Bielefeld and Karlsruhe Lectures on Nonlinear Waves

Jeremy Marzuola [1]
Mathematics Department, University of North Carolina
Phillips Hall, Chapel Hill, NC 27599, USA
marzuola@math.unc.edu

[1] From a course given Summer 2012 at Universität Bielefeld and Summer 2013 at Karlsruhe Institute of Technology.

ABSTRACT. These notes will attempt to introduce the subject of nonlinear dynamics related to ground states (or solitons) in nonlinear dispersive equations. Many of the topics we will discuss relate to things I have learned about soliton stability through my Ph.D. thesis with Daniel Tataru along with graduate school projects I was lucky enough to work on with Justin Holmer and Maciej Zworski, followed by subsequent collaborations as a postdoc with Michael Weinstein and Herbert Koch. From all that they taught me, I was able to look deeper into the topic in further collaborations with Pierre Albin, Hans Christianson, Vincent Duchene, Chris Jones, Jianfeng Lu, Jason Metcalfe, Sarah Raynor, Mikael Rechtsman, Gideon Simpson, Michael Taylor and Laurent Thomann. In addition, throughout the years there have been many helfpul conversations with Wilhelm Schlag. In particular, much of what we will discuss analytically I have learned in much greater depth due to my ongoing collaborations with my colleagues Hans Christianson, Jason Metcalfe and Michael Taylor at the University of North Carolina, to whom I must credit many of the observations presented below. I also must mention that much of my numerical expertise has been developed in my past and ongoing collaborations with Gideon Simpson, who has taught me a great deal and shown me many ways I never thought possible to visualize spectral and stability issues. I also thank Mikael Rechtsman for shaping much of what I know in terms of applications of nonlinear dispersive equations in nonlinear optics. I am indebted to my many collaborators in all the works cited and developed here as many of the insights here are theirs. Finally, a special thanks to Sebastian Herr and the Guest Lectureship Program at Universität Bielefeld for hosting me during the Summer of 2012 and hence allowing for these lecture notes to even exist.

Contents

CHAPTER 1

Bielefeld Lecture 1

Lec1

In this series of lecture notes, we explore some analytic and computational aspects of stability and scattering theory for nonlinear bound states. We begin with a study problems of the form

eqn:bdstate
$$0 = -\Delta u + V(x)u + \lambda u - f(x,|u|^2)u. \tag{1.1}$$

Such problems arise in the study of nonlinear optics, gravitational fields, Bose-Einstein condensates, water waves and more. From a mathematical standpoint, these states act as obstructions to dispersion in the sense that they can be explicit solutions, which serve as critical points of this "infinite dimensional" dynamical system.

1.1. Examples

We will discuss in detail how such states arise, but as a general motivation, take the nonlinear Schrödinger equation

eqn:nls
$$i u_t + \Delta u + V(x)u + |u|^p u = 0,\ u(0,x) = u_0(x) \in H^1, \tag{1.2}$$

the generalized Korteweg-de Vries equation

eqn:kdv
$$u_t + \partial_x^3 u + \partial_x(|u|^p u) = 0,\ u(0,x) = u_0(x) \in H^1, \tag{1.3}$$

and the massive ($m > 0$) nonlinear Klein-Gordon equation

eqn:kg
$$u_{tt} = \Delta u - mu + |u|^p u,\ \ (u,u_t)(0,x) = (u_0,u_1)(x) \in H^1 \times L^2. \tag{1.4}$$

As we will discuss, solutions to the linear versions of these operators, the linear Schrödinger equation

eqn:ls
$$i u_t + \Delta u = 0,\ u(0,x) = u_0(x) \in H^1, \tag{1.5}$$

the Airy equation

eqn:Airy
$$u_t + \partial_x^3 u = 0,\ u(0,x) = u_0(x) \in H^1, \tag{1.6}$$

and the massive ($m > 0$) Klein-Gordon equation

eqn:linkg
$$u_{tt} = \Delta u - mu,\ \ (u,u_t)(0,x) = (u_0,u_1)(x) \in H^1 \times L^2, \tag{1.7}$$

are so-called "dispersive" equations. This means that the solution operators to (1.5), (1.6) and (1.7) tend to push energy towards ∞ in a sense that we will make precise when we study stability and perturbation in time-dependent problems. A nonlinear bound state arises when the solution itself cause energy to be "trapped," which is why we will study so-called "attractive" or "focussing" nonlinearities. This is directly related to the sign of the nonlinear terms being precisely the same as the sign of the Laplacian. In other words, one expects rather different behavior for (12.1) than for the "defocussing" problem

eqn:dnls
$$i u_t + \Delta u + |u|^p u = 0,\ u(0,x) = u_0(x) \in H^1. \tag{1.8}$$

Another key feature of a dispersive equation is the existence of one or many conserved quantities, which will play a major role in our analysis and will be explored in a detail later.

Though these are simply written as initial value problems, we for the most part will look for solutions with some sort of decay at ∞, which makes the H^s, $s > 0$ reasonable spaces in which to analyze the various evolution equations. Note different boundary conditions do arise in mathematical models, see for instance literature on Ginzburg-Landau vortices, but while aspects of the the analysis can be similar, the approach must be modified appropriately to deal with these non-trivial boundary conditions.

For the moment, let us simply proceed with the assumption that we will work in spaces that have some decay. Let us take as an ansatz

$$u(x,t) = e^{i\lambda t}\phi(x)$$

in (12.1),

$$u(x,t) = \phi(x - \lambda t)$$

in (1.3), and

$$u(x,t) = e^{i\sqrt{(m-\lambda)}t}\phi(x)$$

in (12.1), Then, given $\phi : \mathbb{R} \to \mathbb{R}$, we have

$$\partial_{xx}\phi - \lambda\phi + \phi^p = 0,$$

a nice nonlinear elliptic equation as above in (1.1). Note, this problem has studied for a very long time, even going back to the work of Pohozaev, [**Poh65**]. For the version of (1.1) appearing here, note also that the sign of $\lambda > 0$ makes a difference in that by both elliptic regularity and simple asymptotics on the appropriate Green's function, the decay of a function u solving

$$(-\Delta + \lambda)u = f$$

is much stronger for $\lambda > 0$ and indeed if $\lambda < 0$, one expects exponential growth at ∞ for the Green's function, plus one has plan wave solutions that experience no decay.

1.2. A Simple ODE

Let us take the special case of $d = 1$, $\lambda = 1$ and $p = 2$. The problem is now simply a second order ODE and can be solved (up to translational symmetry!) under the assumption that $\phi'(0) = 0$ and the solution decays as $|x| \to \infty$. In particular, there exists a solution of the form

$$\phi(x) = \operatorname{sech}(x).$$

To see this, note

$$\phi_x^2 = \lambda\phi^2 - \frac{1}{2}\phi^4.$$

Using the transformation $w = 1/\phi$,

$$w_x^2 = \lambda w^2 - \frac{1}{2},$$

giving $\phi(x) = (2\lambda)^{\frac{1}{2}}\operatorname{sech}(\lambda^{\frac{1}{2}}x)$.

Generically, one does not have such explicit ODE methods available, but many of the properties implicit to the sech function remain in tact upon other means of finding solutions to the elliptic equation. In particular, if one assumes radial symmetry of the

underlying space, one can frame the solution to such a problem as that of a system of ODEs that has a positive, radial solution. One can apply shooting techniques or other tools of classical dynamical systems to understand if such a solution exists.

1.3. Variational Characterization

To approach the problem variationally, we can write (1.1) as

$$\begin{aligned} 0 &= -\Delta u + V(x)u + \lambda u - f(x,|u|^2)u \\ &= H_1(u) + H_2(u) + H_3(u) + H_4(u), \end{aligned}$$

which allows us to write a Lagrangian energy

eqn:Lagrangian

$$\begin{aligned} \mathscr{L}(u) &= \frac{1}{2}\left(\int_{\mathbb{R}^d} |\nabla u|^2 dx + \int_{\mathbb{R}^d} V(x)|u|^2 dx + \lambda \int_{\mathbb{R}^d} |u|^2 dx - \int_{\mathbb{R}^d} F(x,|u|^2)u dx\right) \\ &= \mathscr{L}_1(u) + \mathscr{L}_2(u) + \mathscr{L}_3(u) + \mathscr{L}_4(u) \end{aligned} \tag{1.9}$$

where

$$\frac{dF}{ds}(x,s) = f(x,s). \tag{1.10}$$

Another way to generally go about solving such problems is through the method of constrained minimization in H^1. Namely, given the Lagrangian $\mathscr{L}$, we solve the problem

$$\min_{u\in H^1}\{\sum_{j\in\sigma} \mathscr{L}_j \mid \sum_{j\in\sigma^*} \mathscr{L}_j = \sigma\},$$

where $\sigma \subset \{1,2,3,4\}$ and $\sigma^* = \{1,2,3,4\}\setminus\sigma$. Using that such a minimizer would be critical point of the constrained Lagrangian along the level set of the constraint Lagrangian, we find the Lagrange multiplier for such a problem gives

$$\sum_{j\in\sigma} H_j = K\left(\sum_{j\in\sigma^*} H_j\right),$$

which if K can be scaled to be -1, gives a solution to the desired elliptic problem.

There are other ways to prove the existence of such nonlinear states however. For instance, the operator $H = -\Delta + V$ has discrete spectrum, at small amplitude one can use perturbation theory and the implicit function theorem to construct a nonlinear solution. These particular methods allow one to really explore symmetry breaking and find interesting and occasionally surprising dynamical systems, and will be explored in this class.

No matter how these states are constructed, some natural questions arise that we will attempt to answer here. First of all, are these states that we have constructed unique? Are they stable? In what sense? Can we control the dynamics about one of these points? Do they serve as global attractors for the infinite dimensional dynamical system? Are there any implicit symmetries to the space over which the evolution is defined and the evolution equation that can be exploited?

To begin, we recall the celebrated concentration compactness lemma of P.L. Lions, [**Lio84a**]:

lem:cc

LEMMA 1.3.1 (Concentration Compactness). *Let $(\rho_n)_{n\geq 1}$ be a sequence in $L^1(\mathbb{R}^d)$ satisfying:*

$$\rho_n \geq 0 \text{ in } \mathbb{R}^d, \int_{\mathbb{R}^d} \rho_n dx = \lambda$$

where $\lambda > 0$ is fixed. Then there exists a subsequence $(\rho_{n_k})_{k\geq 1}$ satisfying one of the three following possibilities:

i. (compactness) there exists $y_k \in \mathbb{R}^d$ such that $\rho_{n_k}(\cdot + y_{n_k})$ is tight, i.e.:

$$\forall \epsilon > 0,\ \exists R < \infty,\ \int_{y_k + B_R} \rho_{n_k}(x)dx \geq \lambda - \epsilon;$$

ii. (vanishing) $\lim_{k\to\infty} \sup_{y\in\mathbb{R}^d} \int_{y+B_R} \rho_{n_k}(x)dx = 0$, for all $R < \infty$;

iii. (dichotomy) there exists $\alpha \in [0, \lambda]$ such that for all $\epsilon > 0$, there exists $k_0 \geq 1$ and $\rho_k^1, \rho_k^2 \in L^1_+(\mathbb{R}^d)$ satisfying for $k \geq k_0$:

$$\begin{cases} \|\rho_{n_k} - (\rho_k^1 + \rho_k^2)\|_{L^1} \leq \epsilon,\ |\int_{\mathbb{R}^d} \rho_k^1 dx - \alpha| \leq \epsilon,\ |\int_{\mathbb{R}^d} \rho_k^2 dx - (\lambda - \alpha)| \leq \epsilon \\ d(Supp(\rho_k^1), Supp(\rho_k^2)) \to \infty. \end{cases}$$

Key Idea: Let X be a metric space and $\{\mu_k\}$ a sequence of Borel probability measures on X. For $R \in (0,\infty)$, $y \in X$, set $B_R(y) = \{x \in X : d(x,y) \leq R\}$ to be the closed ball of radius R, centered at y. Define

$$Q_k(R) = \sup_{y\in X} \mu_k(B_R(y)). \tag{1.11}$$

Of course each Q_k is a monotone increasing function of R on $[0,\infty)$, and

$$\lim_{R\to\infty} Q_k(R) = 1. \tag{1.12}$$

Using a standard diagonalization procedure, we can reduce to a subsequence (which we still denote by $\{\mu_k\}$) such that $Q_k \to Q$ on $\mathbb{Q}^+$. The function Q is monotone increasing, so set

$$\alpha = \lim_{R\to\infty} Q(R) \in [0,1]. \tag{1.13}$$

We examine separately the three cases $\alpha = 0$, $\alpha = 1$, and $0 < \alpha < 1$. We will see that these three cases lead to the phenomena of vanishing, concentration, or splitting respectively. (Observe that Lions labels the third case "dichotomy" rather than splitting.)

1.4. Exercises

CHAPTER 2

Bielefeld Lecture 2

lec2

Much of the following can be found applied to very general geometric settings in [**CMMT12**].

sa1

2.1. The concentration-vanishing-splitting trichotomy of Lions in a general setting

In this section we show the concentration-vanishing-splitting trichotomy of Lions [**Lio84a**, **Lio84b**] can be extended in a natural fashion to a metric space setting.

Case I: $\alpha = 0$. In this case,

$$\lim_{k\to\infty} \sup_{y\in X} \mu_k(B_R(y)) = 0, \quad \forall R < \infty. \tag{2.1}$$

This is precisely the case of *vanishing*.

Case II: $\alpha = 1$. In this case, for each $\mu \in (0,1)$, there exists $R = R(\mu)$ such that, for every k, $Q_k(R(\mu)) > \mu$. That means there exist points $y_k(\mu) \in X$ such that

$$\mu_k(B_{R(\mu)}(y_k(\mu))) > \mu. \tag{2.2}$$

Set $y_k = y_k(1/2)$, and observe that

$$\mu \geq 1/2 \implies d(y_k(\mu), y_k) \leq R(1/2) + R(\mu). \tag{2.3}$$

This follows by definition, since otherwise there would be two disjoint balls in M, each with μ_k-measure exceeding $1/2$, which contradicts μ_k being a probability measure.

As a consequence, with $y_k = y_k(1/2)$ as above, $\mu \in (1/2, 1)$, and

$$\tilde{R}(\mu) = R(1/2) + 2R(\mu), \tag{2.4}$$

we have for all k

$$\mu_k(B_{\tilde{R}(\mu)}(y_k)) > \mu. \tag{2.5}$$

As this holds for each $\mu \in (1/2, 1)$, this is the phenomenon of *concentration*.

Case III: $0 < \alpha < 1$. Pick $\epsilon > 0$. Then choose $R \in (0, \infty)$ such that $Q(R) > \alpha - \epsilon$. There exists k_0 such that for each $k \geq k_0$,

E:Qk-limit

$$\alpha - \epsilon < Q_k(R) < \alpha + \epsilon. \tag{2.6}$$

We can also choose a sequence $R_k \to \infty$ such that

E:Qk-limit

$$Q_k(R_k) \leq \alpha + \epsilon. \tag{2.7}$$

By (2.6), there exist points $y_k \in X$ such that

E:muk-BR

$$\mu_k(B_R(y_k)) \in (\alpha - \epsilon, \alpha + \epsilon). \tag{2.8}$$

Set

E:muk-BR (2.9) $$E_k^\sharp = B_R(y_k), \quad E_k^b = X \setminus B_{R_k}(y_k).$$

Then

(2.10) $$\mathrm{d}(E_k^\sharp, E_k^b) \geq R_k - R,$$

and

E:muk-X (2.11) $$\begin{aligned} \mu_k(X) - \mu_k(E_k^\sharp) - \mu_k(E_k^b) &= \mu_k(B_{R_k}(y_k) \setminus B_R(y_k)) \\ &\leq \alpha + \epsilon - (\alpha - \epsilon) \\ &= 2\epsilon. \end{aligned}$$

This is the phenomenon of *splitting*. Observe that (2.8) and (2.11) imply

(2.12) $$|\mu_k(E_k^\sharp) - \alpha| < \epsilon, \quad |\mu_k(E_k^b) - (1-\alpha)| < 3\epsilon.$$

2.2. Useful Background Lemmas for Applying Concentration Compactness

We recall here a few key useful tools from functional analysis, for generalizations and proofs, see [**Eva98**] or [**Tay97a**], which contains many generalizations to manifolds. We will use here the Sobolev space notation that $u \in W^{k,p}$ if it has k derivatives in the Sobolev space L^p and $u \in C^{k,\alpha}$ if it has k derivaties that α Hölder continuous.

thm:sobolev **Theorem 2.1 (Sobolev Embedding)** *Let U be a bounded, open set in $\mathbb{R}^d$ and $u \in W^{k,p}(U)$. Then,*

- *If $k < \frac{d}{p}$, then $u \in L^q$ for*
$$\frac{1}{q} = \frac{1}{p} - \frac{k}{d}$$
and there exists $C = C(U, d, k, q, p)$ such that
$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}(U)},$$
- *If $k > \frac{d}{p}$, then $u \in C^{k-\left[\frac{d}{p}\right]-1,\gamma}$ for*
$$\gamma = \begin{cases} \left[\frac{d}{p}\right] + 1 - \frac{d}{p} & \text{for } \frac{d}{p} \notin \mathbb{Z}, \\ < 1 & \text{for } \frac{d}{p} \in \mathbb{Z}, \end{cases}$$
and there exists $C = C(U, d, k, q, p)$ such that
$$\|u\|_{C^{k-\left[\frac{d}{p}\right]-1,\gamma}(U)} \leq C\|u\|_{W^{k,p}(U)}.$$

Theorem 2.2 (Rellich Compactness)
Let $1 \leq p \leq d$, and U be a bounded, open set in $\mathbb{R}^d$ such that $\partial U \in C^1$. Then $W^{1,p}(U)$ is compactly embedded in $L^q(U)$, denoted $W^{1,p}(U) \subset\subset L^q$, for
$$1 \leq q \leq \frac{pd}{d-p}.$$
By compactly embedded, we mean here
$$\|u\|_{L^q(U)} \leq C(U, d, p, q)\|u\|_{W^{1,p}}$$
and for any bounded sequence $\{u_n\}$ in $W^{1,p}$, there is a convergent subsequence in L^q.

Theorem 2.3 (Gagliardo-Nirenberg)
Given $u \in H^1(\mathbb{R}^d)$, we have

$$\|u\|_{L^{p+1}} \le C(p,d)\|u\|_{L^2}^{1-\gamma}\|\nabla u\|_{L^2}^{\gamma},$$

where $\gamma(p+1) < 2$ and

$$\gamma = \frac{d}{2} - \frac{d}{p+1}.$$

2.3. Applying Concentration Compactness

The nonlinear Schrödinger equations we consider are given by

E:NLS (2.13)
$$\begin{cases} i v_t + \Delta v + |v|^{p-1} v = 0, \ x \in \mathbb{R}^d \\ v(0,x) = v_0(x). \end{cases}$$

Recall, a nonlinear bound state is a choice of initial condition $u_\lambda(x)$ such that

E:NLS (2.14)
$$v(t,x) = e^{i\lambda t} u_\lambda(x)$$

satisfies (2.13) with initial data $v(0,x) = u_\lambda(x)$. Such a solution is also called a ground state, a standing wave, or a solitary wave, or, sometimes, a soliton. Plugging in this ansatz yields the following elliptic equation for u_λ:

E:SNLS (2.15)
$$-\Delta u_\lambda + \lambda u_\lambda - |u_\lambda|^{p-1} u_\lambda = 0.$$

Similarly, seeking a standing wave solution to a nonlinear Klein-Gordon equation,

E:SNLS (2.16)
$$v_{tt} - \Delta v + \sigma^2 v - |v|^{p-1} v = 0, \quad v(t,x) = e^{i\mu t} u(x),$$

leads to (2.15), with $\lambda = \sigma^2 - \mu^2$.

In studying (2.15), we will assume

E:lam-del (2.17)
$$\lambda > 0,$$

given that the spectrum of $-\Delta$ is contained in a semi-infinite interval

E:delta-spec (2.18)
$$\mathrm{Spec}(-\Delta) \subset [0,\infty).$$

We will analyze two methods of establishing the existence of a solution to (2.15). One is to mimimize the functional

E:F-lambda (2.19)
$$F_\lambda(u) = \|\nabla u\|_{L^2}^2 + \lambda \|u\|_{L^2}^2$$

subject to the constraint

E:J-p (2.20)
$$J_p(u) = \int_M |u|^{p+1} dx = \beta,$$

with $\beta \in (0,\infty)$ fixed. For this, we will require

E:p-range (2.21)
$$p \in \left(1, 1 + \frac{4}{n-2}\right), \quad \text{i.e., } \ p+1 \in \left(2, \frac{2n}{n-2}\right).$$

The other is to minimize the energy

1.0.10 (2.22)
$$E(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{p+1}\int_M |u|^{p+1} dx,$$

subject to the constraint on the "mass"

E:Q-beta (2.23)
$$Q(u) = \|u\|_{L^2}^2 = \beta,$$

with $\beta \in (0,\infty)$ fixed. For this, we require the more stringent condition

E:pp (2.24) $$p \in \Big(1, 1+\frac{4}{n}\Big).$$

The energy functional (2.22) is conserved for sufficiently regular solutions to the nonlinear Schrödinger equation (2.13), which imparts special importance to energy minimizers.

We preview these approaches in more detail.

2.3.1. F_λ **minimizers.** We make the hypotheses (2.17) and (2.21), and desire to minimize $F_\lambda(u)$ over $u \in H^1(\mathbb{R}^d)$, subject to the constraint (2.20).

