STRICHARTZ ESTIMATES FOR THE SCHRÖDINGER EQUATION ON POLYGONAL DOMAINS

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Abstract. We prove Strichartz estimates with a loss of derivatives for the Schrödinger equation on polygonal domains with either Dirichlet or Neumann homogeneous boundary conditions. Using a standard doubling procedure, estimates on the polygon follow from those on Euclidean surfaces with conical singularities. We develop a Littlewood-Paley squarefunction estimate with respect to the spectrum of the Laplacian on these spaces. This allows us to reduce matters to proving estimates at each frequency scale. The problem can be localized in space provided the time intervals are sufficiently small. Strichartz estimates then follow from a recent result of the second author regarding the Schrödinger equation on the Euclidean cone.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a compact polygonal domain in the plane, that is, a compact, connected region in $\mathbb{R}^2$ whose boundary is piecewise linear. Suppose $u(t,x) : [-T,T] \times \Omega \longrightarrow \mathbb{C}$ is a solution to the initial value problem for the Schrödinger equation on Ω,

$$\begin{cases} (D_t + \Delta)\, u(t,x) = 0 \\ u(0,x) = f(x), \end{cases} \tag{1.1}$$

satisfying either Dirichlet or Neumann homogeneous boundary conditions,

$$u\big|_{[-T,T]\times\partial\Omega} = 0 \qquad \text{or} \qquad \partial_n u\big|_{[-T,T]\times\partial\Omega} = 0. \tag{1.2}$$

Here, ∂_n denotes the normal derivative along the boundary, $D_t = \frac{1}{i}\,\partial_t$, and $\Delta = -\partial_{x_1}^2 - \partial_{x_2}^2$ is the nonnegative Laplace operator.

In this note, we are interested in Strichartz estimates for solutions to the aforementioned Schrödinger IBVP (1.1)-(1.2); these are a family of space-time integrability bounds of the form

$$\|u\|_{L^p([-T,T];L^q(\Omega))} \leq C_T\, \|f\|_{H^s(\Omega)} \tag{1.3}$$

with $p > 2$ and $\frac{2}{p} + \frac{2}{q} = 1$. In this estimate, we take the space $H^s(\Omega)$ to be the L^2-based Sobolev space of order s defined with respect to the spectral resolution of either the Dirichlet or Neumann Laplacian. More precisely, this self-adjoint operator possesses a sequence of eigenfunctions forming a basis for $L^2(\Omega)$. We write the eigenfunction and eigenvalue pairs as $\Delta\varphi_j = \lambda_j^2\varphi_j$, where λ_j denotes the

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frequency of vibration. The Sobolev space of order s can then be defined as the image of $L^2(\Omega)$ under $(1+\Delta)^{-s}$ with norm

$$\|f\|_{H^s(\Omega)}^2 = \sum_{j=1}^{\infty} \left(1+\lambda_j^2\right)^s |\langle f, \varphi_j\rangle|^2 . \tag{1.4}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product.

Strichartz estimates are well-established when the domain Ω is replaced by Euclidean space. In that case, one can take $s = 0$ in (1.3), and by scaling considerations, this is the optimal order for the Sobolev space; see for example Strichartz [27], Ginibre and Velo [16], Keel and Tao [17], and references therein. When Ω is a compact domain or manifold, much less is known about the validity and optimality of these estimates. The finite volume of the manifold and the presence of trapped geodesics appear to limit the extent to which dispersion can occur. In addition, the imposition of boundary conditions complicate many of the known techniques for proving Strichartz estimates. Nonetheless, estimates on general compact domains with smooth boundary have been shown by Anton [2] and Blair-Smith-Sogge [4]. Both of these works build on the approach for compact manifolds of Burq-Gérard-Tzvetkov [7].

In the present work, we prove the following

Theorem 1.1. *Let Ω be a compact polygonal domain in $\mathbb{R}^2$, and let Δ denote either the Dirichlet or Neumann Laplacian on Ω. Then for any solution $u = \exp(-it\Delta) f$ to the Schrödinger IBVP (1.1)-(1.2) with f in $H^{\frac{1}{p}}(\Omega)$, the Strichartz estimates*

$$\|u\|_{L^p([-T,T];L^q(\Omega))} \leq C_T \|f\|_{H^{\frac{1}{p}}(\Omega)} \tag{1.5}$$

hold provided $p > 2$, $\infty > q \geq 2$, and $\frac{2}{p} + \frac{2}{q} = 1$.

Remark 1.2. To define the Laplacian, we use the standard Friedrichs extension, which is the canonical self-adjoint extension of a non-negative densely defined symmetric operator as defined in for instance [13]. In this work, the Neumann Laplacian is taken to be the Friedrichs extension of the Laplace operator acting on smooth functions which vanish in a neighborhood of the vertices. In this sense, our Neumann Laplacian imposes Dirichlet conditions at the vertices and Neumann conditions elsewhere. The Dirichlet Laplacian is taken to be the typical Friedrichs extension of the Laplace operator acting on smooth functions which are compactly supported in the interior of Ω.

