$$

For V_n and W_n by the Moser estimate (3.2) we have

$$\|V_n\|_{l^1X^{s-1}} \ll 1, \qquad \|W_n\|_{l^1X^{s-2}} \ll 1.$$

This allows us to estimate the right hand side of (5.8) in l^1Y^{s-1} via (3.4) and (3.3). To estimate $v^{(n+1)}$ we use Proposition 5.2. We obtain

$$\|v^{(n+1)}\|_{l^1X^{s-1}} \ll \|v^{(n)}\|_{l^1X^{s-1}}. \tag{5.9}$$

This implies that our iteration scheme converges in l^1X^{s-1} to some function u. Furthermore, by the uniform bound (5.7) it follows that

$$\|u\|_{l^1X^s} \lesssim \|u_0\|_{l^1H^s}. \tag{5.10}$$

Thus we have established the existence part of our main theorem.

5.5. **Uniqueness via weak Lipschitz dependence.** Consider the difference $v = u^{(1)} - u^{(2)}$ of two solutions. This solves an equation of the form (5.4) where

$$\begin{aligned} V &= V(u^{(1)}, \nabla u^{(1)}, u^{(2)}, \nabla u^{(2)}), \\ W &= h_1(u^{(1)}, u^{(2)}) + h_2(u^{(1)}, u^{(2)})\nabla^2 u^{(1)}. \end{aligned}$$

Applying Proposition 5.2(a) we see that this equation is well-posed in l^1H^{s-1}, and obtain the estimate

$$\|u^{(1)} - u^{(2)}\|_{l^1X^{s-1}} \lesssim \|u^{(1)}(0) - u^{(2)}(0)\|_{l^1H^{s-1}}. \tag{5.11}$$

5.6. **Frequency envelope bounds.** Here we prove a stronger frequency envelope version of the estimate (5.10).

Proposition 5.3. *Let $u \in l^1X^s$ be a small data solution to* (1.3)*, which satisfies* (5.10)*. Let $\{a_j\}$ be an admissible frequency envelope for the initial data u_0 in l^1H^s. Then $\{a_j\}$ is also a frequency envelope for u in l^1X^s.*

Proof. Define an admissible envelope $\{b_j\}$ for u in l^1X^s by

$$b_j = 2^{-\delta j} + \|u\|_{l^1X^s}^{-1} \max_k 2^{-\delta|j-k|}\|S_k u\|_{l^1X^s}. \tag{5.12}$$

We estimate $u_j = S_j u$ using Proposition 5.2 applied to the equation (5.1). For the functions f_j we use Lemma 5.1 to obtain

$$\|f_j\|_{l^1Y^s} \lesssim b_j \|u\|_{l^1X^s}^2 c(\|u\|_{l^1X^s}). \tag{5.13}$$

By Proposition 5.2 applied to the equation (5.1) we obtain

$$\|S_j u\|_{l^1X^s} \lesssim a_j\|u_0\|_{l^1H^s} + b_j\|u\|_{l^1X^s}^2 c(\|u\|_{l^1X^s}).$$

This implies that

$$b_j \lesssim a_j \|u_0\|_{l^1H^s}\|u\|_{l^1X^s}^{-1} + b_j\|u\|_{l^1X^s} c(\|u\|_{l^1X^s}).$$

Since $\|u\|_{l^1X^s}$ is small and $\|u_0\|_{l^1H^s} \lesssim \|u\|_{l^1X^s}^{-1}$, this implies that $b_j \lesssim a_j$, concluding the proof. □

5.7. **Continuous dependence on the initial data.** Here we show that the map $u_0 \to u$ is continuous from l^1H^s into l^1X^s.