If $u, v \in H^1(\mathbb{R}^d)$, we have

1.1.1 (2.25)
$$\begin{aligned} \frac{d}{d\tau} F_\lambda(u+\tau v)\Big|_{\tau=0} &= 2\mathfrak{Re}(-\Delta u + \lambda u, v), \\ \frac{d}{d\tau} J_p(u+\tau v)\Big|_{\tau=0} &= (p+1)\mathfrak{Re}\int |u|^{p-1} u\overline{v}\, dx. \end{aligned}$$

If $u \in H^1(\mathbb{R}^d)$ is a constrained minimizer, then

(2.26)
$$\begin{aligned} &v \in H^1(\mathbb{R}^d) \text{ and } \mathfrak{Re}\int_{\mathbb{R}^d} |u|^{p-1} u\overline{v}\, dx = 0 \\ &\Longrightarrow \mathfrak{Re}(-\Delta u + \lambda u, v) = 0, \end{aligned}$$

so the two elements of $H^{-1}(\mathbb{R}^d)$, $-\Delta u + \lambda u$ and $|u|^{p-1}u$, are linearly dependent over $\mathbb{R}$. Hence there exists a real constant K_0 such that

E:stat-K0 (2.27) $$-\Delta u + \lambda u = K_0 |u|^{p-1} u,$$

with equality holding in $H^{-1}(\mathbb{R}^d)$. To determine K_0, we pair each side of this equation with u and use $H^1 - H^{-1}$ duality:

1.1.4 (2.28) $$\|\nabla u\|_{L^2}^2 + \lambda \|u\|_{L^2}^2 = K_0 \int_{\mathbb{R}^d} |u|^{p+1}\, dx = \beta K_0.$$

Hence

E:K0 (2.29) $$K_0 = \frac{1}{\beta} \inf\{F_\lambda(u) : u \in H^1(\mathbb{R}^d), J_p(u) = \beta\}.$$

Given the existence of such an infimizer, it follows that

E:K0 (2.30) $$K_0 > 0.$$

Now if u solves (2.27), then $u_a(x) = au(x)$ solves

(2.31) $$-\Delta u_a + \lambda u_a = |a|^{-(p-1)} K_0 |u_a|^{p-1} u_a,$$

so that we can solve

(2.32) $$-\Delta u + \lambda u = K|u|^{p-1} u$$

for any $K > 0$.

2.3.2. Energy minimizers. We make the hypothesis (2.24) on p and desire to minimize $E(u)$, subject to the constraint (2.23). If $u, v \in H^1(\mathbb{R}^d)$, we have

1.2.1

$$\begin{aligned} \frac{d}{d\tau} E(u+\tau v)\Big|_{\tau=0} &= \mathfrak{Re}(-\Delta u - |u|^{p-1}u, v), \\ \frac{d}{d\tau} Q(u+\tau v)\Big|_{\tau=0} &= 2\,\mathfrak{Re}(u, v). \end{aligned} \tag{2.33}$$

If $u \in H^1(\mathbb{R}^d)$ is a constrained minimizer, then

1.2.2

$$\begin{aligned} v \in H^1(\mathbb{R}^d), \quad \mathfrak{Re}(u, v) = 0 \\ \Longrightarrow \mathfrak{Re}(\Delta u + |u|^{p-1}u, v) = 0, \end{aligned} \tag{2.34}$$

and it follows that there exists $\lambda \in \mathbb{R}$ such that $\Delta u + |u|^{p-1}u = \lambda u$, or equivalently,

1.2.3

$$-\Delta u + \lambda u = |u|^{p-1}u. \tag{2.35}$$

2.4. Exercises

CHAPTER 3

Bielefeld Lecture 3

Much of the following can be found applied to very general geometric settings in [**CMMT12**].

3.1. F_λ minimizers

s2

We wish to prove finish our goal of minimizing $F_\lambda(u)$, subject to the constraint $J_p(u) = \beta$. Observe that (2.17)–(2.18) imply

$$F_\lambda(u) \simeq \|u\|^2_{H^1(\mathbb{R}^d)}. \tag{3.1}$$

The hypothesis (2.21) on the range of p implies

$$H^1(\mathbb{R}^d) \subset L^{p+1}(\mathbb{R}^d). \tag{3.2}$$

Hence there exists a constant $C > 0$ such that

$$\|u\|^2_{L^{p+1}} \le C F_\lambda(u), \tag{3.3}$$

so that

E:I-beta

$$I_\beta = \inf\{F_\lambda(u) : u \in H^1(\mathbb{R}^d),\, J_p(u) = \beta\} > 0. \tag{3.4}$$

Suppose $\{u_\nu\} \subset H^1(\mathbb{R}^d)$ is a sequence satisfying

2.0.5

$$J_p(u_\nu) = \beta, \quad F_\lambda(u_\nu) \le I_\beta + \frac{1}{\nu}. \tag{3.5}$$

Passing to a subsequence if necessary, we have

$$u_\nu \to u \in H^1(\mathbb{R}^d) \tag{3.6}$$

converging in the weak topology. Rellich's theorem implies

E:Rellich

$$H^1(\mathbb{R}^d) \hookrightarrow L^{p+1}(\Omega) \tag{3.7}$$

is compact provided Ω is relatively compact. This implies that, for such Ω,

E:Rellich
S:s@nt

$$u_\nu \to u \text{ in the } L^{p+1}(\Omega) \text{ norm}. \tag{3.8}$$

3.1.1. Concentration. The enemy of finding a minimizer is the possibility of minimizing sequences escaping to spatial infinity. We will show that some minimizing subsequence concentrates, which we will define shortly.

Let us first record a basic Lemma of concentration compactness. This is given in Lions [**Lio84a**], pp. 115–117.

LEMMA 3.1.1
Fix $\beta \in (0,\infty)$. Let $\{u_\nu\} \in L^{p+1}(\mathbb{R}^d)$ be a sequence satisfying $\int |u_\nu|^{p+1} dx = \beta$. Then, after extracting a subsequence, one of the following three cases holds:

(i) Vanishing. If $B_R(y) = \{x \in \mathbb{R}^d : \mathrm{d}(x,y) \le R\}$ is the closed ball, then for all $0 < R < \infty$,

E:vanishing (3.9) $$\lim_{\nu\to\infty} \sup_{y\in\mathbb{R}^d} \int_{B_R(y)} |u_\nu|^{p+1}\, dx = 0.$$

(ii) Concentration. There exists a sequence of points $\{y_\nu\} \subset M$ with the property that for each $\epsilon > 0$, there exists $R(\epsilon) < \infty$ such that

E:vanishing (3.10) $$\int_{B_{R(\epsilon)}(y_\nu)} |u_\nu|^{p+1}\, dx > \beta - \epsilon.$$

(iii) Splitting. There exists $\alpha \in (0,\beta)$ with the following properties: For each $\epsilon > 0$, there exists $\nu_0 \ge 1$ and sets $E_\nu^\sharp, E_\nu^b \subset \mathbb{R}^d$ such that

E:splitting-1 (3.11) $$\mathrm{d}(E_\nu^\sharp, E_\nu^b) \to \infty \text{ as } \nu \to \infty,$$

and

E:splitting-2 (3.12) $$\left| \int_{E_\nu^\sharp} |u_\nu|^{p+1}\, dx - \alpha \right| < \epsilon, \ \left| \int_{E_\nu^b} |u_\nu|^{p+1}\, dx - (\beta - \alpha) \right| < \epsilon.$$

We now show that for a minimizing sequence in our problem, the vanishing phenomenon cannot occur. The proof uses the following lemma, essentially given in [**Lio84b**].

L:vanishing **LEMMA 3.1.2** *Assume $\{u_\nu\}$ is bounded in $H^1(\mathbb{R}^d)$, and*

E:vanishing-2 (3.13) $$\lim_{\nu\to\infty} \sup_{y\in\mathbb{R}^d} \int_{B_R(y)} |u_\nu|^2\, dx = 0, \text{ for some } R > 0.$$

Then

E:vanishing-3 (3.14) $$2 < r < \frac{2n}{n-2} \implies \|u_\nu\|_{L^r(\mathbb{R}^d)} \to 0.$$

PROOF. This is a special case of Lemma I.1 on p. 231 of [**Lio84b**]. The only two geometric properties used in the proof in [**Lio84b**] are the existence of Sobolev embeddings on balls of radius $R > 0$, and the fact that there exists $m < \infty$ such that $\mathbb{R}^n$ has a covering by balls of radius R in such a way that each point is contained in at most m balls. As an idea for how applicable the concentration compactness machinery is, these two facts hold on every smooth Riemannian manifold with $\mathscr{C}^\infty$ bounded geometry.

In reality, we prove the following general version of 3.1.2.

L:genvanishing **LEMMA 3.1.3** *Assume $\{u_\nu\}$ is bounded in $L^q(\mathbb{R}^d)$ and $\{\nabla u_n\}$ bounded in $L^p(\mathbb{R}^d)$, and*

E:genvanishing-2 (3.15) $$\lim_{\nu\to\infty} \sup_{y\in\mathbb{R}^d} \int_{B_R(y)} |u_\nu|^q\, dx = 0, \text{ for some } R > 0.$$

Then

E:genvanishing-3 (3.16) $$q < r < \frac{dp}{d-p} \implies \|u_\nu\|_{L^r(\mathbb{R}^d)} \to 0.$$

The key ideas go as following:

If $\|u_n\|_{L^\infty} < L$ uniformly, then we can apply Hölder's inequality and the Sobolev embedding theorem repeatedly to observe

$$\sup_{y\in\mathbb{R}^d} \int_{B(y,R)} |u_n|^\rho\, dx \to 0$$

for any $R>0$ and $\rho > \min(q, dp/(d-p))$, which then implies

$$\sup_{y\in\mathbb{R}^d}\int_{B(y,R)} |u_n|^{\bar\rho-1}|\nabla u_n|dx \to 0$$

for $\bar\rho > q$, $(\bar\rho-1)p' > q$.

Hence, using the Sobolev embedding with $k=1$, $p=1$, we have a $\gamma>1$ such that

$$\begin{aligned}\int_{B(y,R)} |u_n|^{\bar\rho\gamma}dx &\le C\left(\int_{B(y,R)}(|u_n|^{\bar\rho}+\bar\rho|u_n|^{\bar\rho-1}|\nabla u_n|)dx\right)^\gamma\\ &\le \epsilon_n^{\gamma-1}\left(\int_{B(y,R)}(|u_n|^{\bar\rho}+\bar\rho|u_n|^{\bar\rho-1}|\nabla u_n|)dx\right),\end{aligned}$$

where from above $\epsilon_n\to 0$, which implies

$$\int_{B(y,R)} |u_n|^{\bar\rho\gamma}dx \to 0.$$

To complete the proof in general, define $v_n(x)=\min\{u_n(x),L\}$. Then, we can do the same procedure on $\{v_n\}$, but now

$$\limsup_{n\to\infty}\int_{\mathbb{R}^d}|u_n|^\alpha dx \le \lim_{n\to\infty}\int|v_n|^\alpha dx + \frac{1}{L^{\rho-\alpha}}\int|u_n|^\rho dx,$$

which is uniformly bounded by $L^{\alpha-\rho}$ for $q<\rho<dp/(d-p)$ and the term on the right vanishes as $L\to\infty$.

□

P:no-vanishing **PROPOSITION 3.1** *Suppose $\{u_\nu\}$ is a minimizing sequence. Then no subsequence can satisfy* (3.9).

PROOF. If (3.9) holds, then so does (3.13), by Hölder's inequality on finite measure balls. Then (3.14) holds with $r=p+1$ (recall (2.21)). This contradicts the assumption that $J_p(u)=\beta>0$. □

To show that splitting is impossible, we start by showing that I_β has the property that, for all $\beta>0$,

2.1.7 (3.17) $$I_\beta < I_\eta + I_{\beta-\eta}, \text{ for any } \eta\in(0,\beta).$$

Lions gives (3.17) a key role in results of [**Lio84a**]– [**Lio84b**], showing that, in various situations, splitting cannot occur. In fact, in this case we have much more structure.

P:subadd **PROPOSITION 3.2** *For all $\beta>0$, we have*

(3.18) $$I_\beta = I_1\beta^{2/(p+1)}.$$

PROOF. Suppose u_ν satisfies (3.5), so

(3.19) $$J_p(u_\nu)=\beta,\quad F_\lambda(u_\nu)\to I_\beta.$$

Then for $a>0$,

(3.20) $$J_p(au_\nu)=a^{p+1}\beta,\quad F_\lambda(au_\nu)\to a^2 I_\beta.$$

Hence

2.1.11 (3.21) $$\gamma = a^{p+1}\beta \Longrightarrow I_\gamma \le a^2 I_\beta.$$

Now if we replace β by γ and a by a^{-1}, we get $I_\beta \le a^{-2} I_\gamma$, i.e., $I_\gamma \ge a^2 I_\beta$, which together with (3.21) implies

$$\gamma = a^{p+1}\beta \implies I_\gamma = a^2 I_\beta, \tag{3.22}$$

and proves the proposition. □

Note that since $CF_\lambda(u) \ge \int |u|^{p+1} dx$, we have $I_1 > 0$, so $I_\beta > 0$ for every $\beta > 0$, and (3.17) follows.

Remark. The proof of Proposition 3.2 implies that F_λ-minimizers scale to other F_λ-minimizers. We state this formally.

c2.1.R **COROLLARY 1** *If $\beta > 0$ and u minimizes F_λ, subject to the constraint $J_p(u) = \beta$, and if $\kappa > 0$, then $u_\kappa = \kappa u$ minimizes F_λ, subject to the constraint $J_p(u_\kappa) = \kappa^{p+1}\beta$.*

Let us now show that splitting cannot occur for a minimizing subsequence. Suppose on the contrary that there exists $\alpha \in (0, \beta)$ and for each $\epsilon > 0$, sets $E_\nu^\sharp, E_\nu^b \subset \mathbb{R}^d$ such that (3.11)-(3.12) occur. Choose $\epsilon > 0$ sufficiently small that

E:C-1-eps
$$I_\beta < I_\alpha + I_{\beta-\alpha} - C_1\epsilon, \tag{3.23}$$

where $C_1 > 0$ is a sufficiently large constant to be fixed later. Since $\|u_\nu\|_{H^1(\mathbb{R}^d)}$ is uniformly bounded, if follows from (3.11) that there exists ν_1 such that $\nu \ge \nu_1$ implies

$$\int_{S_\nu} |u_\nu|^2 \, dx < \epsilon, \tag{3.24}$$

where S_ν is a set of the form

$$S_\nu = \{x \in \mathbb{R}^d : d_\nu < \mathrm{d}(x, E_\nu^\sharp) \le d_\nu + 2\} \subset M \setminus (E_\nu^\sharp \cup E_\nu^b), \tag{3.25}$$

for some $d_\nu > 0$. For $r > 0$ and $\nu \ge \nu_1$, set

$$\tilde{E}_\nu(r) = \{x \in \mathbb{R}^d : \mathrm{d}(x, E_\nu^\sharp) \le r\}, \tag{3.26}$$

so that $S_\nu = \tilde{E}_\nu(d_\nu + 2) \setminus \tilde{E}(d_\nu)$. Define functions $\chi_\nu^\sharp$ and χ_ν^b by

$$\chi_\nu^\sharp(x) = \begin{cases} 1, & \text{if } x \in \tilde{E}_\nu(d_\nu), \\ 1 - \mathrm{d}(x, \tilde{E}_\nu(d_\nu)), & \text{if } x \in \tilde{E}_\nu(d_\nu + 1), \\ 0, & \text{if } x \notin \tilde{E}_\nu(d_\nu + 1), \end{cases} \tag{3.27}$$

and

$$\chi_\nu^b(x) = \begin{cases} 0, & \text{if } x \in \tilde{E}_\nu(d_\nu + 1), \\ \mathrm{d}(x, \tilde{E}_\nu(d_\nu)), & \text{if } x \in \tilde{E}_\nu(d_\nu + 2), \\ 1, & \text{if } x \notin \tilde{E}_\nu(d_\nu + 2), \end{cases} \tag{3.28}$$

These functions are both Lipschitz with Lipschitz constant 1 and almost disjoint supports. Set $u_\nu^\sharp = \chi_\nu^\sharp u_\nu$ and $u_\nu^b = \chi_\nu^b u_\nu$. Since $0 \le \chi_\nu^\sharp + \chi_\nu^b \le 1$, we have

$$J_p(u_\nu^\sharp) + J_p(u_\nu^b) \le \int (\chi_\nu^\sharp + \chi_\nu^b)|u_\nu|^{p+1} \, dV \le J_p(u_\nu) = \beta. \tag{3.29}$$

Also, of course since $\lambda \ge 0$,

$$\lambda \|u_\nu^\sharp\|_{L^2}^2 + \lambda \|u_\nu^b\|_{L^2}^2 \le \lambda \|u_\nu\|_{L^2}^2. \tag{3.30}$$

We have $\nabla u_v^\sharp = \chi_v^\sharp \nabla u_v + (\nabla \chi_v^\sharp) u_v$ and similarly for u_v^b, and $|\nabla \chi_v^\sharp| \le 1$ except for a set of measure zero, so

$$\|\nabla u_v^\sharp\|_{L^2}^2 + \|\nabla u_v^b\|_{L^2}^2 \le \|\nabla u_v\|_{L^2}^2 + \int_{S_v} |u_v|^2 \, dx. \tag{3.31}$$

As a consequence, we have

E:F-lambda-sharp

$$F_\lambda(u_v^\sharp) + F_\lambda(u_v^b) \le F_\lambda(u_v) + \epsilon, \tag{3.32}$$

Using the support properties of $u_v^\sharp$, u_v^b together with (3.11)-(3.12) yields

E:J-p-sharp

$$|J_p(u_v^\sharp) - \alpha|, \quad |J_p(u_v^b) - (\beta - \alpha)| \le 3\epsilon. \tag{3.33}$$

Combining (3.32)-(3.33), and letting $v \to \infty$, we get

$$I_\alpha + I_{\beta-\alpha} \le I_\beta + C_2\epsilon, \tag{3.34}$$

where C_2 depends only on $\delta_0 > 0$, the bottom of the spectrum of $-\Delta$. Hence if $C_1 > C_2$ is chosen sufficiently large in (3.23) (which simply amounts to producing ϵ sufficiently small), we contradict (3.23). This, together with Proposition 3.1 yields the following proposition, which states that for a minimizing sequence u_v, only the concentration phenomenon can occur.

P:only-concentration

PROPOSITION 3.3 *Let $\{u_v\}$ be a minimizing sequence of I_λ. Then every subsequence of the $\{u_v\}$ has a further subsequence (which we will continue to denote by $\{u_v\}$) with the following property. There exists a sequence $\{y_v\} \subset M$ and a function $\tilde{R}(\epsilon)$ such that for all v,*

E:only-concentration

$$\int_{B_{\tilde{R}(\epsilon)}(y_v)} |u_v|^{p+1} \, dx > \beta - \epsilon, \quad \forall\, \epsilon > 0, \tag{3.35}$$

Remark 3.1.4
It is very important to observe the following facts in Proposition 3.3. The sequence of points $\{y_v\}$ is independent of $\epsilon > 0$, and the function $\tilde{R}(\epsilon)$ is independent of the index v.

s2.2

3.1.2. Compactness and existence of minimizers. We retain the hypotheses (2.18)–(2.21). Let $\{u_v\} \subset H^1(\mathbb{R}^d)$ be a minimizing sequence for (3.4), that is,

$$J_p(u_v) = \beta, \ F_\lambda(u_v) \le I_\beta + \frac{1}{v}, \tag{3.36}$$

where F_λ is given by (2.19) and J_p is given by (2.20) as usual. After passing to a subsequence if necessary, Proposition 3.3 shows we have points $\{y_v\} \subset \mathbb{R}^d$ and a function $\tilde{R}(\epsilon)$ such that (3.35) holds.

We now fix a base point, say the origin $0 \in \mathbb{R}^d$ and apply translation invariance: for each v, set $v_v(x) = u_v(x - y_v)$, so that v_v is now concentrated near 0 instead of near y_v. The sequence $\{v_v\}$ satisfies

$$J_p(v_v) = J_p(u_v) = \beta, \ F_\lambda(v_v) = F_\lambda(u_v) \le I_\beta + \frac{1}{v}, \tag{3.37}$$

and

E:conc-int

$$\int_{B_{\tilde{R}(\epsilon)}(0)} |v_v|^{p+1} \, dx > \beta - \epsilon, \ \forall\, \epsilon > 0. \tag{3.38}$$

Passing to a further subsequence, which we continue to denote by $\{v_\nu\}$, we have

2.2.3A (3.39) $$v_\nu \to v \text{ weakly in } H^1(\mathbb{R}^d).$$

Since F_λ is comparable to the $H^1(\mathbb{R}^d)$ norm squared, we have

(3.40) $$F_\lambda(v) \le \liminf_{\nu\to\infty} F(v_\nu) = I_\beta.$$

Similarly

(3.41) $$J_p(v) \le \liminf_{\nu\to\infty} J_p(v_\nu) = \beta.$$

On the other hand, by (3.7), we have for each $\epsilon > 0$,

(3.42) $$v_\nu \to v, \text{ in the } L^{p+1}(B_{\tilde{R}(\epsilon)}(0)) \text{ norm},$$

so that (3.38) implies $J_p(v) \ge \beta$. Hence $J_p(v) = \beta$, which in turn implies

2.2.7 (3.43) $$F_\lambda(v) = I_\beta.$$

Let us summarize this argument in the following Proposition.

PROPOSITION 3.4
If (2.18)–(2.21) *hold, then there exists a minimizer* $v \in H^1(\mathbb{R}^d)$ *of* $F_\lambda(v)$*, subject to the constraint* $J_p(v) = \beta$.

Remark. It follows from (3.43) that convergence $v_\nu \to v$ in (3.39) holds in norm in $H^1(\mathbb{R}^d)$, hence in norm in $L^{p+1}(\mathbb{R}^d)$.

3.2. Exercises

CHAPTER 4

Bielefeld Lecture 4

Much of the following can be found applied to very general geometric settings in [**CMMT12**].

4.1. Energy Minimizers

s3

Now, we approach the topic of minimizing energy with respect to fixed mass, though we must limit ourselves to a slightly smaller range of powers p in the nonlinear term of (2.13). We require

E:subcrit

$$p \in \Big(1, 1+\frac{4}{n}\Big), \tag{4.1}$$

which is the range of L^2 subcritical powers in the standard Euclidean case example. We desire to minimize the functional

3.0.2

$$E(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{p+1}\int |u|^{p+1}\,dx \tag{4.2}$$

over $H^1(M)$, subject to the constraint

3.0.3

$$Q(u) = \|u\|_{L^2}^2 = \beta. \tag{4.3}$$

As seen in (2.33)–(2.35), given $u \in H^1$ a solution to the constrained minimization problem, we must have, for some $\lambda \in \mathbb{R}$,

$$-\Delta u + \lambda u - |u|^{p-1}u = 0. \tag{4.4}$$

The range of powers in (4.1) plays an important role in the Gagliardo-Nirenberg inequality

3.0.5

$$\|u\|_{L^{p+1}(\mathbb{R}^n)} \le C\|u\|_{L^2}^{1-\gamma}\|u\|_{H^1}^{\gamma}, \tag{4.5}$$

where

$$\gamma = \frac{n}{2} - \frac{n}{p-1}, \quad \text{hence } \gamma(p+1) < 2. \tag{4.6}$$

As a result, we have

E:EQbds

$$\begin{aligned} \|u\|_{H^1}^2 &= E(u) + \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1} + Q(u) \\ &\le E(u) + \tilde{C}Q(u)^{(p+1)(1-\gamma)/2}\|u\|_{H^1}^{\gamma(p+1)} + Q(u), \end{aligned} \tag{4.7}$$

which gives a priori bounds at $\|u\|_{H^1}$ in terms of bounds on $E(u)$ and $Q(u)$. Then, take

E:I-beta-Emin

$$\mathscr{I}_\beta = \inf\{E(u) : u \in H^1,\, Q(u) = \beta\}, \tag{4.8}$$

for $\beta > 0$. The a priori bounds in (4.7) show that for a particular β, we have $\mathscr{I}_\beta > -\infty$ since $\gamma(p+1) < 2$. Taking at this point the sequence $u_\nu \in H^1$ such that

3.0.9

$$Q(u_\nu) = \beta,\ E(u_\nu) \le \mathscr{I}_\beta + \frac{1}{\nu}. \tag{4.9}$$

Note, from (4.7), we have that $\|u_\nu\|_{H^1}$ bounded. Then, we apply the concentration-compactness techniques to the L^1 sequence given by $\{|u_\nu|^2\}$.