Remark 1.3. We note that our estimates have a loss of $s = 1/p$ derivatives as in [7], which we believe is an artifact of our methods. Given specific geometries, there are results showing that such a loss is not sharp. For instance, when Ω is replaced by a flat rational torus, the estimate (1.3) with $p = q = 4$ holds for any $s > 0$, as was shown by Bourgain [5]; see also [6] for results in the case of irrational tori. In addition, we note that in [7], they also observe a sharp loss of $s = 1/2$ derivatives for $p = 2$ on the sphere. We also point out that in certain domains with boundary, a loss of derivatives is expected due to the existence of gliding rays, as shown by Ivanovici [22].

Remark 1.4. Using a now standard application of the Christ-Kiselev lemma [12], we can conclude that for a solution u to the inhomogeneous Schrödinger IBVP

$$\begin{cases} (D_t + \Delta)\, u(t,x) = F(t,x) \\ \qquad\quad u(0,x) = f(x) \end{cases} \tag{1.6}$$

satisfying either Dirichlet or Neumann homogeneous boundary conditions, the estimate

$$\|u\|_{L^{p_1}([-T,T];L^{q_1}(\Omega))} \leq C_T \left(\|f\|_{H^{\frac{1}{p_1}}(\Omega)} + \|F\|_{L^{p_2'}([-T,T];W^{\frac{1}{p_1}+\frac{1}{p_2},q_2'}(\Omega))} \right), \tag{1.7}$$

holds for $\frac{2}{p_j} + \frac{2}{q_j} = 1$ for $j = 1, 2$. Here, $(\cdot)'$ denotes the dual exponent, e.g. $\frac{1}{p_1} + \frac{1}{p_1'} = 1$.

We prove Theorem 1.1 by utilizing a doubling procedure to reduce estimates on the polygonal domain Ω to estimates on a Euclidean surface with conical singularities. A Euclidean surface with conical singularities (ESCS) is, loosely speaking, a Riemannian surface (X, g) locally modeled on either Euclidean space or the flat Euclidean cone; for a precise definition, see Section 2. As will be outlined below, any compact planar polygonal domain Ω can be doubled across its boundary to produce a compact ESCS. In this procedure, a vertex of Ω of angle α induces a conic point of X with cone angle 2α. Taking the Laplacian on X to be the Friedrichs extension of the Laplacian on $\mathcal{C}_c^\infty(X_0)$, where X_0 is X less the singular points, and the Sobolev spaces $H^s(X)$ as in (1.4), Theorem 1.1 will follow from the following

Theorem 1.5. *Let X be a compact ESCS, and let Δ_{g} be the Friedrichs extension of $\Delta_{\mathrm{g}}\big|_{\mathcal{C}_c^\infty(X_0)}$. Then for any solution $u = \exp(-it\Delta_{\mathrm{g}})\, f$ to the Schrödinger IVP on X with initial data f in $H^{\frac{1}{p}}(X)$, the Strichartz estimates*

$$\|u\|_{L^p([-T,T];L^q(X))} \leq C_T \, \|f\|_{H^{\frac{1}{p}}(X)} \tag{1.8}$$

hold provided $p > 2$, $\infty > q \geq 2$, and $\frac{2}{p} + \frac{2}{q} = 1$.

The method here is to develop a local parametrix for the operator at frequency-dependent time scales using a Littlewood-Paley decomposition. Since an ESCS locally looks like either the plane or the Euclidean cone, estimates will follow from a recent result of the second author [15], which develops Strichartz estimates on the latter. However, since propagation speed is proportional to frequency, the error in the parametrix is only bounded over time intervals of size inversely proportional to the frequency scale, cp. [7]. The loss of $\frac{1}{p}$ derivatives relative to estimates on the plane thus results from decomposing the time interval $[-T, T]$ into smaller frequency-dependent time intervals over which the error is bounded.

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2. Euclidean surfaces with conic singularities

In this section, we review the definition and properties of Euclidean surfaces X with conical singularities. For more details, we refer the reader to [8], where we believe manifolds of this type to have been first introduced, and [21] and [20], where properties and applications of such surfaces are explored.

We begin by establishing the notation $C(\mathbb{S}^1_\rho) \overset{\text{def}}{=} \mathbb{R}_+ \times \left(\mathbb{R}/2\pi\rho\mathbb{Z}\right)$ for the flat Euclidean cone of radius $\rho > 0$ equipped with the metric $\mathrm{h}(r,\theta) = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$. With this in mind, we have the following

Definition 2.1. A *Euclidean surface with conical singularities* (ESCS) is a topological space X possessing a decomposition $X = X_0 \sqcup P$ for a finite set of singular points $P \subsetneq X$ such that

(1) X_0 is an open, smooth two-dimensional Riemannian manifold with a locally Euclidean metric g, and
(2) each singular point p_j of P has a neighborhood U_j such that $U_j \setminus \{p_j\}$ is isometric to a neighborhood of the tip of a flat Euclidean cone $C(\mathbb{S}^1_{\rho_j})$ with p_j mapped to the cone tip.