Suppose that $u_0^{(n)} \to u_0$ in $l^1H_x^s$. Denote by $a_j^{(n)}$, respectively a_j the frequency envelopes associated to $u_0^{(n)}$, respectively u_0, given by (2.4). If $u_0^{(n)} \to u_0$ in $l^1H_x^s$ then $a_j^{(n)} \to a_j$ in l^2. Then for each $\epsilon > 0$ we can find some N_ϵ so that

$$\|a_{>N_\epsilon}^{(n)}\|_{l^2} \leq \epsilon \qquad \text{for all } n.$$

By Proposition 5.3 we conclude that

$$\|u_{>N_\epsilon}^{(n)}\|_{l^1X^s} \leq \epsilon \qquad \text{for all } n. \tag{5.14}$$

To compare $u^{(n)}$ with u we use (5.11) for low frequencies and (5.14) for the high frequencies,

$$\begin{aligned}\|u^{(n)} - u\|_{l^1X^s} \lesssim& \|S_{<N_\epsilon}(u^{(n)} - u)\|_{l^1X^s} + \|S_{>N_\epsilon}u^{(n)}\|_{l^1X^s} + \|S_{>N_\epsilon}u\|_{l^1X^s} \\ \lesssim& 2^{N_\epsilon}\|S_{<N_\epsilon}(u^{(n)} - u)\|_{l^1X^{s-1}} + 2\epsilon \\ \lesssim& 2^{N_\epsilon}\|S_{<N_\epsilon}(u_0^{(n)} - u_0)\|_{l^1H^{s-1}} + 2\epsilon.\end{aligned}$$

Letting $n \to \infty$ we obtain

$$\limsup_{n\to\infty} \|u^{(n)} - u\|_{l^1X^s} \lesssim \epsilon.$$

Letting $\epsilon \to 0$ we obtain

$$\lim_{n\to\infty} \|u^{(n)} - u\|_{l^1X^s} = 0,$$

which gives the desired result.

5.8. **Higher regularity.** Here we prove that the solution u satisfies the bound

$$\|u\|_{l^1 X^\sigma} \lesssim \|u_0\|_{l^1 H^\sigma}, \qquad \sigma \geq s, \tag{5.15}$$

whenever the right hand side is finite.

The idea is to repeatedly differentiate the equation. The simplest way to do this would be to say that ∇u solves the linearized equation. But this is like the difference equation and is well-posed only in $l^1 H^{s-1}$ not in $l^1 H^s$. Instead we redo the computation as follows. The original equation is

$$(i\partial_t + \partial_j g^{jk}(u)\partial_k)u = F(u, \nabla u).$$

Differentiating we obtain

$$\begin{aligned}(i\partial_t + \partial_j g^{jk}(u)\partial_k)(\partial_l u) = &- (g^{jk})'(u)(\partial_j \partial_l u \partial_k u + \partial_l u \partial_j \partial_k u)\\ &+ F_{z_l}(u, \nabla u)\nabla \partial_l u + F_{z_0}(u, \nabla u)\partial_l u.\end{aligned}$$

We write this in an abbreviated form as

$$(i\partial_t + \partial_j g^{jk}(u)\partial_k)v_1 = G(u, \nabla u)\nabla v_1 + F_1(u, \nabla u)$$

for $v_1 = \nabla u$, where $G(z) = O(|z|)$ and $F_1(z) = O(|z|^2)$ near 0. We know that u is small in $l^1 X^s$, therefore, by Proposition 3.1 we get

$$\|G(u, \nabla u)\|_{l^1 X^{s-1}} \ll 1, \qquad \|F_1(u, \nabla u)\|_{l^1 Y^s} \lesssim \|u\|^2_{l^1 X^s}.$$

Hence using Proposition 5.2(b) we obtain

$$\|v_1\|_{l^1 X^s} \lesssim \|v_1(0)\|_{l^1 H^s} + \|u\|^2_{l^1 X^s},$$

which shows that

$$\|u\|_{l^1 X^{s+1}} \lesssim \|u(0)\|_{l^1 H^{s+1}} + \|u\|^2_{l^1 X^s}.$$

Inductively, we write an equation for $v_n = \nabla^n u$,

$$(i\partial_t + \partial_j g_{jk}(u)\partial_k)v_n = G(u, \nabla u)\nabla v_n + F_n(u, \cdots, \nabla^n u)$$

with the same G as above. This leads to

$$\|v_n\|_{l^1 X^s} \lesssim \|v_n(0)\|_{l^1 H^s} + \|u\|^2_{l^1 X^{s+n-1}},$$

which shows that

$$\|u\|_{l^1 X^{s+n}} \lesssim \|u(0)\|_{l^1 H^{s+n}} + \|u\|^2_{l^1 X^{s+n-1}}.$$

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