It turns out that we need to assume

3.0.9A (4.10) $$\mathscr{I}_\beta < 0,$$

to show in the concentration-compactness argument that splitting and vanishing cannot occur. In connection with this, note that replacing u by au in (4.2) and letting $a \nearrow +\infty$ shows that

E:i-beta (4.11) $$\mathscr{I}_\beta \to -\infty \text{ as } \beta \to +\infty.$$

In particular, $\mathscr{I}_\beta < 0$ for all sufficiently large β. We will see later that $\mathscr{I}_\beta < 0$ for all β, but note it is not guaranteed that $\mathscr{I}_\beta < 0$ for all β in general.

s3.1 **4.1.1. Concentration.** We need to show that there is no vanishing and no splitting. We first establish that there is no vanishing when $\mathscr{I}_\beta < 0$. In fact, The vanishing condition for our sequence implies that given $B_R(y) = \{x \in \mathbb{R}^d : d_M(x,y) \le R\}$, we have

E:van (4.12) $$\lim_{\nu\to\infty} \sup_{y\in\mathbb{R}^d} \int_{B_R(y)} |u_\nu|^2 \, dx = 0, \quad \forall\, R < \infty.$$

Using Lemma 3.1.2, we have the following.

P:van **PROPOSITION 4.1** *Assume $\mathscr{I}_\beta < 0$. For u_ν a sequence minimizing the energy with fixed mass,* (4.12) *cannot occur.*

PROOF. For p as in (4.1), let us assume that (4.12) occurs for $\{u_\nu\}$. Then, Lemma 3.1.2 shows

(4.13) $$\|u_\nu\|_{L^{p+1}} \to 0,$$

implying

(4.14) $$\frac{1}{2}\|\nabla u_\nu\|_{L^2}^2 \to \mathscr{I}_\beta < 0,$$

a contradiction. □

Our next task is to establish that there is no splitting. If there were splitting, we see that for any $\alpha \in (0,\beta)$, for each $\epsilon > 0$ there exists $\nu_0 \ge 1$ and sets $E_\nu^\#,\ E_\nu^b \subset \mathbb{R}^d$ such that

(4.15) $$d(E_\nu^\#, E_\nu^b) \to \infty \text{ as } \nu \to \infty$$

and

(4.16) $$\left|\int_{E_\nu^\#} |u_\nu|^2 \, dx - \alpha\right| < \epsilon, \ \left|\int_{E_\nu^b} |u_\nu|^2 \, dx - (\beta - \alpha)\right| < \epsilon.$$

We record here some subadditivity properties of $\mathscr{I}_\beta$ in order to argue similarly to the splitting argument in Section 3.1.1.

P:subadd0 **PROPOSITION 4.2** *If $\beta > 0$, $\mathscr{I}_\beta < 0$, $\sigma > 1$, then*

(4.17) $$\mathscr{I}_{\sigma\beta} < \sigma \mathscr{I}_\beta.$$

PROOF. Let u_ν be a minimizing sequence as in (4.9). Define

$$w_\nu = \sigma^{1/2} u_\nu, \quad \text{so } \|w_\nu\|_{L^2}^2 = \sigma\beta. \tag{4.18}$$

Hence,

$$\begin{aligned} E(w_\nu) &= \frac{\sigma}{2}\|\nabla u_\nu\|_{L^2}^2 - \frac{\sigma^{p+1}}{p+1}\int |u_\nu|^{p+1}\,dx \\ &= \sigma E(u_\nu) - \frac{\sigma^{p+1}-\sigma}{p+1}\|u_\nu\|_{L^{p+1}}^{p+1}. \end{aligned} \tag{4.19}$$

Passing to the limit gives

$$\mathscr{I}_{\sigma\beta} \le \sigma \mathscr{I}_\beta. \tag{4.20}$$

However, given $\mathscr{I}_\beta < 0$, as in the proof of Proposition 4.1, $\|u_\nu\|_{L^{p+1}}$ does not approach 0 and the result follows. □

Then, we have the following result similar to Proposition 3.2.

P:subadd1 **PROPOSITION 4.3** *Given $0 < \eta < \beta$ and $\mathscr{I}_\beta < 0$, we have*

$$\mathscr{I}_\beta < \mathscr{I}_{\beta-\eta} + \mathscr{I}_\eta. \tag{4.21}$$

PROOF. Assume without loss of generality that $\eta \le \beta - \eta$ and take

$$\beta - \eta = \sigma\eta \tag{4.22}$$

with $\sigma \ge 1$. Hence, using Proposition 4.2 we have

$$\mathscr{I}_{\sigma\eta} \le \sigma\mathscr{I}_\eta,\ \mathscr{I}_\beta = \mathscr{I}_{(\sigma+1)\eta} < \frac{\sigma+1}{\sigma}\mathscr{I}_{\sigma\eta} = \frac{\sigma+1}{\sigma}\mathscr{I}_{\beta-\eta}. \tag{4.23}$$

As a result,

$$\begin{aligned} \mathscr{I}_\beta = \mathscr{I}_{(\sigma+1)\eta} &< \frac{\sigma+1}{\sigma}\mathscr{I}_{\sigma\eta} \\ &= \mathscr{I}_{\beta-\eta} + \frac{1}{\sigma}\mathscr{I}_{\sigma\eta} \\ &\le \mathscr{I}_{\beta-\eta} + \mathscr{I}_\eta. \end{aligned} \tag{4.24}$$

□

Applying this proposition in the same way as in Section 3.1.1, we have the result that no splitting can occur for the sequence $\{u_\nu\}$.

Therefore, upon passing to a subsequence, we have concentration. There exist $y_\nu \in \mathbb{R}^d$ (independent of ϵ) with the following property. For each $\epsilon > 0$, there exists $\widetilde{R}(\epsilon) < \infty$ such that

3.1.16 $$\int_{B_{\widetilde{R}(\epsilon)}(y_\nu)} |u_\nu|^2\,dx > \beta - \epsilon. \tag{4.25}$$

s3.2

4.1.2. Existence of energy minimizers. As long as

4.0 (4.26) $$\mathscr{I}_\beta < 0,$$

we are left with the situation where $u_\nu \in H^1(\mathbb{R}^d)$ satisfies (4.9) and the concentration phenomenon (4.25). We can translate the points y_ν to a subset of some compact $K \subset \mathbb{R}^d$. We relabel the associated translates of u_ν as u_ν. Passing to a subsequence, we have

4.1 (4.27) $$u_\nu \longrightarrow u, \quad \text{weak}^* \text{ in } H^1(\mathbb{R}^d).$$

By Rellich compactness, $u_\nu \to u$ in $L^2(B)$, in norm, for each bounded $B \subset \mathbb{R}^d$. Hence, by (4.25)

4.2 (4.28) $$\|u\|_{L^2}^2 = \beta.$$

Hence,

4.3 (4.29) $$u_\nu \longrightarrow u \quad \text{in } L^2(\mathbb{R}^d) \text{ norm.}$$

Now, as in (4.5), we have

4.4 (4.30) $$\|u - u_\nu\|_{L^{p+1}} \le C\|u - u_\nu\|_{L^2}^{1-\gamma}\|u - u_\nu\|_{H^1}^{\gamma},$$

so

4.5 (4.31) $$u_\nu \longrightarrow u \text{ in } L^{p+1}(\mathbb{R}^d) \text{ norm.}$$

Now

4.6 (4.32) $$\frac{1}{2}\|\nabla u_\nu\|_{L^2}^2 - \frac{1}{p+1}\|u_\nu\|_{L^{p+1}}^{p+1} \longrightarrow \mathscr{I}_\beta,$$

and

4.7 (4.33) $$\|u_\nu\|_{L^{p+1}}^{p+1} \longrightarrow \|u\|_{L^{p+1}}^{p+1}.$$

Also, since (4.28) holds,

4.8 (4.34) $$\frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1} \ge \mathscr{I}_\beta.$$

Hence

4.9 (4.35) $$\|\nabla u\|_{L^2}^2 \ge \liminf \|\nabla u_\nu\|_{L^2}^2,$$

so

4.10 (4.36) $$\nabla u_\nu \longrightarrow \nabla u \text{ in } L^2(\mathbb{R}^d) \text{ norm,}$$

and u minimizes $E(u)$ subject to the constraint (4.3), at least provided (4.26) holds.

4.1.3. Primer on Symmetrization. We recall here some properties of symmetric rearrangements in $\mathbb{R}^d$. Probably the best reference for these results are [**LL01**]. Let us define the symmetric, decreasing rearrangement of a function, f, such that

$$f^*(|x|) = \inf\{t | \mu(\{|f| > t\} \le \mu(B(0,|x|))\},$$

where μ is the standard Lebesgue measure. Another way to define such a rearrangement is

$$f^*(x) = \int_0^\infty \chi^*_{\{y||f|>t\}}(x)\,dt,$$

where χ_A^* is the standard cut-off function to $A^* = B(0,R_A)$, where $Vol(A) = Vol(A^*)$. Note that

$$\{x|f^* > t\} = \{x||f| > t\}^*.$$

Since

$$\int |f|^p \, dx = p \int_0^\infty t^{p-1} \mu(\{x||f| > t\}) dt,$$

it is easy to see that

$$\|f^*\|_{L^p} = \|f\|_{L^p}$$

for any $1 < p < \infty$. Hence, one observes that symmetrization will decrease the energy or F_λ iff

$$\|\nabla u^*\|_{L^2} \leq \|\nabla u\|_{L^2}.$$

The key to this inequality relies upon the observation that

$$(-\Delta u, u) = \lim_{t \to 0} \frac{(u,u) - (e^{\Delta t} u, u)}{t},$$

meaning that one would like to prove

$$(e^{\Delta t} u^*, u^*) \geq (e^{\Delta t} u, u),$$

which follows from the following propostion.

PROPOSITION 4.4 (Lieb-Loss)
Let $f_1, \ldots, f_m$ are non-negative functions on $\mathbb{R}^d$ that vanish at ∞. Letting $k \leq m$, define $B = \{b^{ij}\}$ a $k \times m$ matrix. Then, given

$$I(f_1, \ldots, f_m) = \int \cdots \int \prod_{j=1}^{m} f_j (\sum_{i=1}^{l} b^{ij} x_i) dx_1 \ldots dx_k,$$

then

$$I(f_1, \ldots, f_m) \leq I(f_1^*, \ldots, f_m^*).$$

The proof of this proposition relies upon the fact that heat kernel of the Laplacian is a monotonic, decreasing function and that $\mathbb{R}^d$ is invariant under reflection symmetry.

sa4

4.1.4. Symmetrization approach to F_λ-minimizers. We assume

2 (4.37) $$\lambda > 0, \quad p+1 \in \left(2, \frac{2n}{n-2}\right), \quad \beta \in (0, \infty).$$

We aim to minimize

3 (4.38) $$F_\lambda(u) = \|\nabla u\|_{L^2}^2 + \lambda \|u\|_{L^2}^2,$$

subject to the constraint

4 (4.39) $$J_p(u) = \int |u|^{p+1} \, dx = \beta.$$

We turn to the task of finding the desired minimizer. Note that (4.37) implies

5 (4.40) $$F_\lambda(u) \approx \|u\|_{H^1}^2,$$

and in this setting we have the Sobolev embedding result

6 (4.41) $$H^1 \subset L^q, \quad \forall \, q \in \left[2, \frac{2n}{n-2}\right],$$

if $n \geq 3$, $\forall \, q \in [2, \infty)$ if $n = 2$. The results (4.40)–(4.41) imply

14 (4.42) $$\|u\|_{L^{p+1}}^2 \leq C F_\lambda(u),$$

so

15 (4.43) $$I_\beta = \inf\{F_\lambda(u) : J_p(u) = \beta\} > 0.$$

Let $u_\nu \in H^1$ satisfy

16 (4.44) $$J_p(u_\nu) = \beta, \quad F_\lambda(u_\nu) \le I_\beta + \frac{1}{\nu}.$$

Passing to a subsequence, which we continue to denote (u_ν), we have

17 (4.45) $$u_\nu \longrightarrow u \in H^1, \quad \text{converging weakly.}$$

Rellich's Compactness theorem gives

18 (4.46) $$H^1(\Omega) \hookrightarrow L^{p+1}(\Omega) \ \text{ compact,}$$

for all smoothly bounded $\Omega \subset \mathbb{R}^d$, as long as $p+1$ satisfies (4.37), so

19 (4.47) $$u_\nu \longrightarrow u, \ \text{ in } L^{p+1}(\Omega) \text{ norm,}$$

for all such $\Omega \subset \mathbb{R}^d$. Fix a base poin as the origin, then replace u_ν by its radial decreasing rearrangement. By the rearrangement lemma, this replacement does not increase $\|\nabla u_\nu\|_{L^2}$. On the other hand, such a replacement clearly leaves $\|u_\nu\|_{L^2}$ fixed, hence lowers $F_\lambda(u_\nu)$. It also leaves $J_p(u_\nu)$ fixed. Thus we can assume $u_\nu(x) \ge 0$, that it is rotationally symmetric about 0, that it is monotone in $\text{dist}(x,0)$, and that (4.44), (4.45) and (4.47) hold. We need to show that

20 (4.48) $$J_p(u) = \beta, \quad \text{i.e., } \ \|u\|_{L^{p+1}} = \beta^{1/(p+1)}.$$

Clearly $J_p(u) \le \beta$ and $F_\lambda(u) \le I_\beta$. Given (4.48), it would follow from (4.43) and (4.45) that

21 (4.49) $$F_\lambda(u) = I_\beta,$$

and also that H^1-norm convergence holds in (4.45).

To demonstrate (4.48), let us set $\|u\|_{H^1} = \|\nabla u\|_{L^2} + \|u\|_{L^2}$,

$$H^1_r = \text{ set of radially symmetric functions in } H^1,$$

$M_R = M \setminus B(0,R)$, and

22 (4.50) $$J_R v = v\Big|_{M_R}.$$

We assert the following.

la4.2 **LEMMA 4.1.1** *Given $q > 2$, we have*

23 (4.51) $$\lim_{R\to\infty} \|J_R\|_{\mathscr{L}(H^1_r, L^q)} = 0.$$

Given this lemma, we have for the radial sequence (u_ν) satisfying (4.44) that, for each $\epsilon > 0$, there exists $R < \infty$ such that

24 (4.52) $$\int_{M_R} |u_\nu|^{p+1}\, dV \le \epsilon, \quad \forall\, \nu,$$

and then (4.48) follows from (4.47).

It remains to prove Lemmma 4.1.1. If we show that

25 (4.53) $$\lim_{R\to\infty} \|J_R\|_{\mathscr{L}(H^1_r, L^\infty)} = 0,$$

then, since for $q > 2$

26 (4.54)
$$\begin{aligned}\int_{M_R} |v|^q\,dx &\le \|v\|_{L^\infty(M_R)}^{q-2}\int_{M_R}|v|^2\,dx\\ &\le \|J_R v\|_{L^\infty}^{q-2}\|v\|_{H^1}^2,\end{aligned}$$

we have (4.51).

It remains to prove (4.53). Here is one approach. We can replace R by $R+1$. Take $\chi_R \in \mathrm{Lip}(M)$, $\chi_R(x) = 0$ for $x \in B(0,R)$, $\chi_R(x) = \mathrm{dist}(x, B_R(o))$ for $x \in B_{R+1}(o)$, $\chi_R(x) = 1$ for $x \in M_{R+1}$. Then, for $v \in H^1_r$, we have

(4.55)
$$\chi_R v \in H^1_{0,r}(M_R) = H^1_0(M_R) \cap H^1_r,$$

and

27 (4.56)
$$\|\nabla(\chi_R v)\|_{L^2} \le \|v\|_{H^1}.$$

Hence, in all cases except $\mathbb{R}^2$, (4.53) is a consequence of the following.

la4.3 **LEMMA 4.1.2** *Except for $\mathbb{R}^2$, we have*

28 (4.57)
$$\|v\|_{L^\infty} \le \eta(R)\|\nabla v\|_{L^2}, \quad \forall\, v \in H^1_{0,r}(M_R),$$

with

29 (4.58)
$$\lim_{R\to\infty}\eta(R) = 0.$$

PROOF. Take $v \in H^1_{0,r}(M_R)$. Slightly abusing notation, we write $v(x) = v(r)$. Then

30 (4.59)
$$\|\nabla v\|_{L^2}^2 = \int_R^\infty |v'(r)|^2 A(r)\,dr,$$

where

31 (4.60)
$$A(r) = (n-1)\text{-dimensional area of } \{x \in \mathbb{R}^d : \mathrm{dist}(x,0) = r\}.$$

Now

32 (4.61)
$$\begin{aligned}\|v\|_{L^\infty} &\le \int_R^\infty |v'(r)|\,dr\\ &= \int_r^\infty |v'(r)|A(r)^{1/2}A(r)^{-1/2}\,dr\\ &\le \eta(R)\|\nabla v\|_{L^2},\end{aligned}$$

by Cauchy's inequality, where

33 (4.62)
$$\eta(R) = \left(\int_R^\infty \frac{dr}{A(r)}\right)^{1/2}.$$

This gives (4.57), except in the case of $\mathbb{R}^2$. □

Finally, the case of $\mathbb{R}^2$ in (4.53) follows from the next result, given in [**BL83**] as Radial Lemma A.II, which in turn follows [**Str05**].

la4.4 **LEMMA 4.1.3** *On $\mathbb{R}^n$, $n \ge 2$, then, for $R \ge 1$,*

34 (4.63)
$$\sup_{|x|=R}|v(x)| \le C_n R^{-(n-1)/2}\|v\|_{H^1}, \quad \forall\, v \in H^1_{0,r}(M_1).$$

4.2. Exercises

CHAPTER 5

Bielefeld Lecture 5

Much of the following can be found applied to very general geometric settings in [**CMMT12**].

5.1. Symmetrization approach to energy minimizers

sa5

We fix $\beta > 0$, and pick $u_\nu \in H^1$ such that

C.0A (5.1) $$Q(u_\nu) = \beta, \quad E(u_\nu) \le \mathscr{I}_\beta + \frac{1}{\nu},$$

with

(5.2) $$Q(u) = \|u\|_{L^2}^2, \quad E(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1},$$
$$\mathscr{I}_\beta = \inf\{E(u) : u \in H^1, Q(u) = \beta\}.$$

As seen in §4.1, this leads to bounds

C.1 (5.3) $$\|u_\nu\|_{H^1},\ \|u_\nu\|_{L^{p+1}} \le K < \infty.$$

To proceed, fix the origin as a base point. A symmetric, decreasing replacement leaves $Q(u_\nu)$ invariant. Thus we can assume our minimizing sequence (u_ν) consists of such radial decreasing functions. Passing to a subsequence, we have

C.2 (5.4) $$u_\nu \longrightarrow u, \quad \text{weak}^* \ \text{in} \ H^1,$$

hence weak* in L^2 and in L^{p+1}. The limit u is radial and decreasing. The next result provides valuable information about $E(u)$.

pc.2 **PROPOSITION 5.1** *For such a sequence (u_ν), we have*

C.3 (5.5) $$u_\nu \longrightarrow u \ \textit{in} \ L^{p+1}\textit{-norm.}$$

PROOF. As long as $p+1 < 2n/(n-2)$, Rellich's theorem gives

C.4 (5.6) $$H^1(B(0,R)) \hookrightarrow L^{p+1}(B(0,R)) \ \text{compact},$$

for each $R < \infty$, where $B(0,R)$ is the ball of radius R centered at o. Hence we have

C.5 (5.7) $$u_\nu \longrightarrow u \ \text{in} \ L^{p+1}(B(0,R))\text{-norm}, \quad \forall\, R < \infty.$$

To proceed, denote by H^1_r the space of radially symmetric functions in H^1. Set

C.6 (5.8) $$M_R = M \setminus B(0,R), \quad J_R v = v\Big|_{M_R}.$$

The following complement to (5.7) follows from Lemma 4.1.1. Namely, given $q > 2$, we have, for $v \in H^1_r$,

C.7 (5.9) $$\|J_R v\|_{L^q} \le \delta_q(R)\|v\|_{H^1_r}, \quad \delta_q(R) \to 0 \ \text{as} \ R \to \infty.$$

Consequently, we have for the radial sequence (u_ν) that, for each $\epsilon > 0$, there exists $R < \infty$ such that

C.8 (5.10) $$\int_{M_R} |u_\nu|^{p+1}\, dV \le \epsilon, \quad \forall\, \nu,$$

and then (5.5) follows from (5.7), proving Proposition 5.1. □

From (5.4) we have $\|\nabla u\|_{L^2} \le \liminf \|\nabla u_\nu\|_{L^2}$, and this together with (5.5) gives

C.9 (5.11) $$E(u) \le \liminf_{\nu\to\infty} E(u_\nu) = \mathscr{I}_\beta.$$

We'd like to know that

C.10 (5.12) $$Q(u) = \beta.$$

However, doing so as in the previous section requires $q > 2$, so it is not clear how to establish (5.12) directly.