We stress that the analysis required to prove Theorem 1.5 all occurs on the Riemannian manifold X_0. As remarked previously, we take the Laplacian Δ_{g} on X to be the Friedrichs extension of $\Delta_{\mathrm{g}}\Big|_{\mathcal{C}_c^\infty(X_0)}$. This is a nonnegative, self-adjoint operator on $L^2(X)$ with discrete spectrum tending to infinity, as can be seen from the Rellich-type theorem of [10, Theorem 3.4]. We can thus take the Sobolev spaces $H^s(X)$ to be the images of $L^2(X)$ under $(1+\Delta_{\mathrm{g}})^{-s}$ with norm defined similarly to that in (1.4).

We now discuss how any compact polygonal domain Ω in $\mathbb{R}^2$, possibly with polygonal holes, gives rise to an ESCS X equipped with a flat metric g. Begin with two copies Ω and $\sigma\Omega$ of the polygonal domain, where σ is a reflection of the plane. The double X is obtained by taking the formal union $\Omega \cup \sigma\Omega$, where two corresponding sides are identified pointwise. Taking polar coordinates near each vertex of the polygon, it can be seen that the flat metric g extends smoothly across the sides. In particular, a vertex in Ω of angle α gives rise to a conic point of X locally isometric to $C(\mathbb{S}^1_\rho)$ with $\rho = \frac{\alpha}{\pi}$. This is equivalent to the observation that such a doubling procedure produces a conic point of angle 2α, see [20].

The reflection σ of Ω gives rise to an involution of X commuting with the Laplace operator. This operator Δ_{g} thus decomposes into two operators acting on functions which are odd or even with respect to σ, and these operators are then equivalent to the Laplace operator on Ω with Dirichlet or Neumann boundary conditions respectively. In particular, for any eigenfunction φ_j of the Dirichlet, resp. Neumann, Laplace operator on Ω, we can construct an eigenfunction of the Laplace operator on X by taking φ_j in Ω and $-\varphi_j \circ \sigma$, resp. $\varphi_j \circ \sigma$, in $\sigma\Omega$. As a consequence, the Schrödinger flow over X can be seen to extend that for Ω, and hence the Strichartz estimates in Theorem 1.1 follow from those in Theorem 1.5 as claimed.

3. Strichartz estimates

In this section we prove Theorem 1.5. Our proof relies on a Littlewood-Paley decomposition of our solution with respect to spatial frequencies, followed by a localization of each dyadic piece to time scales inversely proportional to the spatial frequency scale and subsequent localized Strichartz estimates without loss. This strategy of proof when dealing with the Schrdinger equation is due to [7].

We start with a Littlewood-Paley decomposition of our solution u in the spatial frequency domain. Namely, choose a nonnegative bump function β in $\mathcal{C}_c^\infty(\mathbb{R})$ supported in $\left(\frac{1}{4}, 4\right)$ and satisfying $\sum_{k\geq 1} \beta\left(2^{-k}\,\zeta\right) = 1$ for $\zeta \geq 1$. Taking $\beta_k(\zeta) \overset{\text{def}}{=}$

$\beta\big(2^{-k}\,\zeta\big)$ for $k \geq 1$ and $\beta_0(\zeta) \stackrel{\text{def}}{=} 1 - \sum_{k\geq 1} \beta_k(\zeta)$, we define the frequency localization u_k of u in the spatial variable by

$$u_k \stackrel{\text{def}}{=} \beta_k\left(\sqrt{\Delta_{\mathrm{g}}}\right) u, \tag{3.1}$$

where the operator $\beta_k\big(\sqrt{\Delta_g}\big)$ is defined using the functional calculus with respect to Δ_g. Hence, $u = \sum_{k\geq 0} u_k$, and in particular, u_0 is localized to frequencies smaller than 1.