5.2. Symmetrization approach to Weinstein functional maximization

sa4x

We note how the symmetrization procedure allows for a simplified proof of the existence of a maximum for the Weinstein functional $W(u)$, in the Euclidean space setting, $\mathbb{R}^n$ (for $n \ge 2$). Recall,

A.4X.0 (5.13) $$W(u) = \frac{\|u\|_{L^{p+1}}^{p+1}}{\|u\|_{L^2}^{\alpha}\|\nabla u\|_{L^2}^{\beta}},$$

with $\alpha = 2-(n-2)(p-1)/2$, $\beta = n(p-1)/2$. We keep the requirement (4.37) on p. The Gagliardo-Nirenberg estimate implies $W(u)$ is bounded from above. Denote its supremum by $W_{\max}$.

Now, if $u_\nu \in H^1(\mathbb{R}^n)$ and $W(u_\nu) \to W_{\max}$, then $W(u_\nu^*) \ge W(u_\nu)$ if u_ν^* is the radial decreasing rearrangement of u_ν, so we need only maximize $W(u)$ over $H^1_r(\mathbb{R}^n)$. For the next step, we follow the standard argument and use the fact that $W(u)$ is invariant under $u \mapsto au$ and $u(x) \mapsto u(bx)$ to impose the normalization

A.4X.1 (5.14) $$\|u_\nu\|_{L^2} = 1, \quad \|\nabla u_\nu\|_{L^2} = 1,$$

so

A.4X.2 (5.15) $$\|u_\nu\|_{L^{p+1}} \to W_{\max}^{1/(p+1)}.$$

If we pass to a subsequence such that $u_\nu \to u$ weak* in $H^1(\mathbb{R}^n)$, results from §A.4 yield $u_\nu \to u$ in norm in $L^{p+1}(\mathbb{R}^n)$. Also $\|u\|_{L^2} \le 1$ and $\|\nabla u\|_{L^2} \le 1$, so

A.4X.3 (5.16) $$W(u) \ge W_{\max}.$$

This requires $W(u) = W_{\max}$ (hence $\|u\|_{L^2} = \|\nabla u\|_{L^2} = 1$, and therefore $u_\nu \to u$ in norm in $H^1(\mathbb{R}^n)$.) We have the desired maximizer. A computation of

A.4X.4 (5.17) $$\frac{d}{d\tau} W(u+\tau v)\Big|_{\tau=0} = \frac{(N(u), v)}{\|u\|_{L^2}^{2\alpha}\|\nabla u\|_{L^2}^{2\beta}}$$

shows that such a maximizer u solves the equation

A.4X.5 (5.18) $$\Delta u - \lambda u + K u^p = 0,$$

with

A.4X.6 (5.19)
$$\lambda = \frac{\alpha}{\beta}\frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2}, \quad K = \frac{p+1}{\beta}\frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^{p+1}}^{p+1}},$$

hence, with the normalization imposed above,

A.4X.7 (5.20)
$$\lambda = \frac{\alpha}{\beta}, \quad K = \frac{p+1}{\beta W_{\max}}.$$

5.3. More on W_{max} via Scaling

We note here the role of W_{max} following [**Wei83**]. We take $p = 1+4/n$. Let us define

$$u_\lambda = \lambda^\alpha u(\lambda^{\frac{1}{2}}).$$

Then,

$$\int |u_\lambda|^2 dx = \lambda^{2\alpha - n/2}\int |u|^2 dx.$$

Hence, given $\alpha = n/4$, we have that

$$\|u_\lambda\|_{L^2} = \|u\|_{L^2}.$$

Note also, given a solution u to

$$-\Delta\psi + \psi - \psi^p = 0,$$

we have

$$-\Delta\tilde{u} + \lambda\tilde{u} - \tilde{u}^p = 0,$$

when $p = 1+4/n$ giving us a connection between ground states for all λ, provided one has uniqueness of such ground states. Let us diverge momentarily into the nonlinear Schrödinger equation

$$iu_t + \Delta u + |u|^{p-1}u = 0, \quad u(x,0) = u_0(x) \in H^1.$$

A quick calculation shows that

$$E(u(t)) = E(u(0)), \quad Q(u(t)) = Q(u(0))$$

where E, Q are as in our constrained minimization calculations.

Using the energy estimate (4.7), and plugging in the optimal constant provided by the bound state mass, we observe

$$\left(1 - \frac{\|u_0\}_{L^2}}{\|\psi\|_{L^2}}\right)^{\frac{4}{n}} \|\nabla u\|_{L^2}^2(t) \le E(u),$$

giving a priori control on the H^1 norm for all time!!

5.4. Exercises

CHAPTER 6

Bielefeld Lecture 6

Much of the following can be found applied to very general geometric settings in [**CMMT12**].

s2.3

6.1. Smoothness of minimizers

Let $\Omega \subset \mathbb{R}^n$ be an open set, and assume $u \in H^1_{\rm loc}(\Omega)$ solves

E:local-ell

$$-\Delta u + \lambda u = f(u), \quad f(u) = K|u|^{p-1}u, \tag{6.1}$$

with p as in (2.21).

P:local-reg-prop

PROPOSITION 6.1 *Every solution $u \in H^1_{loc}(\Omega)$ to* (6.1) *satisfies*

$$u \in C^{p+2}(\Omega) \tag{6.2}$$

if $p \notin \mathbb{N}$. If p is an odd integer, then $u \in C^\infty(\Omega)$. If p is an even integer, $u \in C^s(\Omega)$ for all $s < p+2$, and if p is an even integer and $u \geq 0$ on Ω then $u \in C^\infty(\Omega)$. Finally, for any p satisfying (2.21)*, if u is nowhere vanishing on Ω, then $u \in C^\infty(\Omega)$.*

PROOF. We have linear elliptic regularity: for $1 < q < \infty$ and $s \geq 0$,

E:local-ell-reg

$$\begin{aligned} -\Delta u + \lambda u = f \in H^{s,q}_{\rm loc}(\Omega) &\implies u \in H^{s+2,q}_{\rm loc}(\Omega), \\ -\Delta u + \lambda u = f \in C^s(\Omega) &\implies u \in C^{s+2}(\Omega) \text{ (if } s \notin \mathbb{Z}), \end{aligned} \tag{6.3}$$

see for instance [**Eva98**], Chapter 6. As in 2.1, we also have Sobolev embeddings, such as

E:loc-Sobolev

$$\begin{aligned} &H^{s,q}_{\rm loc}(\Omega) \subset L^{nq/(n-sq)}_{\rm loc}(\Omega), \quad 0 < s < \frac{n}{q}, \\ &H^{s+n/q,q}_{\rm loc}(\Omega) \subset C^s(\Omega), \quad 0 < s < 1. \end{aligned} \tag{6.4}$$

In order to get started, write

$$p = \frac{1}{\gamma}\frac{n+2}{n-2} \tag{6.5}$$

for some $\gamma > 1$. Then for $f(u)$ as in (6.1),

$$\begin{aligned} u \in H^1_{\rm loc}(\Omega) &\implies u \in L^{2n/(n-2)}_{\rm loc}(\Omega) \quad (\text{if } n \geq 3) \\ &\implies f(u) \in L^{2n\gamma/(n+2)}_{\rm loc}(\Omega), \end{aligned} \tag{6.6}$$

so that (6.3) yields

$$u \in H^{2,2n\gamma/(n+2)}_{\rm loc}(\Omega) \tag{6.7}$$

if $n \geq 3$. If $n = 2$, then $u \in H^{2,q}_{\rm loc}(\Omega)$ for all $q < \infty$. Observe

E:gamma-n
$$\frac{2n\gamma}{n+2} > \frac{n}{2} \Leftrightarrow \gamma > \frac{n+2}{4}. \tag{6.8}$$

If (6.8) holds, we have

E:u-in-Cs
$$u \in C^s(\Omega) \tag{6.9}$$

for some $s \in (0,1)$. In the endpoint case $\gamma = (n+2)/4$, we have

E:u-in-Cs
$$u \in L^q_{\rm loc}(\Omega), \quad \forall\, q < \infty, \tag{6.10}$$

and hence $f(u) \in L^q_{\rm loc}(\Omega)$ for all $q < \infty$ as well. Then by (6.3),

$$u \in H^{2,q}_{\rm loc}(\Omega), \quad \forall\, q < \infty, \tag{6.11}$$

and (6.9) holds in this case as well. If $\gamma < (n+2)/4$, we use the Sobolev embeddings (6.4) to get

$$u \in L^{2n\gamma/(n+2-4\gamma)}_{\rm loc}(\Omega), \tag{6.12}$$

and hence

$$f(u) \in L^{2n\gamma_2/(n+2)}_{\rm loc}(\Omega), \tag{6.13}$$

where

$$\gamma_2 = \gamma^2 \frac{n-2}{n+2-4\gamma} > \gamma^2. \tag{6.14}$$

Inserting this improved regularity for $f(u)$ into the elliptic regularity estimates (6.3) yields now

$$u \in H^{2,2n\gamma_2/(n+2)}_{\rm loc}(\Omega). \tag{6.15}$$

A finite number of iterations, say L, of this procedure yields the property (6.9). This in turn implies $f(u) \in C^s(\Omega)$, hence $u \in C^{s+2}(\Omega)$ for some $s \in (0,1)$.

From this, the conclusions of Proposition 6.1 follow, once we observe that if $p > 1$ is not an odd integer, then one cannot improve the implication

$$u \in C^s(\Omega),\ s \geq p, \implies f(u) \in C^p(\Omega), \tag{6.16}$$

except when p is an even integer and u does not change sign, while if p is an odd integer, we get $f(u) \in C^s(\Omega)$.

□

We now want global estimates for functions $u \in H^1$ satisfying (6.1) on all of $\mathbb{R}^d$. We always have

$$(\lambda - \Delta)^{-1} : H^{s,q} \to H^{s+2,q} \tag{6.17}$$

whenever $\lambda > 0$ and $q = 2$, however when $q \neq 2$ one often needs a stronger bound on λ. Hence we will take a different approach, which will also yield some decay estimates on solutions.

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $\widetilde{\Omega} \Subset \Omega$. Assume $\widetilde{\Omega}$ contains a ball of radius 1. Then we can choose translations T_j such that, if we set $\widetilde{\Omega}_j = T_j(\widetilde{\Omega})$, then the countable collection $\{\widetilde{\Omega}_j\}$ covers $\mathbb{R}^d$, and we can assume that there exists $m < \infty$ such that each point $x \in \mathbb{R}^d$ is in at most m of the $\widetilde{\Omega}_j$.

Now depending on p satisfying (2.21), let L be the number of iterations required in the proof of Proposition 6.1. Choose intermediate nested open sets:

2.3.18 (6.18) $$\widetilde{\Omega} \Subset \Omega^{(L)} \Subset \cdots \Subset \Omega^{(1)} \Subset \Omega,$$

along with the associated translates

2.3.19 (6.19) $$\Omega_j^{(l)} = T_j(\Omega^{(l)}).$$

We set

E:Aj (6.20) $$A_j = \|u\|_{H^1(\Omega_j)},$$

so that

E:H1-norm-equiv (6.21) $$\|u\|_{H^1}^2 \simeq \sum_j A_j^2.$$

Applying the proof of Proposition 6.1 on Ω and its translates by isometries yields a similar statement on each Ω_j with constants independent of j. From (6.20), we have

(6.22) $$\|u\|_{L^{2n/(n-2)}(\Omega_j)} \le C_1 A_j$$

with C_1 independent of j. As usual, if $n = 2$, this holds for $L^q(\Omega_j)$, $q < \infty$. Hence

(6.23) $$\begin{aligned} \|f(u)\|_{L^{2n\gamma/(n+2)}(\Omega_j)} &\le C_2 \|u^p\|_{L^{2n/(n-2)p}(\Omega_j)} \\ &\le C_2 C_1^p A_j^p. \end{aligned}$$

The local elliptic regularity then gives

E:C4 (6.24) $$\|u\|_{H^{2,2n\gamma/(n+2)}(\Omega_j^{(1)})} \le C_3(C_2 C_1^p A_j^p + C_1 A_j) \le C_4 A_j.$$

Iterating this argument L times, we obtain

E:local-reg-j (6.25) $$\|u\|_{C^s(\widetilde{\Omega}_j)} \le C_\star A_j,$$

for a constant $C_\star$ independent of j, and where s satisfies similar properties to that in Proposition 6.1. We record the result in the following Proposition.

P:local-reg-j **PROPOSITION 6.2** *If $u \in H^1$ is a solution to* (6.1)*, then* (6.25) *holds with A_j given by* (6.20)*, $C_\star$ independent of j, and $s = 2 + p$ if $p \notin 2\mathbb{N}$, and for every $s < 2 + p$ if $p \in 2\mathbb{N}$.*

Remark 6.1.1
Observe that the second inequality in (6.24) *uses that $A_j \le \|u\|_{H^1}$ for each j. Hence $C_\star$ is independent of j, but depends on $\|u\|_{H^1}$ in a nonlinear fashion. In light of* (6.21)*, we conclude*

E:local-reg-sum (6.26) $$\sum_j \|u\|_{C^s(\widetilde{\Omega}_j)}^2 \le C\left(\|u\|_{H^1}\right)\|u\|_{H^1}^2.$$

s2.4

6.1.1. Positivity of Minimizers. In this subsection we examine the question of positivity of minimizers. If $u \in H^1$ is a minimizer of F_λ, subject to the constraint $J_p(u) = \beta$, set $v(x) = |u(x)|$. Of course we have

$$\|v\|_{L^2} = \|u\|_{L^2}, \quad \|\nabla v\|_{L^2} = \|\nabla u\|_{L^2}, \quad \|v\|_{L^{p+1}} = \|u\|_{L^{p+1}}, \tag{6.27}$$

so v is a solution to the same constrained minimization problem. But then

2.4.2

$$v \geq 0, v \in H^1, \quad -\Delta v + \lambda v = K_0 |v|^{p-1} v, \tag{6.28}$$

with $K_0 = I_\beta/\beta$ as in (2.29). Then Proposition 6.1 implies $v \in C^{2+p}$, and $v \in C^\infty$ if p is an integer. We improve this in the next Proposition.

P:posip2v4ty

PROPOSITION 6.3 *The function $v = |u(x)|$ satisfies*

$$v(x) > 0 \tag{6.29}$$

for all $x \in M$, and hence $v \in C^\infty$.

This result is a consequence of the following Harnack inequality, which follows from Theorem 8.20 – Corollary 8.21 of [**GT01**]. If $v \geq 0$ solves (6.28), and if $\Omega_j^{(l)}$ are as in (6.18)–(6.19), then (in light of the bounds on $|v|^{p-1}$ established in §6.1) there exists a constant C_0, independent of j, such that

2.4.4

$$\sup_{x \in \Omega_j^{(1)}} u(x) \leq C_0 \inf_{x \in \Omega_j^{(1)}} u(x), \tag{6.30}$$

Given (6.30), if $v \geq 0$ solves (6.28) and is not $\equiv 0$, strict positivity is immediate, and smoothness follows from Proposition 6.1, so Proposition 6.3 is proven. This in turn immediately gives the following.

COROLLARY 2
Every real-valued F_λ-minimizer u satisfies either $u > 0$ or $u < 0$.

We now wish to extend the global regularity estimates on u beyond $s = 2 + p$, when $u > 0$. The issue is that although $u > 0$, u must decay at infinity, and since $f(u)$ is singular at $u = 0$, there is some work to be done. Again the Harnack inequality (6.30) (with u in place of v) provides the key to success. With this, we can establish the following improvement of Proposition 6.2.

p2.4.4

PROPOSITION 6.4 *If $u \in H^1$ solves* (6.1) *and $u > 0$, then* (6.25) *and* (6.26) *hold for every $s < \infty$ with constants depending on s but not on j.*

PROOF. It suffices to prove the statment for $s \in \mathbb{N}$. Proposition 6.2 implies that the assertions are true when $s = 3$, and we proceed by induction. Let $k \in \mathbb{N}$ and suppose (6.25) holds for $s = k$. Then the covering property of the $\{\widetilde{\Omega}_j\}$ implies

$$\|u\|_{C^k(\Omega_j)} \leq C_k A_j \tag{6.31}$$

for some C_k independent of j. We want to show

E:u-in-Ck1

$$\|u\|_{C^{k+1}(\widetilde{\Omega}_j)} \leq C_{k+1} A_j \tag{6.32}$$

for some C_{k+1} independent of j.

We need to estimate the C^k norm of $f(u)$. The chain rule applied to $f(u)$ gives

E:u-in-Ck1 (6.33) $$D^\alpha f(u) = \sum_{\gamma_1+\gamma_2\cdots+\gamma_\nu=\alpha,\nu,|\gamma_\mu|\geq 1} C_\gamma u^{(\gamma_1)}\cdots u^{(\gamma_\nu)} f^{(\nu)}(u).$$

Now

(6.34) $$|f^{(\nu)}(u)| \leq C|u|^{p-\nu},$$

and from the Harnack inequality and our induction hypothesis, there exists C_1, C_2 such that

(6.35) $$C_1 A_j \leq u \leq C_2 A_j \text{ on } \Omega_j^{(1)},$$

so that

(6.36) $$|f^{(\nu)}| \leq C A_j^{p-\nu}.$$

Hence for $|\alpha| \leq k$ (so that, in particular, $\nu, |\gamma_\mu| \leq k$), we have

2.4.13 (6.37) $$\begin{aligned} |D^\alpha f(u)| &\leq C\sum_\gamma A_j^\nu A_j^{p-\nu} \\ &\leq C' A_j^p \\ &\leq C'' A_j, \end{aligned}$$

where again C'' is allowed to depend nonlinearly on the quantity $\|u\|_{H^1}$. The last inequality in (6.37) uses the global bound $A_j \leq \|u\|_{H^1}$. Hence

(6.38) $$\|f(u)\|_{C^k(\Omega_j^{(1)})} \leq C A_j.$$

Then the local elliptic regularity applied to (6.1) yields (6.32), completing the proof.

□

6.1.2. Exponential Decay in the case of Symmetric Minimizers. Using the properties of regularity and integrability we have now computed, let us write the bound state now solving the radial ODE

$$u_{rr} + \frac{d-1}{r} u_r - \lambda u + u^p = 0, u'(0) = 0, \lim_{r\to\infty} u = 0.$$

Setting $v = r^{(d-1)/2} u$ gives

$$v_{rr} = (q(u) + \beta/r^2) v,$$

where $q(u) = \lambda - u^{p-1}$ and $\beta = (d-1)(d-3)/4$. Note, this transformation "blows down geometrically at $r = 0$ and is related to the Emden-Fowler transformation to be discussed later. Then, since u is monotonically decreasing as a symmetrized minimizer and $u \in H^1$, we have for $r \geq r_0$ sufficiently large that

$$q(r) + \beta/r^2 \geq \lambda/2.$$

Setting $w = v^2 \geq 0$, we have

$$\frac{1}{2} w_{rr} = v_r^2 + (q(u) + \beta/r^2) w,$$

giving $w_{rr} \geq \lambda w$ for $r \geq r_0$. Given $z = e^{-\sqrt{\lambda} r}(w_r + \sqrt{\lambda} w)$, we then have

$$z_r \geq 0$$

for $r \geq r_0$. Hence, z is non-decreasing. Should there exist $r_1 > r_0$ such that $z(r_1) > 0$, we observe also that

$$\int_{r_1}^{\infty} (w_r + \lambda w) > \infty.$$

This is a contradiction to the fact that $u \in H^1_r$ implies $w \in W^{1,1}$. Hence,

$$\left(e^{\sqrt{\lambda}r} w\right)_r \leq 0,$$

giving $w \leq Ce^{-\sqrt{\lambda}r}$ hence

$$|u| \leq r^{-(d-1)/2} e^{-\sqrt{\lambda}/2r}.$$

Integrating $(r^{d-1}u_r)_r = r^{d-1}(\lambda u - u^p)$ gives that u_r decays exponentially, hence plugging into the equation gives u_{rr} decays exponentially.

s2.5

6.1.3. Further decay estimates. The following decay estimates are an observation of Michael Taylor. We continue to study properties of a positive solution $u \in H^1$ to the elliptic equation

$$-\Delta u + \lambda u = f(u), \tag{6.39}$$

where

$$f(u) = K|u|^{p-1}u. \tag{6.40}$$

We also continue to assume this equation holds on a manifold M that is weakly homogeneous in the sense described in the previous sections. So far we have shown that $u \in L^q$ for every $q \in [2,\infty]$, and for each $s < \infty$,

2.5.2A

$$\|u\|_{C^s(\widetilde{\Omega}_j)} \leq C_s A_j, \quad \forall\, j, \tag{6.41}$$

where $\{\widetilde{\Omega}_j\}$ is an open cover of M by sets which are images under isometries of a fixed set $\widetilde{\Omega}$ and $A_j = \|u\|_{H^1(\Omega_j)}$, so

2.5.2B

$$\sum_j A_j^2 \approx \|u\|^2_{H^1}. \tag{6.42}$$

These are varieties of decay results. In this section we seek stronger decay results. Here, we replace the hypothesis $\lambda > -\delta_0$ by

$$\lambda > 0, \tag{6.43}$$

which for $\delta_0 = 0$ involves no change.

Since $\{e^{t\Delta} : t \geq 0\}$ is a contraction semigroup on L^q for each $q \geq 1$, we have

$$(-\Delta + \lambda)^{-1} = \int_0^{\infty} e^{-\lambda t} e^{t\Delta} dt, \tag{6.44}$$

which implies

$$(-\Delta + \lambda)^{-1} : L^q \to L^q \tag{6.45}$$

for every $q \in [1,\infty]$, with operator norm bounded by λ^{-1}. Our previously estabished L^q estimates on u imply

$$f(u) \in L^q, \quad \forall\, q \in [1,\infty]. \tag{6.46}$$

Since $u = (-\Delta + \lambda)^{-1} f(u)$, we hence have $u \in L^q$ for every $q \in [1,\infty]$.

Now set

E:Bj-def

$$B_j = \|u\|_{L^1(\Omega_j)} \tag{6.47}$$

so that

E:Bj-def (6.48) $$\sum_j B_j \simeq \|u\|_{L^1}.$$

Comparing to (6.42), we see the collection $\{B_j\}$ satisfy "stricter bounds" than $\{A_j\}$. In this sense, the following result improves (6.41).

p2.5.1 **PROPOSITION 6.5** *For each $\epsilon \in (0,1)$, and each $s < \infty$, there exists $C_{\epsilon,s} < \infty$ such that*

(6.49) $$\|u\|_{C^s(\tilde{\Omega}_j)} \leq C_{\epsilon,s} A_j^{\epsilon} B_j^{1-\epsilon}, \quad \forall\, j.$$

PROOF. To start, (6.41) implies

(6.50) $$\|u\|_{L^\infty(\tilde{\Omega}_j)} \leq C A_j$$

for every j, with constants independent of j. Interpolating with (6.47), for each $\epsilon > 0$ we can produce $q > 1$ such that

E:u-first-int (6.51) $$\|u\|_{L^q(\tilde{\Omega}_j)} \leq C A_j^{\epsilon/2} B_j^{1-\epsilon/2},$$

for every j, with constants indepenent of j. Next, (6.41) implies that for each $k < \infty$ there exists a constant $C_k < \infty$ such that

E:u-first-int (6.52) $$\|u\|_{H^{k,q}(\tilde{\Omega}_j)} \leq C_k A_j,$$

for every j, where again C_k is independent of j. Then interpolation with (6.51) implies that, for each $\epsilon \in (0,1)$, $\sigma < \infty$, there exists $C_{\epsilon,\sigma}$ such that

(6.53) $$\|u\|_{H^{\sigma,q}(\tilde{\Omega}_j)} \leq C_{\epsilon,\sigma} A_j^{\epsilon} B_j^{1-\epsilon},$$

where again the constants are independent of j. Taking $\sigma > 0$ sufficiently large proves the Proposition.

□

s3.4 **6.1.4. Smoothness, positivity, and decay of energy minimizers.** If $u \in H^1$ minimizes (4.2), subject to the constraint (4.3), so does $v = |u|$, so v solves

(6.54) $$v \in H^1,\ v \geq 0, \quad -\Delta v + \lambda v - |v|^{p-1} v = 0,$$

for some $\lambda \in \mathbb{R}$. Boundedness (and decay) results of §6.1 hold. Then, as in §6.1.1, the Harnack inequality (6.30) implies $v > 0$, and we get:

p3.4.1 **PROPOSITION 6.6** *Every real-valued energy minimizer u satisfies either $u > 0$ on M or $u < 0$ on M, and belongs to $\mathscr{C}^\infty$.*

Given this, the decay results Proposition 6.4 and Proposition 6.5 apply to these energy minimizers.

6.2. Exercises

CHAPTER 7

Bielefeld Lecture 7

Much of the following can be found applied to very general geometric settings in [**CMMT12**].

7.1. Second variation of energy

s3.5

With $\beta \in (0,\infty)$, let

3.5.1 (7.1) $$X = \{u \in H^1 : Q(u) = \beta\}, \quad \mathscr{I}_\beta = \inf\{E(u) : u \in X\},$$

and

3.5.2 (7.2) $$Y = \{u \in X : E(u) = \mathscr{I}_\beta\}.$$

Conditions guaranteeing that Y is nonempty have been given in §4.1.2. Recall that $E(u)$ and $Q(u)$ are given by (4.2)–(4.3). Here we study

3.5.4 (7.3) $$\frac{d^2}{ds^2}E(w(s)),$$

when $w(s)$ is a smooth path in X satisfying $w(0) = u \in Y$. To be definite, take $u \in Y$,

3.5.6 (7.4) $$\psi \in T_u X = \{\psi \in H^1 : \mathfrak{Re}(u,\psi) = 0\},$$

and set

3.5.7 (7.5) $$w(s) = a\,\frac{u + s\psi}{\|u + s\psi\|}, \quad a = \beta^{1/2}.$$

In light of the discussion in §6.1.4, we can assume

3.5.8 (7.6) $$u > 0 \text{ on } M,$$

but we cannot assume ψ is real valued. Set

3.5.9 (7.7) $$\psi = \psi_0 + i\psi_1, \quad \psi_0, \psi_1 \text{ real valued.}$$

Then the condition (7.4) is equivalent to

3.5.10 (7.8) $$(u, \psi_0) = 0,$$

with no constraint on ψ_1.

The chain rule gives

3.5.11 (7.9) $$\frac{d}{ds}E(w(s)) = DE(w(s))\,w'(s),$$

and in particular

3.5.12 (7.10) $$\frac{d}{ds}E(w(s))\Big|_{s=0} = DE(u)\,w'(0).$$

Differentiating (7.9) gives

3.5.13 (7.11) $$\frac{d^2}{ds^2}E(w(s)) = D^2E(w(s))(w'(s), w'(s)) + DE(w(s))\,w''(s),$$

and in particular

3.5.14 (7.12) $$\frac{d^2}{ds^2}E(w(s))\Big|_{s=0} = D^2E(u)(w'(0),w'(0)) + DE(u)w''(0).$$

We turn to the computation of $w'(0)$ and $w''(0)$. Rewrite (7.5) as

3.5.15 (7.13) $$\begin{aligned} w(s) &= aF(s)(u+s\psi),\\ F(s) &= \|u+s\psi\|^{-1} = (a^2+s^2\|\psi\|^2)^{-1/2}. \end{aligned}$$

Then

3.5.16 (7.14) $$\begin{aligned} F'(s) &= -s(a^2+s^2\|\psi\|^2)^{-3/2}\|\psi\|^2,\\ F''(s) &= -(a^2+s^2\|\psi\|^2)^{-3/2}\|\psi\|^2 - s\|\psi\|^2\,\frac{d}{ds}(a^2+s^2\|\psi\|^2)^{-3/2}, \end{aligned}$$

so

3.5.17 (7.15) $$F'(0) = 0, \quad F''(0) = -a^{-3}\|\psi\|^2.$$

We have

3.5.18 (7.16) $$\begin{aligned} w'(s) &= aF(s)\psi + aF'(s)(u+s\psi), \quad \text{so}\\ w'(0) &= aF(0)\psi = \psi, \end{aligned}$$

and

3.5.20 (7.17) $$\begin{aligned} w''(s) &= 2aF'(s)\psi + aF''(s)u, \quad \text{so}\\ w''(0) &= aF''(0)u = -\frac{\|\psi\|^2}{a^2}u. \end{aligned}$$

Thus (7.10) and (7.12) become

3.5.22 (7.18) $$\frac{d}{ds}E(w(s))\Big|_{s=0} = DE(u)\psi,$$

and

3.5.23 (7.19) $$\frac{d^2}{ds^2}E(w(s))\Big|_{s=0} = D^2E(u)(\psi,\psi) - \frac{1}{a^2}\|\psi\|^2 DE(u)u.$$

Also

3.5.24 (7.20) $$DE(u)\psi = \frac{d}{ds}E(u+s\psi)\Big|_{s=0},$$

and

3.5.25 (7.21) $$D^2E(u)(\psi,\psi) = \frac{d^2}{ds^2}E(u+s\psi)\Big|_{s=0}.$$

Our next task is to compute the right sides of (7.20) and (7.21). It is convenient to set

3.5.25A (7.22) $$E(u) = T(u) - \tilde{J}(u),$$

with

3.5.25B (7.23) $$T(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2, \quad \tilde{J}(u) = \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}.$$

First, the calculation

3.5.26 (7.24) $$\begin{aligned} T(u+s\psi) &= \frac{1}{2}\|\nabla u + s\nabla\psi\|^2\\ &= \frac{1}{2}\|\nabla u\|^2 + s\mathfrak{Re}(\nabla u,\nabla\psi) + \frac{s^2}{2}\|\nabla\psi\|^2 \end{aligned}$$

gives

3.5.27 (7.25) $$\frac{d}{ds}T(u+s\psi)\Big|_{s=0} = \mathfrak{Re}(\nabla u, \nabla\psi) = \mathfrak{Re}(-\Delta u, \psi),$$

and

3.5.28 (7.26) $$\frac{d^2}{ds^2}T(u+s\psi)\Big|_{s=0} = \|\nabla\psi\|^2 = \|\nabla\psi_0\|^2 + \|\nabla\psi_1\|^2.$$

Next, write

3.5.29 (7.27) $$\tilde{J}(u+s\psi) = \frac{1}{p+1}\int (u+s\psi)^{(p+1)/2}(\overline{u}+s\overline{\psi})^{(p+1)/2}\,dx.$$

Then

3.5.30 (7.28) $$\begin{aligned}\frac{d}{ds}\tilde{J}(u+s\psi) = \frac{1}{2}\int\Big[&(u+s\psi)^{(p-1)/2}(\overline{u}+s\overline{\psi})^{(p+1)/2}\psi\\ &+(u+s\psi)^{(p+1)/2}(\overline{u}+s\overline{\psi})^{(p-1)/2}\overline{\psi}\Big]\,dx,\end{aligned}$$

and

3.5.31 (7.29) $$\begin{aligned}\frac{d^2}{ds^2}\tilde{J}(u+s\psi) = \frac{1}{2}\int\Big[&\frac{p+1}{2}(u+s\psi)^{(p-1)/2}(\overline{u}+s\overline{\psi})^{(p-1)/2}\psi\overline{\psi}\\ &+\frac{p-1}{2}(u+s\psi)^{(p-3)/2}(\overline{u}+s\overline{\psi})^{(p+1)/2}\psi^2\\ &+\frac{p+1}{2}(u+s\psi)^{(p-1)/2}(\overline{u}+s\overline{\psi})^{(p-1)/2}\psi\overline{\psi}\\ &+\frac{p-1}{2}(u+s\psi)^{(p+1)/2}(\overline{u}+s\overline{\psi})^{(p-3)/2}\overline{\psi}^2\Big]\,dx.\end{aligned}$$

In particular,

3.5.32 (7.30) $$\begin{aligned}\frac{d}{ds}\tilde{J}(u+s\psi)\Big|_{s=0} &= \frac{1}{2}\int |u|^{p-1}(\overline{u}\psi + u\overline{\psi})dx\\ &= \mathfrak{Re}(|u|^{p-1}u, \psi).\end{aligned}$$

Before evaluating (7.29) at $s=0$, let us record that (7.25) and (7.30) imply

3.5.33 (7.31) $$\frac{d}{ds}E(u+s\psi)\Big|_{s=0} = \mathfrak{Re}(-\Delta u - |u|^{p-1}u, \psi).$$

(This calculation does not use (7.6).) For $u\in Y$, i.e., a minimizer of $E|_X$, this must vanish for all $\psi\in T_uX$, described by (7.4). Consequently, given $u\in Y$,

3.5.34 (7.32) $$\psi\in H^1,\ \mathfrak{Re}(u,\psi)=0 \Longrightarrow \mathfrak{Re}(\Delta u + |u|^{p-1}u, \psi) = 0.$$

It follows that there exists $\lambda\in\mathbb{R}$ such that

3.5.35 (7.33) $$\Delta u + |u|^{p-1}u = \lambda u,$$

and we recover (2.34)–(2.35).

We also note that the last term in (7.19) is

3.5.36 (7.34) $$\begin{aligned}&\frac{1}{a^2}\|\psi\|^2\frac{d}{ds}E(u+su)\Big|_{s=0}\\ &\quad= \frac{1}{a^2}\|\psi\|^2\,\mathfrak{Re}(-\Delta u - |u|^{p-1}u, u)\\ &\quad= \frac{1}{a^2}\|\psi\|^2(-\lambda u, u)\\ &\quad= -\lambda\|\psi\|^2,\end{aligned}$$

the second identity by (7.33).

We now evaluate (7.29) at $s = 0$. For this, we will use (7.6), and write $\psi = \psi_0 + i\psi_1$, as in (7.7). We have

$$\begin{aligned}\frac{d^2}{ds^2}\tilde{J}(u+s\psi)\Big|_{s=0} &= \frac{1}{2}\int\Big[(p+1)|u|^{p-1}|\psi|^2 \\ &\quad + \frac{p-1}{2}u^{(p-3)/2}\overline{u}^{(p+1)/2}\psi^2 \\ &\quad + \frac{p-1}{2}u^{(p+1)/2}\overline{u}^{(p-3)/2}\overline{\psi}^2\Big]\,dx \\ &= \frac{1}{2}((p+1)|u|^{p-1}\psi,\psi) + \frac{p-1}{2}\mathfrak{Re}(|u|^{p-3}u^2\overline{\psi},\psi) \\ &= \frac{p+1}{2}(u^{p-1}\psi,\psi) + \frac{p-1}{2}\mathfrak{Re}(u^{p-1}\overline{\psi},\psi),\end{aligned} \tag{7.35}$$

the last identity by (7.6). Now

$$(u^{p-1}\psi,\psi) = (u^{p-1}\psi_0,\psi_0) + (u^{p-1}\psi_1,\psi_1), \tag{7.36}$$

and

$$\begin{aligned}\mathfrak{Re}(u^{p-1}\overline{\psi},\psi) &= \mathfrak{Re}\int_M u^{p-1}(\psi_0 - i\psi_1)^2\,dx \\ &= (u^{p-1}\psi_0,\psi_0) - (u^{p-1}\psi_1,\psi_1),\end{aligned} \tag{7.37}$$

so

$$\begin{aligned}\frac{d^2}{ds^2}\tilde{J}(u+s\psi)\Big|_{s=0} &= \frac{p+1}{2}(u^{p-1}\psi_0,\psi_0) + \frac{p-1}{2}(u^{p-1}\psi_0,\psi_0) \\ &\quad + \frac{p+1}{2}(u^{p-1}\psi_1,\psi_1) - \frac{p-1}{2}(u^{p-1}\psi_1,\psi_1) \\ &= p(u^{p-1}\psi_0,\psi_0) + (u^{p-1}\psi_1,\psi_1).\end{aligned} \tag{7.38}$$

Together with (7.26), this gives

$$\begin{aligned}\frac{d^2}{ds^2}E(u+s\psi)\Big|_{s=0} &= (-\Delta\psi_0 - pu^{p-1}\psi_0,\psi_0) \\ &\quad + (-\Delta\psi_1 - u^{p-1}\psi_1,\psi_1).\end{aligned} \tag{7.39}$$

This, together with (7.19) and (7.34), yields

$$\begin{aligned}\frac{d^2}{ds^2}E(w(s))\Big|_{s=0} &= ((-\Delta - pu^{p-1})\psi_0,\psi_0) + \lambda(\psi_0,\psi_0) \\ &\quad + ((-\Delta - u^{p-1})\psi_1,\psi_1) + \lambda(\psi_1,\psi_1),\end{aligned} \tag{7.40}$$

when $w(s)$ is given by (7.5). In other words,

$$\frac{d^2}{ds^2}E(w(s))\Big|_{s=0} = (L_+\psi_0,\psi_0) + (L_-\psi_1,\psi_1), \tag{7.41}$$

with $L_\pm : H^1 \to H^{-1}$ given by

$$\begin{aligned}L_+\psi_0 &= (-\Delta + \lambda - p|u|^{p-1})\psi_0, \\ L_-\psi_1 &= (-\Delta + \lambda - |u|^{p-1})\psi_1.\end{aligned} \tag{7.42}$$

The Friedrichs method defines L_+ and L_- as self-adjoint operators on L^2, with domain H^2.

s3.a

7.1.1. Second variation of F_λ. In this setting, we take p as in (2.21), J_p as in (2.20), and F_λ as in (2.19). With $\beta \in (0,\infty)$, set

3.A.1 (7.43) $$\widetilde{X} = \{u \in H^1 : J_p(u) = \beta\}, \quad I_\beta = \inf\{F_\lambda(u) : u \in \widetilde{X}\},$$

and

3.A.2 (7.44) $$\widetilde{Y} = \{u \in \widetilde{X} : F_\lambda(u) = I_\beta\}.$$

Conditions guaranteeing that $\widetilde{Y}$ is nonempty have been given in §3.1. Here we compute

3.A.3 (7.45) $$\frac{d^2}{ds^2} F_\lambda(w(s)),$$

where $w(s)$ is a smooth path in $\widetilde{X}$ satisfying $w(0) = u \in \widetilde{Y}$. To be definite, take $u \in \widetilde{Y}$ (we can and will assume $u > 0$), take

3.A.4 (7.46) $$\psi \in T_u\widetilde{X} = \{\psi \in H^1 : \mathrm{Re}\,(u^p, \psi) = 0\},$$

and set

3.A.5 (7.47) $$w(s) = a\,\frac{u + s\psi}{\|u + s\psi\|_{L^{p+1}}}, \quad a = \|u\|_{L^{p+1}} = \beta^{1/(p+1)}.$$

A calculation parallel to that done for $(d/ds)^2 E(w(s))$ in §7.1, which this time we leave to the reader, gives

3.A.6 (7.48) $$\frac{1}{2}\,\frac{d^2}{ds^2} F_\lambda(w(s))\Big|_{s=0} = (L_+\psi_0, \psi_0) + (L_-\psi_1, \psi_1),$$

where $\psi = \psi_0 + i\psi_1$, with ψ_0, ψ_1 real valued, and $L_\pm$ as in (7.42), i.e.,

3.A.7 (7.49) $$\begin{aligned} L_+\psi_0 &= (-\Delta + \lambda - p|u|^{p-1})\psi_0, \\ L_-\psi_1 &= (-\Delta + \lambda - |u|^{p-1})\psi_1, \end{aligned}$$

provided a certain rescaling, described below, is performed. In this case, the condition that $\psi \in H^1$ belong to $T_u\widetilde{X}$ becomes

3.A.8 (7.50) $$(u^p, \psi_0) = 0,$$

with no further condition on ψ_1. Contrast (7.50) with (7.8).

We describe the rescaling of u that yields (7.49). If $\widetilde{X}$ is as in (7.43) and if $u \in \widetilde{Y}$ as in (7.44) and is ≥ 0, then there exists $K \in \mathbb{R}$ such that such that

3.A.9 (7.51) $$-\Delta u + \lambda u = K u^p.$$

Taking the inner product with u yields

3.A.10 (7.52) $$K = \beta^{-1} I_\beta.$$

We can rescale, replacing u by κu, to arrange that $K = 1$ in (7.51), so, with a different β, the new u minimizes F_λ subject to the constraint $J_p(u) = \beta$, and satisfies

3.A.11 (7.53) $$-\Delta u + \lambda u - u^p = 0.$$

Cf. Corollary 1. It is for this rescaled u that (7.48)–(7.49) hold.

s3.6

7.1.2. Properties of $L_\pm$. We have defined operators L_+ and L_- in §§7.1–7.1.1, as

3.6.0
$$\begin{aligned} L_+\psi_0 &= (-\Delta + \lambda - p|u|^{p-1})\psi_0, \\ L_-\psi_1 &= (-\Delta + \lambda - |u|^{p-1})\psi_1, \end{aligned} \tag{7.54}$$

arising when $u \in H^1$ is either an energy minimizer of an F_λ-minimizer, satisfying (7.53). We also assume $u > 0$. In light of decay results on u established in §3.1, these are self-adjoint operators on L^2, with domain H^2. In these respective cases, we have seen that, with $w(s)$ respectively as in (7.5) or (7.47),

3.6.1
$$\frac{d^2}{ds^2} E(w(s))\Big|_{s=0} = (L_+\psi_0, \psi_0) + (L_-\psi_1, \psi_1), \tag{7.55}$$

3.6.2
$$\frac{d^2}{ds^2} F_\lambda(w(s))\Big|_{s=0} = (L_+\psi_0, \psi_0) + (L_-\psi_1, \psi_1), \tag{7.56}$$

with $\psi_j \in H^1$ real valued, ψ_1 otherwise arbitrary, and

3.6.3
$$(u, \psi_0) \quad = 0 \quad \text{in case (7.55)}, \tag{7.57}$$

3.6.4
$$(u^p, \psi_0) \quad = 0 \quad \text{in case (7.56)}. \tag{7.58}$$

Since u is a minimizer, we know that (7.55) (resp., (7.56)) is ≥ 0 for all such paths $w(s)$. We deduce that

3.6.5
$$(L_+\psi_0, \psi_0) \geq 0, \tag{7.59}$$

for all real-valued $\psi_0 \in H^1$ satisfying (7.57) when u is an energy minimizer, and for all ψ_0 satisfying (7.58) when u is an F_λ-minimizer (satisfying (7.53)). Also, in both cases,

3.6.6
$$(L_-\psi_1, \psi_1) \geq 0, \tag{7.60}$$

for all real valued $\psi_1 \in H^1$. Since L_+ and L_- are reality preserving, these results extend readily to the case where ψ_0 and ψ_1 are allowed to be complex valued.

As we have seen, if $u \in H^1$ minimizes $E(u)$, subject to the constraint $\|u\|_{L^2}^2 = \beta$, then there exists $\lambda \in \mathbb{R}$ such that

3.6.7
$$\Delta u - \lambda u + |u|^{p-1}u = 0. \tag{7.61}$$

(This is also the PDE satisfied by the rescaled F_λ-minimizer, discussed in §7.1.1.) From (7.60), we have the following information about λ.

p3.6.1 **PROPOSITION 7.1** *If u is an energy minimizer satisfying (7.61), then*

3.6.8
$$\mathit{Spec}(-\Delta + \lambda) \subset [0, \infty). \tag{7.62}$$

PROOF. In fact, for all $\psi \in H^1$,

3.6.9
$$\begin{aligned} ((-\Delta + \lambda)\psi, \psi) &= (L_-\psi, \psi) + (|u|^{p-1}\psi, \psi) \\ &\geq (|u|^{p-1}\psi, \psi) \\ &\geq 0. \end{aligned} \tag{7.63}$$

□

By contrast with the positivity of L_-, note that

3.6.9A (7.64)
$$\begin{aligned}(L_+u,u) &= -(\Delta u - \lambda u + p|u|^{p-1}u, u)\\ &= -(p-1)\int |u|^{p+1}\,dx\\ &< 0,\end{aligned}$$

the second identity by (7.61) if u is an energy minimizer and by (7.53) if u is an (appropriately rescaled) F_λ-minimizer. This result, together with (7.59), implies:

p3.6.2 **PROPOSITION 7.2** *If u is either an energy minimizer or an F_λ-minimizer, satisfying (7.53), then L_+ has exactly one negative eigenvalue.*

Returning to L_-, we note that

3.6.10 (7.65)
$$L_-u = -(\Delta u - \lambda u + |u|^{p-1}u) = 0,$$

so

3.6.11 (7.66)
$$u \in \mathcal{N}(L_-).$$

We have the following more precise result.

p3.6.3 **PROPOSITION 7.3** *If u is either an energy minimizer or an F_λ-minimizer, satisfying (7.53), then*

3.6.12 (7.67)
$$\mathcal{N}(L_-) = \mathit{Span}(u).$$

PROOF. It suffices to show that any nonzero, real-valued element of $\mathcal{N}(L_-)$ must be either everywhere >0 or <0, since then no two such can be orthogonal to each other. Now, if $v \in \mathcal{N}(L_-)$ is real valued and $\|v\|_{L^2} = 1$, then v minimizes

3.6.13 (7.68)
$$\{\|\nabla v\|_{L^2}^2 + \lambda\|v\|_{L^2}^2 - (|u|^{p-1}v, v) : \|v\|_{L^2} = 1\}.$$

Then $|v|$ is also minimizing, so $|v| \in \mathcal{N}(L_-)$. Then the Harnack inequality implies $|v| > 0$, so indeed either $v > 0$ or $v < 0$. □

We turn to some comments on $\mathcal{N}(L_+)$. Since derivatives commute with Δ we have (assuming $u > 0$)

3.6.14 (7.69)
$$\Delta(\partial u) - \lambda(\partial u) + p|u|^{p-1}(\partial u) = 0,$$

hence

3.6.15 (7.70)
$$\partial_j u \in \mathcal{N}(L_+), \quad \text{if for } j = 1,\dots,d,$$

given estimates on u assuring that $\partial u \in \mathscr{D}(L_+)$.