With this decomposition, we have the following squarefunction estimate for elements a of $L^q(X)$,

$$\left\| \left(\sum_{k\geq 0} \left| \beta_k\left(\sqrt{\Delta_{\mathrm{g}}}\right) a \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q(X)} \approx \|a\|_{L^q(X)}\,, \tag{3.2}$$

with implicit constants depending only on q. Delaying the proof of (3.2) to Section 4, we have by Minkowski's inequality that

$$\|u\|_{L^p([-T,T];L^q(X))} \lesssim \left(\sum_{k\geq 0} \|u_k\|^2_{L^p([-T,T];L^q(X))} \right)^{\frac{1}{2}} \tag{3.3}$$

since we are under the assumption that $p, q \geq 2$. We now claim that for each $k \geq 0$,

$$\|u_k\|_{L^p([-T,T];L^q(X))} \lesssim 2^{\frac{k}{p}} \|u_k(0,\cdot)\|_{L^2(X)}\,. \tag{3.4}$$

Assuming this for the moment, we have by orthogonality and the localization of β that

$$\begin{aligned} 2^{\frac{2k}{p}} \|u_k(0,\cdot)\|^2_{L^2(X)} &= 2^{\frac{2k}{p}} \sum_{j=1}^{\infty} \beta_k(\lambda_j)^2 \left|\langle u(0,\cdot), \varphi_j\rangle\right|^2 \\ &\lesssim \sum_{j=1}^{\infty} \big(1 + \lambda_j^2\big)^{1/p} \beta_k(\lambda_j)^2 \left|\langle u(0,\cdot), \varphi_j\rangle\right|^2 . \end{aligned} \tag{3.5}$$

We now sum this expression over k; after exchanging the order of summation in k and j, we obtain

$$\sum_{k\geq 0} 2^{\frac{2k}{p}} \|u_k(0,\cdot)\|^2_{L^2(X)} \lesssim \|u(0,\cdot)\|^2_{H^{\frac{1}{p}}(X)}\,. \tag{3.6}$$

Combining this with (3.3), we have reduced the proof of Theorem 1.5 to showing the claim (3.4).

We now observe that (3.4) follows from

$$\|u_k\|_{L^p([0,2^{-k}];L^q(X))} \lesssim \|u_k(0,\cdot)\|_{L^2(X)}. \tag{3.7}$$

Indeed, if this estimate holds, then time translation and mass conservation imply the same estimate holds with the time interval $[0, 2^{-k}]$ replaced by $[2^{-k}m, 2^{-k}(m+1)]$. Taking a sum over all such dyadic intervals in $[-T, T]$ then yields (3.4).

Next, we localize our solution in space using a finite partition of unity $\sum_\ell \psi_\ell \equiv 1$ on X such that $\mathrm{supp}(\psi_\ell)$ is contained in a neighborhood U_ℓ isometric to either an open subset of the plane $\mathbb{R}^2$ or a neighborhood of the tip of a Euclidean cone $C(\mathbb{S}^1_\rho)$.

It now suffices to see that if ψ is an element of this partition and U denotes the corresponding open set in $\mathbb{R}^2$ or $C(\mathbb{S}^1_\rho)$, then

$$\|\psi\, u_k\|_{L^p([0,2^{-k}];L^q(U))} \lesssim \|u_k(0,\cdot)\|_{L^2(U)}\,. \tag{3.8}$$

Here and in the remainder of the section, $L^q(U)$ is taken to mean the space of functions on U which are q-integrable with respect to the Riemannian measure over $\mathbb{R}^2$ or $C(\mathbb{S}^1_\rho)$, depending on where U lies.

Observe that $\psi\, u_k$ solves the equation

$$(D_t + \Delta_{\mathrm{g}})\,(\psi\, u_k) = [\Delta_{\mathrm{g}}, \psi]\, u_k \tag{3.9}$$

over $\mathbb{R}^2$ or $C(\mathbb{S}^1_\rho)$. Letting $\mathbf{S}(t)$ denote the Schrödinger propagator either on Euclidean space or the Euclidean cone, depending on which space U lives in, we have for $t \geq 0$ that

$$\psi\, u_k(t,\cdot) = \mathbf{S}(t)\big(\psi\, u_k(0,\cdot)\big) + \int_0^{2^{-k}} \mathbf{1}_{\{t>s\}}(s)\, \mathbf{S}(t-s)([\Delta_{\mathrm{g}}, \psi]\, u_k(s,\cdot))\, \mathrm{d}s. \tag{3.10}$$

Here, $\mathbf{1}_{\{t>s\}}(s)$ is the indicator of the set $t > s > 0$. By Minkowski's inequality,

$$\begin{aligned} \|\psi\, u_k\|_{L^p([0,2^{-k}];L^q(U))} &\lesssim \big\|\, \mathbf{S}(\cdot)\big(\psi\, u_k(0,\cdot)\big)\big\|_{L^p([0,2^{-k}];L^q(U))} \\ &\quad + \int_0^{2^{-k}} \|\, \mathbf{S}(\cdot - s)([\Delta_{\mathrm{g}}, \psi]\, u_k(s,\cdot))\|_{L^p([0,2^{-k}];L^q(U))}\, \mathrm{d}s. \end{aligned} \tag{3.11}$$

We now apply known Strichartz estimates on $\mathbf{S}(t)$. When U is a subset of the plane, the estimates on the propagator are well-known and contained in the references listed in the introduction. When U is a subset of the flat Euclidean cone $C(\mathbb{S}^1_\rho)$, the estimates are due to the following result of the second author.

Theorem 3.1 (Theorem 5.1 of [15])**.** *Suppose $p > 2$ and $q \geq 2$ satisfy $\frac{2}{p} + \frac{2}{q} = 1$. Then the Schrödinger solution operator $\mathbf{S}(t) = \exp(-it\Delta_{\mathrm{h}})$ on $C(\mathbb{S}^1_\rho)$ satisfies the Strichartz estimates*

$$\|\mathbf{S}(t) f\|_{L^p(\mathbb{R};L^q(C(\mathbb{S}^1_\rho)))} \lesssim \|f\|_{L^2(C(\mathbb{S}^1_\rho))}. \tag{3.12}$$

We now conclude that

$$\begin{aligned} \|\psi\, u_k\|_{L^p([0,2^{-k}];L^q(U))} &\lesssim \|\psi(\cdot)\, u_k(0,\cdot)\|_{L^2(U)} \\ &\quad + \int_0^{2^{-k}} \|([\Delta_{\mathrm{g}}, \psi(\cdot)]\, u_k(s,\cdot))\|_{L^2(U)}\, \mathrm{d}s. \end{aligned} \tag{3.13}$$

The estimates (3.7) will then follow provided

$$\begin{aligned} 2^{-k}\, \|[\Delta_{\mathrm{g}}, \psi]\, u_k\|_{L^\infty L^2(X)} &\lesssim 2^{-k}\, \|\nabla_{\mathrm{g}} u_k\|_{L^\infty L^2(X)} + 2^{-k}\, \|u_k\|_{L^\infty L^2(X)} \\ &\lesssim \|u_k(0,\cdot)\|_{L^2(X)}\,, \end{aligned} \tag{3.14}$$

with ∇_{g} denoting the Riemannian gradient. The first inequality here follows by a simple computation of the commutator. For the second inequality, first observe that the term $2^{-k}\|u_k\|_{L^\infty L^2}$ is easily controlled by mass conservation. We then claim that the bound on the gradient term follows from

$$2^{-2k}\, \|\Delta_{\mathrm{g}} u_k\|_{L^\infty L^2(X)} \lesssim \|u_k(0,\cdot)\|_{L^2(X)}\,. \tag{3.15}$$

Indeed, if this holds we have that

$$
\begin{aligned}
2^{-2k}\left\|\nabla_{\mathrm{g}} u_k\right\|_{L^\infty L^2(X)}^2 &\lesssim \sup_t 2^{-2k}\left\langle \Delta_{\mathrm{g}} u_k(t,\cdot), u_k(t,\cdot)\right\rangle \\
&\lesssim \sup_t 2^{-2k}\left\|\Delta_{\mathrm{g}} u_k(t,\cdot)\right\|_{L^2(X)}\left\|u_k(t,\cdot)\right\|_{L^2(X)} \qquad (3.16)\\
&\lesssim \left\|u_k(0,\cdot)\right\|_{L^2(X)}^2 .
\end{aligned}
$$

We next observe that since the Schrödinger propagator $\mathbf{S}(t)=\exp(-it\Delta_{\mathrm{g}})$ commutes with Δ_{g}, mass conservation implies that (3.15) further reduces to showing the bound

$$
2^{-2k}\left\|\Delta_{\mathrm{g}}\,\beta_k\left(\sqrt{\Delta_{\mathrm{g}}}\right)a\right\|_{L^2(X)} \lesssim \|a\|_{L^2(X)} \tag{3.17}
$$

for elements a of $L^2(X)$.

Finally, to see that (3.17) holds, define $\Psi(t)$ to be the Schwartz class function satisfying

$$
\zeta^2\,\beta(\zeta)=\int_{-\infty}^{\infty} e^{it\zeta}\,\Psi(t)\,\mathrm{d}t. \tag{3.18}
$$

This implies

$$
2^{-2k}\,\Delta_{\mathrm{g}}\,\beta_k\left(\sqrt{\Delta_{\mathrm{g}}}\right)=2^k\int_{-\infty}^{\infty} e^{it\sqrt{\Delta_{\mathrm{g}}}}\,\Psi\left(2^k t\right)\,\mathrm{d}t. \tag{3.19}
$$

We now use that $\exp\left(it\sqrt{\Delta_{\mathrm{g}}}\right)$ is an isometry on $L^2(X)$ to obtain

$$
\begin{aligned}
\left\|2^{-2k}\,\Delta_{\mathrm{g}}\,\beta_k\left(\sqrt{\Delta_{\mathrm{g}}}\right)a\right\|_{L^2(X)} &\le 2^k\int_{-\infty}^{\infty}\left\|e^{it\sqrt{\Delta_{\mathrm{g}}}}a\right\|_{L^2(X)}\Psi\left(2^k t\right)\mathrm{d}t \qquad (3.20)\\
&\lesssim \|a\|_{L^2(X)} ,
\end{aligned}
$$

showing (3.17) and thus, moving backwards through the reductions, the claim (3.4).