It is useful to regard L_+ and L_- as two operators in a continuum, defined by

3.6.16 (7.71)
$$L_a\psi = -\Delta\psi + \lambda\psi - a|u|^{p-1}\psi,$$

for $a \in \mathbb{R}$, particularly for $a \in [1,p]$.. Note that

3.6.17 (7.72)
$$L_1 = L_-, \quad L_p = L_+.$$

For each $a \in \mathbb{R}$, L_a is self-adjoint on L^2, with domain $\mathscr{D}(L_a) = H^2$. The following result extends Proposition 7.2.

p3.6.4 **PROPOSITION 7.4** *Assume $u > 0$ is either an energy minimizer or an F_λ-minimizer, satisfying (7.53). Then, if $1 < a \le p$, L_a has exactly one negative eigenvalue, and it has multiplicity one.*

PROOF. First note that

3.6.18 (7.73)
$$\begin{aligned}(L_a u, u) &= (L_1 u + (1-a)u^{p-1}u, u)\\ &= -(a-1)\int u^{p+1}\,dx,\end{aligned}$$

which is < 0 if $a > 1$. Next, for $\psi \in H^1$,

3.6.19 (7.74)
$$\begin{aligned}(L_a\psi, \psi) &= (L_p\psi + (p-a)u^{p-1}\psi, \psi)\\ &= (L_+\psi, \psi) + (p-a)\int u^{p-1}|\psi|^2\,dx,\end{aligned}$$

so, by (7.59), if u is an energy minimizer,

3.6.20 (7.75)
$$(u,\psi) = 0 \Longrightarrow (L_a\psi,\psi) \ge 0 \text{ if } a \le p,$$

while if u is an F_λ-minimizer satisfying (7.53),

3.6.21 (7.76)
$$(u^p,\psi) = 0 \Longrightarrow (L_a\psi,\psi) \ge 0 \text{ if } a \le p.$$

These results prove the proposition. □

The result (7.60) implies $\mathrm{Spec} L_1 \subset [0,\infty)$, and Proposition 7.4 implies that $\mathrm{Ess\,Spec}\, L_a \subset [0,\infty)$ for $1 < a \le p$. We can say more about the essential spectrum.

p3.6.5 **PROPOSITION 7.5** *If u is either an energy minimizer or an F_λ-minimizer satisfying (7.53), then, for all $a \in \mathbb{R}$,*

3.6.22 (7.77)
$$\mathrm{Ess\,Spec}\, L_a = \mathrm{Ess\,Spec}(-\Delta + \lambda).$$

PROOF. Given $a \in \mathbb{R}$, pick $\mu > 0$ so large that $L_a + \mu$ and $-\Delta + (\lambda + \mu)$ are both invertible. By Weyl's essential spectrum theorem ([**RS78a**], p. 112) it suffices to note that

3.6.23 (7.78)
$$S_a = (L_a + \mu)^{-1} - (-\Delta + (\lambda+\mu))^{-1} \text{ is compact.}$$

Recalling the formula (7.71) for L_a, we have, by the resolvent identity,

3.6.24 (7.79)
$$S_a = -a(L_a + \mu)^{-1} M_{|u|^{p-1}} (-\Delta + (\lambda+\mu))^{-1},$$

whose compactness follows readily from the decay results given in §6.1.3 and §6.1.4, plus the Rellich theorem. □

For the next result, we assume the following:

3.6.25 (7.80)
$$\mathrm{Spec}(-\Delta + \lambda) \subset [\delta, \infty), \quad \delta > 0.$$

For F_λ-minimizers, this is equivalent to the hypothesis (2.17)–(2.18). For energy minimizers, (7.80) is slightly stronger than (7.62), and it can be expected to hold for almost all (if not all) energy minimizers.

p3.6.6 **PROPOSITION 7.6** *Let u be either an energy minimizer or an F_λ-minimizer satisfying (7.53), and assume (7.80) holds. Then*

3.6.26 (7.81)
$$1 < a < p \Longrightarrow \mathcal{N}(L_a) = 0.$$

PROOF. By Proposition 7.5 and (7.80), for each a,

3.6.27 (7.82) $$\text{Ess Spec}\, L_a \subset [\delta, \infty), \quad \delta > 0.$$

Suppose $a_0 \in (1,p)$ and $\dim \mathcal{N}(L_{a_0}) = m > 0$. The Kato-Rellich theorem ([**RS78a**], p. 22) implies there exist analytic functions $\lambda_j(a), 1 \le j \le m$, for a close to a_0, with $\lambda_j(a_0) = 0$, such that these are all the eigenvalues of L_a near 0. Also ([**RS78a**], p. 71) there are corresponding eigenfunctions ψ_{ja}, analytic in a:

3.6.28 (7.83) $$L_a \psi_{ja} = \lambda_j(a)\psi_{ja}, \quad (\psi_{ja}, \psi_{ka}) \equiv \delta_{jk},$$

the orthonormality holding for a real (and close to a_0).

Let us denote by $\psi_a^{\#}$ the (normalized) eigenfunction of L_a given by Proposition 7.4. We have

3.6.29 (7.84) $$(\psi_{ja}, \psi_a^{\#}) = 0,$$

for $j \in \{1, \dots, m\}$, real a close to a_0. Now apply d/da to (7.83). We get

3.6.30 (7.85) $$-u^{p-1}\psi_{ja} + L_a \xi_{ja} = \lambda_j'(a)\psi_{ja} + \lambda_j(a)\xi_{ja},$$

where

3.6.31 (7.86) $$\xi_{ja} = \frac{d}{da}\psi_{ja}.$$

The normalization in (7.83) implies

3.6.32 (7.87) $$(\psi_{ja}, \xi_{ja}) = 0,$$

so taking the inner product of (7.85) with ψ_{ja} gives

3.6.33 (7.88) $$\lambda_j'(a)\|\psi_{ja}\|^2 = -\int u^{p-1}|\psi_{ja}|^2\, dx,$$

since

3.6.34 (7.89) $$(L_a\xi_{ja}, \psi_{ja}) = (\xi_{ja}, \lambda_j(a)\psi_{ja}) = 0.$$

Hence, for a close to 0,

3.6.35 (7.90) $$\lambda_j'(a) < 0,$$

and if $\lambda_j(a_0) = 0$, we get

3.6.36 (7.91) $$\lambda_j(a) < 0 \ \text{ for } \ a_0 < a < a_0 + \epsilon,$$

for some positive ϵ. This contradicts Proposition 7.4, and completes the proof. □

s3.8

7.1.3. Conditional orbital stability result. As in §7.1, we fix $\beta > 0$ and set

3.8.1 (7.92) $$X = \{u \in H^1 : Q(u) = \beta\}, \quad \mathscr{I}_\beta = \inf\{E(u) : u \in X\},$$
$$Y = \{u \in X : E(u) = \mathscr{I}_\beta\}.$$

Under these hypotheses, the nonlinear Schrödinger equation

3.8.2 (7.93) $$i v_t + \Delta v + |v|^{p-1}v = 0, \quad v(0) = v_0,$$

is globally solvable, given $v_0 \in H^1$, via an argument given in [**SS99**], §3.2.2. Conservation of mass and energy imply that X and Y are invariant under the solution operator to (7.93). We investigate the following question concerning orbital stability. Assume

3.8.2A (7.94) $$v_0 \in X$$

is close to Y (distance measured in H^1-norm). We then ask whether the solution $v(t)$ to (7.93) can be shown to be close to Y, for all $t \in \mathbb{R}$. Since energy is conserved for solutions to (7.93):

3.8.3 (7.95) $$E(v(t)) \equiv E(v_0),$$

a positive result would follow if one could show that if $u \in X$ and $E(u)$ is close to $\mathscr{I}_\beta$, then u is close to Y.

We establish such a result, under the following two assumptions. The first is an essential uniqueness hypothesis:

3.8.4 (7.96) If u_1, u_2 are positive functions in Y, there is an isometry $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ such that $u_2 = u_1 \circ \varphi$.

Recall that if $u \in Y$, there exists $\lambda \in \mathbb{R}$ such that

3.8.5 (7.97) $$-\Delta u + \lambda u - |u|^{p-1}u = 0.$$

The hypothesis (7.96) implies that (7.97) holds with the same λ for all $u \in Y$. Our second hypothesis is that (7.80) hold, i.e.,

3.8.6 (7.98) $$\mathrm{Spec}(-\Delta + \lambda) \subset [\delta, \infty), \quad \text{for some } \delta > 0,$$

which, recall, is slightly stronger than (7.62).

To state our first result, let $\mathscr{G}$ denote the group of operators on functions on $\mathbb{R}^d$ of the form

3.8.7 (7.99) $$u(x) \mapsto e^{i\theta} u(\varphi(x)), \quad \theta \in \mathbb{R}, \ \varphi : \mathbb{R}^d \to \mathbb{R}^d \text{ isometry}.$$

Thus $\mathscr{G}$ acts as a group of isometries on L^2 and on H^1, preserving X and Y. The following is immediate.

p3.8.1 **PROPOSITION 7.7** *Under the hypothesis (3.8.4), $\mathscr{G}$ acts transitively on Y, and Y is a smooth, finite dimensional submanifold of X.*

To proceed, for $\epsilon > 0$, set

3.8.8 (7.100) $$\mathscr{O}_\epsilon = \{u \in X : \mathrm{dist}_{H^1}(u, Y) \le \epsilon\}.$$

Then $\mathscr{O}_\epsilon$ is invariant under the action of $\mathscr{G}$. By Proposition 7.7, if ϵ is sufficiently small, given $u \in Y$, $\mathscr{O}_\epsilon$ is swept out by the $\mathscr{G}$-action on a codimension-m space Σ, normal to Y at u (with $m = \dim Y$).

The following is an orbital stability result.

p3.8.2 **PROPOSITION 7.8** *Assume hypotheses (7.96) and (7.98). For $\epsilon > 0$ sufficiently small, the following holds. If $v_\nu \in \mathscr{O}_\epsilon$ and $E(v_\nu) \to \mathscr{I}_\beta$, then*

3.8.9 (7.101) $$\mathrm{d}_{H^1}(v_\nu, Y) \to 0.$$

Note that we can take $\tilde{v}_\nu \in \Sigma$ such that $E(\tilde{v}_\nu) = E(v_\nu)$, and $\mathrm{dist}(\tilde{v}_\nu, Y) = \mathrm{dist}(v_\nu, Y)$, so without loss of generality we can assume $v_\nu \in \Sigma$. We will parametrize an appropriate space Σ by a neighborhood of 0 in an $\mathbb{R}$-linear subspace V of $T_u X$, of codimension m, as follows. We set

3.8.10 (7.102) $$V = \{\psi \in T_u X : \psi \perp T_u Y\}.$$

Recall the characterization of T_uX in (7.4), supplemented by (7.7)–(7.8). V is an $\mathbb{R}$-linear subspace of H^1, of codimension $m+1$, a Hilbert space with the H^1-norm.

To proceed, we define a function F on a neighborhood of $0 \in V$ by

3.8.11 (7.103) $$F(\psi) = E\Big(a\,\frac{u+\psi}{\|u+\psi\|}\Big).$$

We have

3.8.12 (7.104) $$F(0) = E(u) = \mathscr{I}_\beta, \quad DF(0) = 0,$$

and calculations of §3.5 give

3.8.13 (7.105) $$D^2F(0)(\psi,\psi) = (L_+\psi_0,\psi_0) + (L_-\psi_1,\psi_1).$$

In light of this, Proposition 7.8 is a consequence of the following.

p3.8.3 **PROPOSITION 7.9** *Let V be a real Hilbert space, $\mathscr{B}_r$ a ball of radius r centered at $0 \in V$, and $F : \mathscr{B}_r \to \mathbb{R}$ a C^2 function satisfying the following conditions:*

3.8.14 (7.106) $$F(0) = \mathscr{I}_\beta, \quad \psi \in \mathscr{B}_r \setminus 0 \Rightarrow F(\psi) > \mathscr{I}_\beta$$

(so $DF(0) = 0$). Also assume there exists $C > 0$ and an orthogonal projection $P : V \to V$, with range of finite codimension, such that, for $\psi \in V$,

3.8.15 (7.107) $$D^2F(0)(\psi,\psi) \geq C\|P\psi\|_V^2.$$

Then, if $\rho \in (0,r)$ is small enough,

3.8.16 (7.108) $$\psi_\nu \in \mathscr{B}_\rho,\ F(\psi_\nu) \to \mathscr{I}_\beta \Longrightarrow \|\psi_\nu\|_V \to 0.$$

PROOF. Taylor's formula with remainder gives

3.8.17 (7.109) $$F(\psi) = \mathscr{I}_\beta + \frac{1}{2}D^2F(0)(\psi,\psi) + R(\psi),$$

with

3.8.18 (7.110) $$R(\psi) = \int_0^1 [D^2F(t\psi) - D^2F(0)](\psi,\psi)\,(1-t)\,dt = o(\|\psi\|_V^2).$$

Hence, if $\psi \in \mathscr{B}_\rho$ and ρ is small enough,

3.8.19 (7.111) $$\begin{aligned} F(\psi) &\geq \mathscr{I}_\beta + \frac{C}{2}\|P\psi\|_V^2 - o(\|\psi\|_V^2) \\ &\geq \mathscr{I}_\beta + \frac{C}{4}\|P\psi\|_V^2 - o(\|P^\perp\psi\|_V^2), \end{aligned}$$

where $P^\perp = I - P$ has finite dimensional range. Hence the hypothesis (7.108) on ψ_ν implies

3.8.20 (7.112) $$\|P\psi_\nu\|_V \longrightarrow 0.$$

We need to show that $P^\perp\psi_\nu \to 0$ in $W = \text{Range}\,P^\perp \subset V$. The sequence $(P^\perp\psi_\nu)$ is a bounded sequence in W, so (ψ_ν) has a subsequence (which we continue to denote (ψ_ν)) such that $P^\perp\psi_\nu \to \tilde{\psi}$. Hence $\psi_\nu \to \tilde{\psi}$. Now $F(\psi_\nu) \to \mathscr{I}_\beta$ implies $F(\tilde{\psi}) = \mathscr{I}_\beta$. The hypothesis (7.106) then gives $\tilde{\psi} = 0$, and completes the proof. □

Remark. In the setting of Proposition 7.9, the range of $P^{\perp}$ is the orthogonal complement of $T_u Y$ in

3.8.21 (7.113) $$\{\psi = \psi_0 + i\psi_1 \in T_u X : (L_+\psi_0, \psi_0) + (L_-\psi_1, \psi_1) = 0\},$$

which is a linear space, by (7.59)–(7.60), and is finite dimensional, given (7.98), by (7.82).

7.2. Exercises

CHAPTER 8

Bielefeld Lecture 8

Much of the following can be found in the seminal papers of Weinstein [**Wei85b**, **Wei86**].

8.1. Lyapunov Stability

As discussed in Lecture 1, using the ansatz $u(x,t) = e^{i\lambda^2 t}R(x)$ in 12.1, we get a semi-linear elliptic equation for R:

$$-\Delta R + \lambda^2 R - |R|^{2\sigma}R = 0.$$

Using variational techniques discussed above, we can prove there exists a minimizer to the resulting Lagrangian, say R_λ, the soliton solution, which is positive, radially symmetric, and exponentially decaying. In the case above with a monomial nonlinearity, by scaling we can actually write $R_\lambda = \lambda^{\alpha(\sigma)}R_1(\lambda x)$. Set $R = R_1$ for simplicity. We call R the "ground state" solution. From the work of McLeod [**McC93**], we know that solitons are unique in the case of the nonlinearity given above. As proved above and by Weinstein in [**Wei83**], R minimizes the functional

$$J[u] = \frac{\|\nabla u\|_{L^2}^N \|u\|_{L^2}^{4-N}}{\|u\|_{L^4}^4}.$$

The result we desire to prove is that these soliton solutions are stable. Namely, that if we begin at time 0 close to a soliton, we should stay close to a soliton for all subsequent times. In what follows, we present the work of M. Weinstein [**Wei83**]. Let us introduce the following norm:

$$\left[\rho_\lambda(\phi(t), \mathcal{G}_{R_\lambda})\right]^2 = \inf_{\gamma, x_0}\left\{\|\nabla\phi(\cdot + x_0, t)e^{i\gamma} - \nabla R_\lambda\|_{L^2}^2 + \lambda^2\|\phi(\cdot + x_0, t)e^{i\gamma} - R_\lambda\|_{L^2}^2\right\},$$

where $\mathcal{G}_R$ is the set of functions generated by the symmetries of NLS applied to R. Hence, for all times t, we measure how the profile of ϕ compares to that of R_λ.

For solutions to NLS, we have conservation of mass

$$\mathcal{M}(\phi) = \int |\phi|^2 dx,$$

and conservation of energy

$$\mathcal{H}(\phi) = \int \frac{1}{2}|\nabla\phi|^2 - \frac{|\phi|^{p+1}}{p+1}dx.$$

From these, we define the Lyapunov functional

$$\mathcal{E}(\phi) = \mathcal{H}(\phi) + \lambda^2\mathcal{M}(\phi).$$

We are now ready to state the main result.

8.1.1. Main Result. Without loss of generality, we set $\lambda = 1$.

Theorem 8.1
Let $R(x)$ be the (assumed to be unique here) ground state soliton. For any $\epsilon > 0$, there exists $\delta > 0$ such that if $\rho(\phi_0, \mathcal{G}_R) < \delta$, then for all $t > 0$, we have $\rho(\phi(t), \mathcal{G}_R) < \epsilon$.

PROOF. Set $\phi(x + x_0, t)e^{i\gamma} = R(x) + w$, $w = u + iv$. Then, we have

$$\begin{aligned}
\Delta\mathcal{E} &= \mathcal{E}(\phi_0(\cdot)) - \mathcal{E}(R(\cdot)) \\
&= \mathcal{E}(\phi(t)) - \mathcal{E}(R(\cdot)) \\
&= \mathcal{E}(\phi(\cdot + x_0, t)e^{i\gamma}) - \mathcal{E}(R) \\
&= \mathcal{E}(R + w) - \mathcal{E}(R) \\
&\geq (L_+ u, u) + (L_- v, v) - C_1 \|w\|_{H^1}^{2+\theta} - C_2 \|w\|_{H^1}^6,
\end{aligned}$$

for $\theta > 0$, $L_- = -\Delta + 1 - R^{p-1}$, $L_+ = -\Delta + 1 - pR^{p-1}$. This follows from keeping the quadratic terms from Taylor series expansion about R.

The result then depends upon the following lemma.

LEMMA 8.1.1
If x_0, γ are chosen to minimize

$$\left\{ \|\nabla\phi(\cdot + x_0, t)e^{i\gamma} - \nabla R_\lambda\|_{L^2}^2 + \lambda^2 \|\phi(\cdot + x_0, t)e^{i\gamma} - R_\lambda\|_{L^2}^2 \right\},$$

then we have the following estimate

$$(L_+ u, u) + (L_- v, v) \geq C_3 \|w\|_{H^1}^2 - C_4 \|w\|_{H^1}^3 - C_5 \|w\|_{H^1}^4.$$

Since according to our ansatz, $\rho(\phi, \mathcal{G}_R) \sim \|w\|_{H^1}$ we have that $\Delta\mathcal{E} \geq g(\rho)$ for $g(x) = cx^2(1 - ax^\theta - bx^4)$. Since $\mathcal{E}$ is continuous for functions in H^1, choose δ such that $\rho_0 < \delta$ implies $\Delta\mathcal{E}(0) < g(\epsilon)$. Since $\Delta\mathcal{E}$ is constant, $g(\rho(t)) < g(\epsilon)$ for all times t. Hence, since g is increasing on a small interval around 0, we get that $\rho < \epsilon$ for all times t. This gives us the result. □

All that remains is to prove the lemma.

PROOF. Proof of Lemma

PROPOSITION 8.1
The operator L_- is a non-negative self-adjoint operator with null space given by span(R_ω) and

$$(L_- g, g) \geq (g, g)$$

for all $g \perp X_j R, \partial_\omega R$.

PROOF. If L_- is non-negative and has kernel generated only by R, then let us take

$$\mu = \inf_{\|g\|_{L^2} = 1} (L_- g, g)$$

with g having the assumed orthogonality properties. If $\mu = 0$, using energy estimates there exists a minimizing sequence of functions u_ν such that

$$0 < \int |\nabla u_\nu|^2 dx + \omega \int |u_\nu|^2 dx \leq \int R_\omega^p |u_\nu|^2 dx + \eta$$

for any $\eta > 0$. Hence, $\|u_\nu\|_{H^1}$ is uniformly bounded. Hence, a subsequence converges weakly to $u_* \in H^1$ again with the correct orthogonality conditions by the regularity and decay properties of R. Similarly, since $\|u_\nu\|_{L^2} = 1$, we have

$$\int R_\omega^p |u_\nu|^2 dx \to \int R^p |u_*|^2 dx \neq 0.$$

Suppose $\|u_*\|_{L^2} < 1$, then define $g_* = u_*/\|u_*\|_{L^2}$. Since by weak convergence we have

$$\|\nabla u_*\|_{L^2} \leq \liminf_{\nu\to\infty} \|\nabla u_\nu\|_{L^2},$$

we have

$$(L_- u_*, u_*) \leq \liminf_{\nu\to\infty} (L_- u_\nu, u_\nu)$$

and hence

$$(L_- g_*, g_*) \leq 0.$$

Since L_- is nonnegative, the minimum is attained at some u_* such that $\|u_*\|_{L^2} = 1$, making it admissible and in this case clearly such that

$$u_* = R/\|R\|_{L^2}.$$

given the nature of the kernel. However, $(R, R_\lambda) \neq 0$ by the slope condition that can be derived by scaling, meaning that $u_* = 0$, a contradition.