4. The Littlewood-Paley squarefunction estimate

In this section, we prove the Littewood-Paley squarefunction estimate (3.2) for Euclidean surfaces with conic singularities, which is the last remaining piece of the proof of Theorem 1.5. To our knowledge, such squarefunction estimates are new in the case of Euclidean surfaces with conic singularities. However, for polygonal domains, the proof is considerably simpler as the heat kernel bounds (4.1) below follow from [13]. As we shall see, the inequality (3.2) is actually valid for any exponent $1<q<\infty$. If X_0 were compact, the results from [25] or the upper bound in [7, Corollary 2.3] would suffice for our purpose. Here, though, care must be taken as X_0 is an incomplete manifold. The method we employ is very robust and applies in a variety of contexts where Gaussian upper bounds on the heat kernel are valid. It is outlined in [23] and treated in the thesis of the first author [3, §7.2 and §7.3].

We emulate the approach in [3], which utilizes a spectral multiplier theorem due to Duong, Ouhabaz, and Sikora [14, Theorem 3.2]. In our case it is sufficient to use results of Alexopolous [1, Theorem 6.1], with less general, but simpler, hypotheses. The crucial one for our purposes is that the heat kernel $P(t,x,y)$ generated by Δ_{g} should satisfy a Gaussian upper bound of the form

$$
P(t,x,y)\lesssim \frac{1}{\left|B(x,\sqrt{t})\right|}\,\exp\left(-\frac{b\,\mathrm{dist}_{\mathrm{g}}(x,y)^2}{t}\right), \tag{4.1}
$$

where $\left|B(x,\sqrt{t})\right|$ is the volume of the ball of radius $\sqrt{t}$ about x. We postpone the proof of this estimate until the end of the section.

Given (4.1), Alexopolous' theorem implies that any spectral multiplier $F\big(\sqrt{\Delta_{\mathrm{g}}}\big)$ satisfying the usual Hörmander condition maps $L^q(X) \longrightarrow L^q(X)$ for any $1 < q < \infty$. Moreover, this boundedness is true for functions F in $\mathcal{C}^N(\mathbb{R})$ which satisfy the weaker Mihlin-type condition

$$\sup_{0\le k\le N}\sup_{\zeta\in\mathbb{R}}\left|\left(\zeta\frac{d}{d\zeta}\right)^k F(\zeta)\right| \le C < \infty, \tag{4.2}$$

where N is taken so that $N \ge \frac{n}{2}+1$.

We now want to apply this theorem to a family of multipliers $F_\theta\big(\sqrt{\Delta_{\mathrm{g}}}\big)$, $0 \le \theta \le 1$, defined using the Rademacher functions $\{r_m\}_{m=0}^{\infty}$. Begin by taking

$$r_0(\theta) \overset{\text{def}}{=} \begin{cases} +1, & 0 \le \theta \le \frac{1}{2} \\ -1, & \frac{1}{2} < \theta < 1, \end{cases} \tag{4.3}$$

and then extend r_0 to the real line by periodicity, i.e. $r_0(\theta+1) = r_0(\theta)$. We then define the functions r_m by $r_m(\theta) \overset{\text{def}}{=} r_0(2^m\theta)$. Given any square integrable sequence of scalars $\{b_m\}_{m\ge 0}$, we consider the function $G(\theta) \overset{\text{def}}{=} \sum_{m\ge 0} b_m\, r_m(\theta)$. By a lemma in [26, Appendix D], for any q in the interval $(1,\infty)$ there exist constants c_q and C_q such that

$$c_q \left\|G\right\|_{L^q([0,1])} \le \left\|G\right\|_{L^2([0,1])} = \left(\sum_{m\ge 0}\left|b_m\right|^2\right)^{\frac{1}{2}} \le C_q \left\|G\right\|_{L^q([0,1])}. \tag{4.4}$$

Define the functions $\widetilde{\beta}_k(\zeta) \overset{\text{def}}{=} \beta_{k-1}(\zeta)+\beta_k(\zeta)+\beta_{k+1}(\zeta)$, and let $F_\theta(\zeta)$ and $\widetilde{F}_\theta(\zeta)$ be the functions

$$F_\theta(\zeta) \overset{\text{def}}{=} \sum_{k\ge 0} r_k(\theta)\,\beta_k\Big(\sqrt{\zeta}\Big) \qquad \text{and} \qquad \widetilde{F}_\theta(\zeta) \overset{\text{def}}{=} \sum_{k\ge 0} r_k(\theta)\,\widetilde{\beta}_k\Big(\sqrt{\zeta}\Big). \tag{4.5}$$

It can be checked that $F_\theta(\zeta)$ and $\widetilde{F}_\theta(\zeta)$ satisfy the condition (4.2), and the constant C appearing on the right of (4.2) can be taken independent of θ. We thus have that for $1 < q < \infty$ and a in $L^q(X)$

$$\begin{aligned} \left\|\left(\sum_{k\ge 0}\left|\beta_k\Big(\sqrt{\Delta_{\mathrm{g}}}\Big)\,a\right|^2\right)^{\frac{1}{2}}\right\|_{L^q(X)}^q &\lesssim \int_X\int_{\theta=0}^1\left|\sum_{k\ge 0} r_k(\theta)\,\beta_k\Big(\sqrt{\Delta_{\mathrm{g}}}\Big)\,a(x)\right|^q \,\mathrm{d}\theta\,\mathrm{d}x \\ &\lesssim \left\|a\right\|_{L^q(X)}^q, \end{aligned} \tag{4.6}$$

and the same holds when the β_k are replaced by the $\widetilde{\beta}_k$.

To see the other inequality in (3.2), consider a_1 in $L^q(X)$ and a_2 in $L^{q'}(X)$, and observe that

$$\begin{aligned}\left|\int_X a_1\, a_2 \,\mathrm{d}x\right| &= \left|\int_X \sum_{k\geq 0} (\beta_k a_1)\left(\widetilde{\beta}_k a_2\right)\,\mathrm{d}x\right| \\ &\leq \left\|\left(\sum_{k\geq 0}|\beta_k\, a_1|^2\right)^{\frac12}\right\|_{L^q(X)}\left\|\left(\sum_{k\geq 0}\left|\widetilde{\beta}_k\, a_2\right|^2\right)^{\frac12}\right\|_{L^{q'}(X)} \\ &\leq C\left\|\left(\sum_{k\geq 0}|\beta_k\, a_1|^2\right)^{\frac12}\right\|_{L^q(X)} \|a_2\|_{L^{q'}(X)}\,.\end{aligned} \tag{4.7}$$

Hence, by duality, we see that (3.2) is valid.

Returning to the proof of (4.1), we use a theorem of Grigor'yan [18, Theorem 1.1] that establishes Gaussian upper bounds on arbitrary Riemannian manifolds. His result implies that if $P(t,x,y)$ satisfies on-diagonal bounds

$$P(t,x,x) \lesssim \max\left(\frac{1}{t}, C\right) \tag{4.8}$$

for some constant $C>0$ then there exists $b>0$ such that

$$P(t,x,y) \lesssim \max\left(\frac{1}{t}, C\right)\exp\left(-\frac{b\,\mathrm{dist}_{\mathrm{g}}(x,y)^2}{t}\right). \tag{4.9}$$

Since $\left|B\left(x,\sqrt{t}\right)\right| \approx t$ for bounded t, this is equivalent to (4.1). In order to verify (4.8), it suffices to prove the estimate

$$P(t,x,x) \lesssim \frac{1}{t}$$

for all $t<1$ as the heat kernel is monotonically decreasing, see [19]. Hence, we restrict our discussion below to the case of small times.

In order to verify (4.8), we take the usual finite cover of the manifold with coordinate charts that are either isometric to a neighborhood of the Euclidean cone or the plane. We adapt an argument of Cheeger [9, §1] to see that within each chart, the heat kernel $\widetilde{P}(t,x,y)$ of the model space is a good approximation to the intrinsic heat kernel on X_0. Adjusting the cover if necessary, we may assume that any chart Z is contained in a slightly larger neighborhood Z'' where the isometry is defined. We may then take an intermediate neighborhood Z' so that $\bar{Z} \subset Z' \subset \bar{Z}' \subset Z''$.

Let ψ be a smooth cutoff supported in Z'' such that $\psi(z)\equiv 1$ in a neighborhood of $\bar{Z}'$. The function $z\mapsto \psi(z)\widetilde{P}(t,z,y)$ can be seen to lie in the domain of the Laplacian Δ_{g} on X_0, and for fixed y we may consider the inhomogeneous heat equation it satisfies on that space. For x and y in Z we have that $\psi(x)\widetilde{P}(t,x,y) = \widetilde{P}(t,x,y)$, so Duhamel's principle shows that

$$\widetilde{P}(t,x,y) - P(t,x,y) = \int_0^t\int_{Z''} P(t-s,x,z)(\partial_s+\Delta_{\mathrm{g}})(\psi(z)\widetilde{P}(s,z,y))\,\mathrm{d}z\,\mathrm{d}s. \tag{4.10}$$

We now recall that for fixed y, $\widetilde{P}(t,x,y)$ satisfies a homogeneous heat equation on the isometric space, which allows us to replace $-\partial_s\widetilde{P}$ by the Laplacian on that space. Applying the divergence theorem shows that (4.10) is equal to

$$\int_0^t \int_{Z''\setminus Z'} \langle \nabla_{\mathrm{g}} P(t-s,x,z), (\nabla_{\mathrm{g}}\psi)\widetilde{P}(s,z,y)\rangle \,\mathrm{d}z\,\mathrm{d}s \\ - \int_0^t \int_{Z''\setminus Z'} \langle (\nabla_{\mathrm{g}}\psi) P(t-s,x,z), \nabla_{\mathrm{g}}\widetilde{P}(s,z,y)\rangle \,\mathrm{d}z\,\mathrm{d}s. \tag{4.11}$$

Indeed, there is cancelation between the terms which have derivatives on both P and $\widetilde{P}$. Also the support conditions on ψ and $1-\psi$ mean that the boundary terms vanish and that the domain of integration can indeed be restricted to $Z''\setminus Z'$.