To verify that $L_- \geq 0$ and $\ker(L_-) = R$, we have first of all that for $R > 0$, $L_- R = 0$ using standard arguments for elliptic operators.

□

From the minimization of ρ over x_0, γ implies the following integral identities:

$$\int R^{p-1} \partial_j R u(x,t) = 0,$$

and

$$\int R^p v(x,t) dx = 0.$$

We arrive at these equations simply by differentiating ρ with respect to x_0 and γ respectively, then using that w is the minimum solution by assumption.

We have that $L_- R = 0$ and R is a positive ground state solution, hence by oscillation theory, we must have $(L_- v, v) \geq (v, v)$ since $v \notin \text{span}(R)$ by the second integral identity above. Hence, using this plus standard elliptic estimates, we see $(L_- v, v) \geq C\|v\|_{H^1}^2$.

Let us now make a simplifying assumption that we will remove later. Let us make the assumption

$$\int |\phi|^2 dx = \int R^2 dx.$$

Hence, we have that $(u, R) = -\frac{1}{2}[(u, u) + (v, v)]$.

Now, we need some results about the operator L_+, which will require several spectral assumptions to be satisfied.

8.2. Exercises

CHAPTER 9

Bielefeld Lecture 9

Much of the following can be found applied in the seminal works of Weinstein [**Wei85b**, **Wei86**].

9.1. Lyapunov Stability Continued

PROPOSITION 9.1
Given that L_+ has a unique negative eigenvalue and that $\partial_\lambda M(R_\lambda) > 0$, we have $\inf_{(f,R)=0}(L_+ f, f) = 0$.

PROOF. This can be proved possibly two ways. Given

$$J[u] = \frac{\|\nabla u\|_{L^2}^N \|u\|_{L^2}^{2+p(1-\frac{d}{2})}}{\|u\|_{L^{p+1}}^{p+1}},$$

if one can prove that R is the minimizer of this functional then the relation follows from analysis of the second variation of the functional J, see [**Wei85a**]. Note here also that if indeed R is the minimizer of this functional, then the second variation shows that $L_+ + r_1$ is a non-negative operator where r_1 is rank-one, which implies there is only one negative eigenvalue. Indeed, in Euclidean space

$$\frac{d^2}{d\epsilon^2} J|_{\epsilon=0}(R + \epsilon w) \geq 0$$

is such that

$$(Tw, w) \geq 0$$

where

$$\begin{aligned}(2 + \frac{p}{2}(2-d))^{-1} Tz &= L_+ z + (2-d) a_{p,d}(R,z) R \\ &\quad - b_{p,d}(R,z)\Delta R + (\frac{pd}{2} - 2) c_{p,d}(\Delta R, z)\Delta R.\end{aligned}$$

Hence,

$$(L_+ f, f) \geq (2 - \frac{pd}{2}) c_{p,d}(\Delta R, f)^2$$

provided $(f, R) = 0$. If such a result can be established for monomial nonlinearities, it can be established for more general nonlinearities assuming uniqueness of the ground state and nondegeneracy of the kernel of L_+ using continuity of eigenvalues for $L_+(\tau)$, the linearization of the equation

$$\Delta u - \lambda u + (1-\tau) u^{p+1} + \tau f(u^2) u = 0.$$

A more general proof also in [**Wei85a**] uses the general result that if A is a self-adjoint operator with exactly one negative eigenvalue, λ_0, with corresponding eigenfunction $\psi_0 \geq 0$, then given a function R such that $(R, f_0) \neq 0$ and $R \perp \mathrm{Ker}(A)$, we have

$$\alpha = \min_f (Af, f) \geq 0$$

for $(f, R) = 0$ if

$$(A^{-1}R, R) \leq 0.$$

Note, this explains the slope stability condition of the soliton curve since $L_+^{-1}R = -\partial_\lambda R$. □

PROPOSITION 9.2
Ker$(L_+) = span\{\partial_j R\}$.

PROOF. This will follow given the knowledge that R is the unique, positive radial solution, which follows from [**McC93**] and a partial result following the ODE analysis from [**Wei85a**].

Given the structure of L_+, any L^2 solution of $L_+ v = 0$ can be decomposed into harmonics of the form $v = f(r)Y(\theta)$. The kernel functions generated by $partial_j R$ correspond to the $k = 1$ harmonic. By oscillation theory, this is the complete description of the kernel on this harmonic.

For $k \geq 2$, since $\partial_r R$ has no interior zeros, it is the ground state of $(L_+)_1$, the radial ODE operator L_+ restricted to the $k = 1$ harmonic, which implies that $(L_+)_1$ is a non-negative operator. Hence, $(L_+)_k$ for $k > 1$ is a positive operator and $(L_+)_k f = 0$, $f \in L^2$ implies $f = 0$.

For $k = 0$, the result would be implied by uniqueness of the radial bound state R_λ. Indeed, $(L_+)_0 f(r) = 0$ would then be forced to correspond with $\partial_a R_a$, where R_a solves the initial value problem

$$-\Delta_g R_a + \lambda R_a - R_a^p = 0,\ R_a(0) = a,\ R_a'(0) = 0.$$

However, ODE techniques show that $|\partial_r(\partial_a R_a|_{a_0}(r))| \to \infty$ as $r \to \infty$ for $R_\lambda = R_{a_0}$. Hence, $f \notin L^2$.

Note, as the $k = 0$ state is the only state in which further kernel functions could arise, we see that $\mathrm{Ker}(L_+) \leq d + 1$ again using oscillation theory. □

Let us take $\phi = u + iv$.

PROPOSITION 9.3
We have $(L_+ u, u) \geq D\|u\|_{H^1}^2 - D'\|\nabla\phi\|_{L^2}^2\|\phi\|_{L^2}^2 - D''\|\phi\|_{H^1}^4$.

PROOF. The proof of this is somewhat intricate. To begin, we write $u = u_{perp} + u_{par}$ where

$$u_{par} = (u, R)R = -\frac{1}{2}[(u, u) + (v, v)]R,$$

and

$$u_{perp} = u - (u, R)R = u + \frac{1}{2}[(u, u) + (v, v)]R.$$

(WLOG, assume $(R, R) = 1$. If not, there is simply a factor of $\frac{1}{(R,R)}$ floating around in the constants).

Then, $(L_+u,u) = (L_+u_{par},u_{par}) + (L_+u_{perp},u_{perp}) + 2(L_+u_{par},u_{perp})$. We wish to analyze each term separately.

By assumption, $(u_{perp},R) = 0$, hence $(L_+u_{perp},u_{perp}) \geq c(u_{perp},u_{perp})$. Otherwise, $u_{perp} = c\cdot\nabla R$ which would violate the integral identity above. Hence,

$$(L_+u_{perp},u_{perp}) \geq c(u_{perp},u_{perp}) = d[(u,u) - \frac{1}{4}[(u,u)+(v,v)]^2].$$

Now,

$$(L_+u_{par},u_{par}) = \frac{1}{4}(L_+R,R)[(u,u)+(v,v)]^2,$$

where (L_+R,R) is some negative constant.

Finally,

$$\begin{aligned}(L_+u_{perp},u_{par}) &= (u,R)(L_+u_{perp},R)\\ &\geq -d'\|\phi\|^2\|\nabla\phi\| - d''\text{h.o.t.}(\|\phi\|),\end{aligned}$$

by expanding the above expression and using the fact the regularity of R.

Putting all of these estimates gives us the desired inequality. □

So, if in fact $\|\phi\|_{L^2}^2 = \|R\|_{L^2}^2$, then we are done. If not, find λ such that $\|\phi\|_{L^2}^2 = \|R_\lambda\|_{L^2}^2$ and use the above result along with the fact that $\|R-R_\lambda\|_{H^1}$ is small. Then, we have the result in general.

□

9.1.1. Modulation of Parameters. For simplicity, let us briefly diverge from the Schrödinger equation to study the KdV equation, which implicitly has less symmetry since it is real and scalar. The generalized KdV equation

eqn:gkdv

$$\begin{cases}\partial_t\psi + \partial_x(\partial_x^2\psi + \psi^p) = 0,\ t,x\in\mathbb{R}\\ \psi(0,x) = \psi_0(x)\end{cases} \tag{9.1}$$

has an explicit soliton solution

$$\psi_c(x,t) = Q_{p,c,c^2t+x_0}(x) := c^{\frac{2}{p-1}}Q_p(c(x-(x_0+c^2t)))$$

with $c>0$, $x_0\in\mathbb{R}$ and

Qps

$$Q_p = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}}\operatorname{sech}^{\frac{2}{p-1}}\left(\frac{p-1}{2}x\right). \tag{9.2}$$

he solutions $Q_{c,y}$ are called traveling waves or solitons. These are minimizers of the constrained variational problem

constraintL2

$$\min\{E(w) : w\in H^1,\ \|w\|_{L^2} = \mu > 0\}, \tag{9.3}$$

where

$$E(u) = \int\left[\frac{1}{2}u_x^2 - \frac{1}{p+1}u^{p+1}\right]dx.$$

Minimizers also are extremals of the Lagrangian

globalmin

$$S(u) = E(u) + \frac{\lambda}{2}\int u^2\,dx, \tag{9.4}$$

where λ is a Lagrangian multiplier.

We wish to see that given a perturbation $u(x,t) = \psi_1(x-x_0) + w(x,t)$, at each time there exists a minimum translation x_0 for instance. Note, we of course must also ask about the scaling parameter, but for the moment let us simplify and only consider the translation. We will follow an argument of Bona [**Bon75**]. Let $\|w\|_{H^1} \ll 1$, hence $\|u(x,t)\|_{H^1} \sim$

$\|Q_p\|_{H^1}$. Since KdV (1.3) gives a continuous flow in H^1 for $p < \infty$ (more to come in the evolution equation section later), there exists some $T > 0$ such that

$$\|w\|_{H^1} < 2\|Q_p\|.$$

CLAIM 9.1.1. *There exists $x_0 < \infty$ such that*

$$x_0 = \inf_{y\in\mathbb{R}} \int_{\mathbb{R}} |u - \psi(x+y)| dx.$$

PROOF. Let us restrict ourselves to time interval $I = [0, T]$, where T is defined above such that $\|w\|_{H^1}$ cannot grow too large. Define

$$\rho_t(y) = \int_{-\infty}^{\infty} (u(x,t) - \psi(x+y))^2 dx.$$

As $y \to \infty$, the scales separate and we have

$$\rho_\infty = \int |u|^2 + \int |Q_p|^2 \approx 2\|Q_p\|_{L^2}^2.$$

However, $\rho(x_0) < 2\|Q_p\|_{L^2}$ by assumption, hence x_0 must be finite. □

Similar considerations hold with the scaling parameter noting that in (9.1), scaling separates spatial scales as well by changing the velocity of ψ.

9.1.2. The Implicit Funtion Theorem. Staying with (9.1) for the moment, let us also explore why defining $x_0(t), c(t)$ can be seen a continuous, even differentiable flow satisfying chosen "non-degenerate" orthogonality conditions as for instance those in our orbital stability argument. For an application of the following result to bound state dynamics, see [**Hol11**].

LEMMA 9.1.2
For $\delta > 0$, let $\delta \le c \le \delta^{-1}$. There exists $\epsilon > 0, C > 0$ such that if the solution to (9.1) *$u(x,t) = \psi_c(x - x_0) + w$, $\|w\|_{H^1} < \epsilon$, there there exists a unique $\tilde{c}, \tilde{x}_0$ such that*

$$|x_0 - \tilde{x}_0| \le C\|w\|_{H^1}, \quad |c - \tilde{c}| \le C\|w\|_{H^1},$$

and $w = u - \psi_c(x - x_0)$ such that

$$\langle w, \psi_c(x - x_0)\rangle = \langle w, \psi_c(x - x_0)\rangle = 0.$$

Before we proceed, note this lemma is using alternative orthogonality conditions to those chosen above for a particular application and simplicity of illustration, but note that one function is even and the other odd since we know implicitly it is important to project away from the symmetric bound state and its anti-symmetric derivative.

PROOF. Define the map $\Phi(u, y, c) : H^1 \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^2$ by

$$\Phi(u, y, c) = \begin{bmatrix} \langle u - \psi_c(x-y), \psi_c(x-y)\rangle \\ \langle u - \psi_c(x-y), (x-y)\psi_c(x-y)\rangle \end{bmatrix}.$$

Looking at the Jacobian in the y, c parameters, we get

$$D_{y,c} = -\begin{bmatrix} 0 & \langle \partial_c \psi_c, \psi_c\rangle \\ -\frac{1}{2}\|\psi_c\|_{L^2}^2 & 0 \end{bmatrix},$$

which is invertible! Hence, we can define a continuous map in H^1 of $y(t), c(t)$. □

sol:cont

9.1.3. A Note on Continuity in λ. We present here a collection of results from [**Sha83**], [**Sha85**] and [**SS85**].

A soliton solution

$$-\Delta R_\omega + \omega R_\omega - f(R_\omega),$$

minimizes the Lagrangian given by

$$J_\omega(R) = \frac{1}{2}\int |\nabla R| dx + \frac{\omega}{2}\int |R|^2 dx - \int F(R) dx,$$

for $F' = f$.

Using a rescaling argument, we see that if R_ω minimizes J_ω, then

$$K_\omega(R_\omega) = \frac{d-2}{2}\int |R_\omega|^2 dx + d(\frac{\omega^2}{2}\int |R_\omega|^2 dx + \int F(R_\omega) dx) = 0.$$

Define $M_\omega = \{R \in H^1_r(\mathbb{R}^d), K_\omega(R) = 0, R \neq 0\}$.

LEMMA 9.1.3
M is a C^1 hypersurface in $H^1_r(\mathbb{R}^d)$ bounded away from 0.

PROOF. Since f is continuous, $K = K_\omega$ is a C^1 functional. If we take a function $u_0 \in H^1_r$ such that $\delta K(u_0) = 0$, it satisfies the equation

$$-(d-2)\Delta u_0 + d\omega^2 u_0 - d f(u_0) = 0.$$

Once again using a rescaling argument, we have

$$\frac{(d-2)^2}{2}\int |\nabla u_0|^2 dx + d^2\omega^2 \int |u_0|^2 dx - d^2 \int F(|u_0|) dx = 0.$$

Hence, if $K(u_0) = 0$, then

$$\frac{(d-2)^2}{2}\int |\nabla u_0|^2 dx - \frac{d(d-2)}{2}\int |\nabla u_0|^2 dx = 0.$$

Hence, $u_0 = 0$, so for $u_0 \in M$, $\delta K(u_0) \neq 0$ and M is a C^1 hypersurface. By the restrictions on the nonlinearity, we have

$$K(u) \geq \frac{d-2}{2}\int |\nabla u|^2 dx + \frac{1}{4}\int |u|^2 dx - C\int |u|^{l+1} dx,$$

where $l = 1 + 4/(d-2)$. Using Gagliardo-Niremberg for $d > 2$, we have

$$\frac{\|u\|^2_{H^1}}{4} - C\|u\|^\alpha_{H^1}$$

with $\alpha > 2$. For $\|u\|_{H^1} < \epsilon$, ϵ small, we have $K(u) > 0$, so M is bounded away from 0. □

Theorem 9.1
For some $v \neq 0$, there exists

$$d(\omega) = \inf_{v \in M_\omega} J_\omega(v),$$

and

$$d(\omega) = \{\inf \frac{1}{d}\int |\nabla v|^2 dx,\ K_\omega(v) \leq 0,\ v \neq 0\}.$$

Moreover, v satisfies a (possibly rescaled) soliton equation.

PROOF. Consider any function $v \in H_r^1$ such that $K_\omega(v) < 0$. Let $v_\beta(x) = v(x/\beta)$. Then, we have

$$K_\omega(v_\beta) = \beta^{d-2}\frac{d-2}{2}\int |\nabla v|^2 dx + \beta^d d(\frac{\omega^2}{2}\int |v|^2 dx + \int G(|v|)dx).$$

For $\beta = 1$, $K_\omega(v_1) < 0$ and for β small, $K_\omega(v_\beta) > 0$. Hence, there exists $\beta_0 \in (0,1)$ such that $K_\omega(v_{\beta_0}) = 0$ and

$$\frac{1}{d}\int |\nabla v_{\beta_0}|^2 dx = \frac{\beta_0^{d-2}}{d}\int |\nabla v|^2 dx < \frac{1}{d}\int |\nabla v|^2 dx.$$

Since $J_\omega(v) = 1/d(\int |\nabla v|^2 dx + K_\omega(v))$, then

$$\begin{aligned} d(\omega) &= \inf_{v\in M_\omega} J_\omega(v) = \inf\{1/d\int |\nabla v|^2 dx,\ K_\omega(v) = 0, v \neq 0\} \\ &= \inf\{1/d\int |\nabla v|^2 dx,\ K_\omega(v) \leq 0,\ v \neq 0\}. \end{aligned}$$

Hence, we have equivalence of the minimization problems.

Let v_k be a minimizing sequence. Then, $\int |\nabla v_k|^2 dx$ is bounded and by the properties of the nonlinearity, we have $G(\eta) > -\epsilon/2\eta^2 - C(\epsilon)\eta^{l+1}$, for $l < 1 + 4/(d-2)$. Since $K_\omega(v_k) \leq 0$,

$$0 \geq K_\omega(v_k) = \frac{d-2}{2}\int |\nabla v_k|^2 dx + d(\frac{\omega^2}{2}\int |v_k|^2 dx + \int G(|v_k|^2)dx),$$

or

$$0 \geq \frac{d-2}{2}\int |\nabla v_k|^2 dx + d(\frac{\omega^2-\epsilon}{2}\int |v_k|^2 dx - C(\epsilon)\int |v_k|^{l+1} dx).$$

Since H_r^1 is compactly embedded in L_r^p for $2 < p < 2 + 4/(d-2)$, we see that $\|v_K\|_{H^1} < \infty$ and there exists a subsequence also denoted v_k such that

$$v_k \rightharpoonup^w v_0 \in H_r^1,\ v_k \to v_0 \in L^p \text{ for } 2 < p < 2 + 4/(d-2).$$

Since weak limits are lower semicontinuous, we have

$$\begin{aligned} K_\omega(v_0) &= \frac{d-2}{2}\int |\nabla v_0|^2 dx + d(\frac{\omega^2}{2}\int |v_0|^2 dx + \int G(|v_0|^2)dx) \\ &\leq \frac{d-2}{2}\int |\nabla v_k|^2 dx + d(\frac{\omega^2}{2}\int |v_k|^2 dx + \int G(|v_k|^2)dx) = 0. \end{aligned}$$

Hence, the weak limit is actually strong and

$$d(\omega) = \inf_{v\in M_\omega} J_\omega(v) = J_\omega(v_0).$$

By Lagrange multipliers for J_ω and K_ω,

$$-\Delta v_0 + \omega^2 v_0 + f(|v_0|)v_0 = \lambda(-(d-2)\Delta v_0 + d\omega^2 v_0 + d f(|v_0|)v_0).$$

Integrating, using a minimizing/scaling argument, the applying that $K_\omega(v_0) = 0$, we have that $\lambda = 0$, proving the result. □

It is clear that $d(\omega) = 1/d\int |\nabla v|^2 dx$ where v is the minimizer from the previous theorem. For each ω, call the minimizer R_ω.

LEMMA 9.1.4
Let $\omega_1 < \omega_2$ be such that $[\omega_1,\omega_2] \subset (0,\infty)$, then $d(\omega)$ and $\int |R_\omega|^2 dx$ are uniformly bounded in $\omega \in [\omega_1,\omega_2]$.

PROOF. Since K is continuous in in ω, $d(\omega)$ is bounded for $\omega \in [\omega_1,\omega_2]$. We have $K_\omega(R_\omega) = 0$. Since $G(\eta) \geq -c\eta^{l+1}$ for η large and $G(0) = G'(0) = G''(0) = 0$, we have

$$\frac{a}{2}\eta^2 + G(\eta) > -C_a\eta^{l_0+1},$$

for and $a > 0$ and $l_0 = 1 + 4/(d-2)$. Hence,

$$\int (\frac{a}{2}|v|^2 + G(|v|))dx > -C_a \int |v|^{l_0+1} dx.$$

As $K_\omega(R_\omega) = 0$, using Gagliardo-Niremberg, we have

$$\frac{d-2}{2}\int |\nabla R_\omega|^2 dx + d\frac{(\omega^2-a^2)}{2}\int |R_\omega|^2 dx - C_a(\int |\nabla R_\omega|^2 dx)^\alpha \leq 0.$$

As a result, $\|R_\omega\|_{L^2}$ is bounded in terms of the bound on $\|\nabla R_\omega\|_{L^2}$. □

sol:lemuni **LEMMA 9.1.5** *We have that $d(\omega)$ is increasing in ω and if $\omega_1 < \omega_2$, then*

$$\begin{aligned} d(\omega_2) &< d(\omega_1) - \frac{(\omega_2^2-\omega_1^2)}{2}\int |R_{\omega_1}|^2 dx + o(\omega_1-\omega_2), \\ d(\omega_1) &< d(\omega_2) + \frac{(\omega_2^2-\omega_1^2)}{2}\int |R_{\omega_2}|^2 dx + o(\omega_1-\omega_2). \end{aligned}$$

Consequently, $d(\omega)$ is continuous in $\omega \in (0,\infty)$.

PROOF. Let $\omega_1 < \omega_2$. Then,

$$K_{\omega_1}(R_{\omega_2}) = K_{\omega_2}(R_{\omega_2}) - d\frac{(\omega_2^2-\omega_1^2)}{2}\int |R_{\omega_2}|^2 dx.$$

Since $K_{\omega_2}(R_{\omega_2}) = 0$, we have

$$K_{\omega_1}(R_{\omega_2}) < 0,$$

and

$$d(\omega_2) = \frac{1}{d}\int |\nabla R_{\omega_2}|^2 dx > \inf\{\frac{1}{d}\int |\nabla v|^2 dx,\ K_{\omega_1(v)} \leq 0,\ v \neq 0\}.$$

Hence, we have $d(\omega_2) > d(\omega_1)$.