We now observe Cheeger's estimate [9, (1.1)], which can be written as

$$\|d^j P(t,x,\cdot)\|_{L^2(Z''\setminus Z')} + \|d^j \widetilde{P}(t,\cdot,y)\|_{L^2(Z''\setminus Z')} \leq K_N\, t^N \qquad \text{as } t\to 0, \tag{4.12}$$

where $j=0$ or 1 and the points x and y lie in Z. The bound (4.8) now follows by proving the same estimate for $\widetilde{P}(t,x,x)$. Indeed, for small t, (4.12) shows that the difference between the two kernels is negligible.

To establish the small time on-diagonal bound for the heat kernel on the Euclidean cone, we use an approach suggested by Li [24, p. 284]. A more general bound is actually announced in Theorem 2.1 of that work, but since the authors are unaware of any published proof, a weaker version of it is verified here. In particular, we emphasize that the approach below establishes on-diagonal bounds for the Euclidean cone only. We use $x=(r,\theta)$ to denote coordinates on the cone and to remain consistent with the notation established in [10], [11], [24], we use ν to denote the square root of the nonnegative Laplacian on the flat torus $\mathbb{R}/2\pi\rho\mathbb{Z}$. In [24, (1)], Li states the identity

$$\widetilde{P}(t,(r,\theta),(r,\theta)) = \\ \frac{1}{2\pi t}\left[\int_0^{\pi} e^{-(1-\cos y)r^2/2t}\cos y\nu\,\mathrm{d}y - \sin(\pi\nu)\int_0^{+\infty} e^{-(1+\cosh y)r^2/2t}e^{-y\nu}\mathrm{d}y\right](\theta,\theta), \tag{4.13}$$

where (θ,θ) means that we are integrating the kernels of $\cos y\nu$ and $\sin\pi\nu\cdot e^{-y\nu}$ evaluated at (θ,θ). This identity can be verified by using Cheeger's functional calculus on cones (see e.g. [9, Example 3.1]) and integral representations of modified Bessel functions (see e.g. Watson [28, §6.22(4)]).

It now suffices to obtain a uniform bound on the two integrals in brackets. For the first we use a formal identity observed by Cheeger and Taylor [11, (4.1), (4.8)] which states that for points θ_1, θ_2 inside a chart on the torus, $\cos y\nu(\theta_1,\theta_2)$ is the $2\pi\rho$-periodic extension of

$$\frac{1}{2}[\delta(\theta_1-\theta_2+y)+\delta(\theta_1-\theta_2-y)]. \tag{4.14}$$

When this formal identity is made rigorous, it is subject to the proviso that if the point masses are integrated against a function with jump discontinuities, it returns the average of the left and right hand limits of the function at the center of the point mass. Integrating $\cos y\nu(\theta,\theta)$ against the function $\mathbf{1}_{[0,\pi]}(y)\,e^{-(1-\cos y)r^2/2t}$ thus yields

$$\frac{1}{2}+\sum_{k=1}^{m} e^{-(1-\cos y_k)r^2/2t}, \tag{4.15}$$

where $\{y_k\}_{k=1}^m$ is the collection of real numbers in $(0,\pi]$ that are equivalent to 0 modulo $2\pi\rho$.

For the second integral in (4.13), we use the following identity in Cheeger-Taylor [11, (4.11)] (observing that $\rho = 1/\gamma$)

$$\sin\nu\pi \cdot e^{-\nu y}(\theta,\theta) = \frac{1}{2\pi\rho}\left(\frac{\sin(\pi/\rho)}{\cosh(y/\rho)-\cos(\pi/\rho)}\right). \tag{4.16}$$

This gives rise to the integral

$$\frac{1}{2\pi\rho}\int_0^\infty e^{-(1+\cosh y)r^2/2t}\frac{\sin(\pi/\rho)}{\cosh(y/\rho)-\cos(\pi/\rho)}\,dy. \tag{4.17}$$

Note that this integral vanishes when $\rho = 1/N$ for N a positive integer; this corresponds to the absence of diffraction on cones of these radii. Otherwise, the integrand is bounded near 0 and rapidly decaying at infinity. This provides uniform bounds on the second integral.

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