Set $\psi_\beta(x) = R_{\omega_2}(x/\beta)$. Then,

$$\begin{aligned} K_{\omega_1}(\psi_\beta) &= \frac{(d-2)\beta^{d-2}}{2}\int |\nabla R_{\omega_2}|^2 dx \\ &+ d\beta^d(\frac{\omega_1^2}{2}\int |R_{\omega_2}|^2 dx + \int G(|R_{\omega_2}|)dx) \\ &= \frac{(d-2)\beta^{d-2}}{2}\int |\nabla R_{\omega_2}|^2 dx - \beta^d(\frac{(d-2)}{2}\int |\nabla R_{\omega_2}|^2 dx \\ &+ d\frac{(\omega_2^2-\omega_1^2)}{2}\int |R_{\omega_2}|^2 dx. \end{aligned}$$

Define $\Delta_{12} = (\omega_2^2 - \omega_1^2)/2 \int |R_{\omega_2}|^2 dx$, then

$$K_{\omega_1}(\psi_\beta) = \frac{d(d-2)}{2} d(\omega_2)\beta^{d-2} - d(\frac{d-2}{2} d(\omega_2) + \Delta_{12})\beta^d,$$

and setting

$$\beta_2 = \frac{1}{(1 + \frac{2\Delta_{12}}{(d-2)d(\omega_2)})}^{\frac{1}{2}},$$

we have $K_{\omega_1}(\psi_{\beta_2}) = 0$ and hence

$$d(\omega_1) \leq \frac{1}{d} \int |\psi_{\beta_2}|^2 dx = \beta_2^{d-2} d(\omega_2).$$

For $\omega_2 - \omega_1$ small, we have $|\Delta_{12}| < C(\omega_2^2 - \omega_1^2$ since $\|R_\omega\|_{L^2}$ is bounded. So,

$$\beta_2^{d-2} = 1 - \frac{\Delta_{12}}{d(\omega_2)} + o(\Delta_{12}),$$

and

eqn:cont1 (9.5) $$d(\omega_1) \leq d(\omega_2) - \frac{(\omega_2^2 - \omega_1^2)}{2} \|R_{\omega_2}\|_{L^2}^2 + o(\omega_2 - \omega_1),$$

which gives the first inequality in the lemma. A similar procedure using a rescaling of R_{ω_1} in K_{ω_2} gives

eqn:cont2 (9.6) $$d(\omega_2) \leq d(\omega_1) + \frac{(\omega_2^2 - \omega_1^2)}{2} \|R_{\omega_1}\|_{L^2}^2 + o(\omega_2 - \omega_1).$$

Continuity is a direct consequence of (9.5) and (9.6). □

Theorem 9.2
Select $\omega_0 \in (0,\infty)$. For ω in a neighborhood of ω_0, let R_ω be the unique ground state solution satisfying the equation

$$-\Delta R_\omega + \omega^2 R_\omega - f(R_\omega) = 0.$$

*Assume also that $L = -\Delta + \omega_0^2 - f'(R_{\omega_0})$ does not have an eigenvalue at 0. If $f \in C^2$ satisfying conditions for existence, see [**BL76**], then the map $\omega \to R_\omega$ is a C^2 map from $\mathbb{R} \to H_r^1$.*

PROOF.

CLAIM 9.1.6. *We have that the map $\omega \to R_\omega$ is continuous in a neighborhood of ω_0 with values taken in in H_r^1.*

PROOF. From Lemma 9.1.5, we know that $d(\omega)$ is continuous with respect to ω and that $\|R_\omega\|_{L^2}$ is bounded locally in terms of ω. Take (ω_j) to be a sequence converging to ω_0. Then (R_{ω_j}) is bounded in H_r^1 and a subsequence can be chosen to converge weakly in H_r^1 to some v. Then,

$$0 = K_\omega(R_\omega) = \frac{d-2}{d} \int |\nabla R_\omega|^2 dx + \frac{\omega^2}{2} \int |R_\omega|^2 + \int F(R_\omega) dx.$$

Letting $\omega_j \to \omega_0$, we see that from the lower semicontinuity of the weak limit and assumptions on F that $K_{\omega_0}(v) \leq 0$. Hence,

$$\int |\nabla v|^2 dx \geq \int |\nabla R_{\omega_0}|^2 dx.$$

By lower semicontinuity of the weak limit again, we have $\|\nabla v\|_{L^2} = \|\nabla R_{\omega_0}\|_{L^2}$, which by uniqueness implies $v = R_{\omega_0}$. Hence $R_\omega \rightharpoonup^w R_{\omega_0}$ in H^1_r. By continuity of d, we can take a strong limit in the expression for K_ω to see that $R_\omega \to R_{\omega_0}$ in L^2. Hence, $R_\omega \to R_{\omega_0}$ in H^1. □

CLAIM 9.1.7. *In a neighborhood of $R_{\omega_0} \in H^1_r$, all solutions lie on a C^1 curve.*

PROOF. We have $-\Delta R + \omega^2 R + f(R) = 0$. Define a function

$$G(\omega, u) = u - (\omega^2 - \Delta)^{-1} f(u),\ u \in H^1_r.$$

The function $G(\omega, u) \in H^1_r$ since $u \in H^1_r$, $f(u) \in L^{2d/(d+2)}$ using the assumptions on the nonlinearity and $(\omega^2 - \Delta)^{-1} f(u) \in H^1$ by Sobolev embeddings. The function G is actually a C^1 operator from $(0,\infty) \times H^1_r$ into H^1_r. Clearly, $G(\omega_0, R_{\omega_0}) = 0$.

Look at the operator given by $L = -\Delta + \omega_0^2 + f'(R_{\omega_0})$. Using Weyl's criteria, we see the continuous spectrum is precisely (ω_0^2, ∞) and L has only discrete spectrum to the left of ω_0^2. Via our uniqueness assumption, we have assumed that L is invertible. Therefore, the compact operator

$$(\lambda_0 - \Delta)^{-\frac{1}{2}} f'(R_{\omega_0})(\lambda_0 - \Delta)^{-\frac{1}{2}}$$

on L^2_r does not have -1 in its spectrum. So,

$$\frac{\partial G}{\partial u}(\omega_0^2, R_{\omega_0}) = I + (\omega_0^2 - \Delta)^{-1} f'(R_{\omega_0}),$$

acting from H^1_r to H^1_r is also invertible. Using the implicit function theorem, the solutions of $G(\omega, u) = 0$ in a neighborhood of ω_0, R_{ω_0} form a C^1 curve in $(0,\infty) \times H^1_r$. □

Using the claims, we have that $\omega \to R_\omega$ is a C^1 curve near $\omega = \omega_0$, $R_\omega = R_{\omega_0} \in H^1_r$. Since R_ω is bounded in H^1_r, by the aymptotic analysis at ∞, on a neighborhood of ω_0 the functions

$$|x|^{\frac{d-1}{2}} R_\omega(x)$$

are bounded independently of ω and x. Bootstrapping using the soliton equation, we have that R_ω is uniformly bounded. Since

eqn:diff

$$[-\Delta + \omega^2 + f'(R_\omega)] \frac{\partial R_\omega}{\partial \omega} = -2\omega R_\omega. \tag{9.7}$$

Since L is invertible, $\frac{\partial R_\omega}{\partial \omega}$ is uniformly bounded. Hence $[\frac{\partial R_\omega}{\partial \omega}]^2$ is bounded in H^1. Taking difference quotients in (9.7), we have that R_ω is C^2 with values in H^1. Then, $\frac{\partial^2 R_\omega}{\partial \omega^2}$ exists and satisfies

$$[-\Delta + \omega^2 + f'(R_\omega)] \frac{\partial^2 R_\omega}{\partial \omega^2} = -4\omega R_\omega - 2R_\omega - f''(R_\omega)[\frac{\partial R_\omega}{\partial \omega}]^2.$$

□

9.2. Exercises

CHAPTER 10

Dispersive Estimates - Resolvent Estimates

The crucial result for proving the Strichartz estimates for the Schrödinger equation is the following lemma. This is also utilized heavily to get bounds for the distorted Fourier bases. The discussion we present here comes from [**Ste93a**].

LEMMA 10.0.1
Given $0 < \gamma < d$, $1 < p < q < \infty$, *and* $\frac{1}{q} = \frac{1}{p} - \frac{d-\gamma}{d}$, *then the following holds*

$$\|f * (|y|^{-\gamma})\|_{L^q(\mathbb{R}^d)} \leq A_{p,q}\|f\|_{L^p(\mathbb{R}^d)}.$$

PROOF. We have

$$\begin{aligned}[f * (|y|^{-\gamma})](x) &= \int f(x-y)|y|^{-\gamma}dy \\ &= \int_{|y|<R} f(x-y)|y|^{-\gamma}dy + \int_{|y|\geq R} f(x-y)|y|^{-\gamma}dy.\end{aligned}$$

The first integral is convolution of f with the function $|y|^{-\gamma}\chi_{B(R)}(y)$, for χ a characteristic function. This function is radial, decreasing and integrable.

CLAIM 10.0.2. *For* ϕ *radial and decreasing,*

$$\sup_{t>0}|f * \phi_t(x)| \leq Mf(x)\int \phi(y)dy,$$

where $\phi_t(x) = t^{-d}\phi(x/t)$ *and* M *is the standard maximal operator such that*

$$(Mf)(x) = \sup_{r>0} c_d r^{-d}\int_{|y|<r}|f(x-y)|dy.$$

PROOF. It suffices to prove

$$|f * \phi(x)| \leq Mf(x),$$

where $\int \phi dy = 1$ since the result follows from scaling. To begin, let $\phi = \sum_{j=1}^N a_j\chi_{B_j}$, where each a_j is positive and each B_j is a ball centered at the origin. Since $\sum_j a_j|B_j| = 1$ and $(f * \chi_{B_j})(x) \leq |B_j|Mf(x)$, the result follows immediately. Since any radial, decreasing ϕ can be approximated by finite sums of this type, the result follows in the limit. □

Using the claim, we have

$$\int_{|y|<R} f(x-y)|y|^{-\gamma}dy \leq (Mf)(x)\int_{|y|<R}|y|^{-\gamma}dy = cR^{d-\gamma}(Mf)(x).$$

Using Hölder's Inequality, we have

$$\int_{|y|\geq R} f(x-y)|y|^{-\gamma}dy \leq \|f\|_{L^p(\mathbb{R}^d)}\||y|^{-\gamma}\chi_{B(R)^c}\|_{L^{p'}(\mathbb{R}^d)},$$

where $\| |y|^{-\gamma} \chi_{B(R)^c} \|_{L^{p'}} = cR^{-\frac{n}{q}}$ by the requirement that $\frac{1}{q} = \frac{1}{p} - \frac{n-\gamma}{n}$.

Summing the two integrals, we have

$$|(f * |y|^{-\gamma})(x)| \leq C[Mf(x)R^{n-\gamma} + \|f\|_{L^p} R^{-\frac{n}{q}}.$$

Let R be chosen such that

$$\frac{Mf(x)}{\|f\|_{L^p}} = R^{-n+\gamma-\frac{n}{q}} = R^{-\frac{n}{p}}.$$

Then,

$$|(f * |y|^{-\gamma})(x)| \leq [(Mf)(x)]^{\frac{p}{q}} \|f\|_{L^p}^{1-\frac{p}{q}}.$$

and the result follows from standard L^p bounds on maximal operators, as proved below. □

10.0.1. Maximal Functions.

DEFINITION 10.0.3. *We define the centered maximal function of f:*

$$(Mf)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.$$

Similarly, we define the uncentered maximal function:

$$(\widetilde{M}f)(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy.$$

Observe $\widetilde{M}f(x) \geq Mf(x)$.

LEMMA 10.0.4 (Properties of Maximal Functions). *(i) For $f \in L^p$,*

max-i (10.1) $$\|\widetilde{M}f\|_{L^p} \leq C\|f\|_{L^p}, \ 1 < p \leq \infty.$$

(ii) For $f \in L^1$,

max-ii (10.2) $$\left|\{|\widetilde{M}f| > \alpha\}\right| \leq \frac{C}{\alpha}\|f\|_{L^1}.$$

In order to prove these results, we need the following lemma.

l:v LEMMA 10.0.5 (Vitali Covering Lemma). *Suppose $\{B_k\}$ is a finite family of balls, then there exists a disjoint subset of the balls $\{B_\alpha\}$ such that*

bk-lem (10.3) $$\left|\bigcup B_k\right| \leq C \sum_\alpha |B_\alpha|.$$

PROOF. Begin by choosing the largest ball, say B_{α_1}. Then, choose B_{α_2} the next largest ball such that $B_{\alpha_1} \cap B_{\alpha_2} = \emptyset$. Continue this process.

We have

$$\bigcup B_k \subset \bigcup_i 4B_{\alpha_i}.$$

Suppose that B_j is in the set $\{B_k\}$, but that $B_j \neq B_{\alpha_i}$ for any i. Then, B_j must intersect a larger ball B_α. By the triangle inequality, we have $B_j \subset 4B_\alpha$ since for $x \in B_\alpha$, $y \in B_j$, we have:

$$d(x,y) \leq d(x,x_\alpha) + d(x_\alpha, y) \leq R + d(y,y_j) + d(x_\alpha, y_j) \leq 4R.$$

Thus, Equation (10.3) follows from properties of measures. □

PROOF. **(ii)** $x \in \{|\widetilde{M}f| \geq \alpha\}$ implies there exists B_x such that $x \in B_x$ and

$$\frac{1}{|B_x|}\int_{B_x} |f|dy \geq \alpha.$$

Then

$$\{|\widetilde{M}f| \geq \alpha\} \subset \bigcup_{x\in\{|\widetilde{M}f|\geq\alpha\}} B_x.$$

By the Vitali Covering Lemma, locally there exists $\{B_{x_k}\}$ such that $B_{x_i} \cap B_{x_j} = \emptyset$ and

$$\begin{aligned} |\{|\widetilde{M}f| \geq \alpha\}| &\leq 4\sum_k |B_{x_k}| \\ &\leq \frac{4}{\alpha}\sum_k \int_{B_{x_k}} |f|dy \\ &\leq \frac{4}{\alpha}\|f\|_{L^1}. \end{aligned}$$

This result can then be extended by looking at $\{|\widetilde{M}f| \geq \alpha\}$ as the limit of an increasing sequence of compact sets.

(i) Let $f \in L^p$. Write $f = f_1 + f_2$ where:

$$\begin{aligned} (a) \quad f_1 &= f \cdot \chi_{\{f\leq\frac{\alpha}{2}\}} \in L^p, \text{ and} \\ (b) \quad f_2 &= f \cdot \chi_{\{f\geq\frac{\alpha}{2}\}} \in L^1. \end{aligned}$$

Then $\widetilde{M}f(x) > \alpha \Rightarrow \widetilde{M}(f_1+f_2)(x) > \alpha$, so $\widetilde{M}f_2(x) > \frac{\alpha}{2}$. Thus,

$$\begin{aligned} \left|\{|\widetilde{M}f| > \alpha\}\right| &\leq \left|\{|\widetilde{M}f_2| > \frac{\alpha}{2}\}\right| \\ &\leq \frac{C}{\alpha}\|f_2\|_{L^1} \\ &\leq \frac{C}{\alpha}\int_{\frac{\alpha}{2}}^{\infty} \left|\{|f| > h\}\right| dh + \left|\{|f| > \alpha\}\right| \end{aligned}$$

which implies

$$\begin{aligned} \left\|\widetilde{M}f\right\|_{L^p}^p &= \int_0^\infty p\alpha^{p-1}\left|\{\widetilde{M}f(x) > \alpha\}\right| d\alpha \\ &\leq \int_0^\infty p\alpha^{p-2}\int_{\frac{\alpha}{2}}^\infty \left|\{|f| > h\}\right| dh + \int_0^\infty p\alpha^{p-1}\left|\{|f| > \alpha\}\right| \\ &\leq \|f\|_{L^p} + \int_0^\infty \int_0^h p\alpha^{p-2} d\alpha \left|\{|f| > h\}\right| dh \\ &= C\|f\|_{L^p}^p. \end{aligned}$$

□

We present here several results from [**RS78b**], [**HS96**] and [**Tay97b**].

DEFINITION 10.0.6. *Let A be a linear operator on a Banach space X. The spectrum of A, denoted $\sigma(A)$ is the set of all $\lambda \in \mathbb{C}$ such that $A-\lambda$ is not invertible. This can happen in the following ways:*
(1) $Ker(A-\lambda) \neq 0$,
(2) $Ker(A-\lambda) = 0$, and $Ran(A-\lambda)$ is dense,
(3) $Ker(A-\lambda) = 0$, but $Ran(A-\lambda)$ is not dense.

The resolvent set of an operator A, denoted $\rho(A)$ is the set of all $\lambda \in \mathbb{C}$ such that $A-\lambda$ is invertible. If $\lambda \in \rho(A)$, the resolvent of A is given by $R_A(\lambda) = (A-\lambda)^{-1}$. The discrete spectrum of A, $\sigma_d(A)$ is given by the set of finite multiplicity, isolated points of $\sigma(A)$. The residual spectrum of an operator A, denoted $\sigma_{res}(A)$ is given by the set $\{\lambda \in \mathbb{C} | Ker(A-\lambda) = \{0\}$ and $Ran(A-\lambda)$ is not dense$\}$. The essential spectrum is given by $\sigma_{ess} = \sigma(A) - \sigma_d(A)$.

Theorem 10.1
The resolvent set $\rho(A)$ is an open subset of $\mathbb{C}$ ($\sigma(A)$ is then closed) and $R_A(\lambda)$ is an analytic operator-valued funtion of λ on $\rho(A)$.

PROOF. To show that $\rho(A)$ is open, assume $\lambda_0 \in \rho(A)$ and let $\lambda \in \mathbb{C}$ such that

$$A - \lambda = \|(A-\lambda_0)^{-1}\|^{-1}.$$

Then, we have

spec:eqn1 (10.4) $$A - \lambda = (A-\lambda)(1-(A-\lambda_0)^{-1}(\lambda-\lambda_0)).$$

By construction, $\|(A-\lambda_0)^{-1}(\lambda-\lambda_0)\| < 1$, hence using Neumann series and operator norms in Banach spaces, we have $1-(A-\lambda_0)^{-1}(\lambda-\lambda_0)$ is invertible, so $A-\lambda$ is invertible. Thus, for any $\lambda_0 \in \rho(A)$, there exists an open, symmetric ϵ-neighborhood, I, such that for any $\lambda \in I$, $A-\lambda$ is invertible. Note, ϵ can be taken to be $\|(A-\lambda_0)^{-1}\|^{-1}$. Hence, the resolvent set $\rho(A)$ is open.

To show that $R_A(\lambda)$ is analytic at λ_0, use (10.4) to say that

$$(A-\lambda)^{-1} = (A-\lambda_0)^{-1}(1-(\lambda-\lambda_0)(A-\lambda_0)^{-1})^{-1}.$$

Taking

$$|\lambda-\lambda_0|\|(A-\lambda_0)^{-1}\|^{-1} < 1,$$

we again use Neumann series to say

$$\begin{aligned}(A-\lambda)^{-1} &= (A-\lambda_0)^{-1}\sum_{k=0}^{\infty}(\lambda-\lambda_0)^k(A-\lambda_0)^{-k} \\ &= \sum_{k=0}^{\infty}(\lambda-\lambda_0)^k(A-\lambda_0)^{-k-1}.\end{aligned}$$

Since this is norm convergent, we have a power series representation and hence analyticity. □

DEFINITION 10.0.7. *We define the adjoint of an operator A with domain $D(A)$, denoted A^*, as a map $A^* : D(A*) \to \mathcal{H}$ satisfying the following relation*

$$\langle Ax, y\rangle = \langle y, A^*x\rangle,$$

for all $y \in D(A)$, $x \in D(A^*)$, where

$$D(A^*) = \{x \in \mathcal{H} | |\langle Ay, x\rangle| \le C_x\|y\| \text{ for } C_x \text{ indepent of } y \text{ and all } y \in D(A)\}.$$

Theorem 10.2
Let $\mathcal{H}$ be a Hilbert space. For any densely defined linear operator A,

$$\overline{Ran(A)} \oplus Ker(A^*) = \mathcal{H}.$$

PROOF. It suffices to prove that $Ker(A^*)$ is the orthogonal complement of $Ran(A)$, then use the following lemma.

LEMMA 10.0.8
Let M be any closed subspace of $\mathcal{H}$, and let $M^\perp$ be its orthogonal complement. Then, any $x \in \mathcal{H}$ can be written uniquely as $x = y + z$ with $y \in M$ and $z \in M^\perp$.

PROOF OF LEMMA. Let y be the projection of x onto M, set $z = x - y$. Let $t \in \mathbb{R}$ and $m \in M$. If $d = \|x - y\| = \inf_{n\in M} \|x - n\|$, then

$$d^2 \leq \|x - (y + tm)\|^2 = \|x - y\|^2 + t^2\|m\|^2 - 2t\mathfrak{Re}\langle x - y, m\rangle.$$

Hence, for all t,

$$t^2\|m\|^2 - 2t\mathfrak{Re}\langle x - y, m\rangle \geq 0.$$

However, this implies that $\langle z, m\rangle = 0$. Uniqueness follows from standard arguments. □

Now, let $u \in Ran(A)$ and $v \in Ker(A^*)$. Then, there exists $f \in D(A)$ such that $u = Af$ and

$$\langle u, v\rangle = \langle Af, v\rangle = \langle f, A^* v\rangle = 0.$$

Hence, $Ker(A^*) \subset (Ran(A))^\perp$. Now, let $w \in (Ran(A))^\perp$. For any $u = Af \in Ran(A)$, we have

$$0 = \langle u, w\rangle = \langle Af, w\rangle = \langle f, A^* w\rangle.$$

As $D(A)$ is dense, $A^* w = 0$, so $(Ran(A))^\perp \subset Ker(A^*)$. □

PROPOSITION 10.0.9. *Let A be a densely defined operator. Then,*
(1) A^* is always closed,
(2) if A is closed, then $D(A^*)$ is dense,
(3) if A is closed, then $A^{**} = A$.

Theorem 10.3
Let A be self-adjoint. If for some $M > 0$ and for all $u \in D(A)$,

$$\|(\lambda - A)u\| \geq M\|u\|,$$

then $\lambda \in \rho(A)$. Moreover,

$$\{z \in \mathbb{C} | |z - \lambda| < M\} \subset \rho(A).$$

Theorem 10.4
Let A be self-adjoint. Then,
(1) $\sigma(A) \subset \mathbb{R}$,
(2) $\sigma_{res}(A) = \emptyset$,
(3) eigenvectors corresponding to distinct eigenvalues are orthogonal.

COROLLARY 10.0.10. *Let A be self-adjoint and $z \in \mathbb{C}$, $\mathfrak{Im}(z) \neq 0$. Then,*

$$\|R_A(z)\| \leq |\mathfrak{Im} z|^{-1}.$$

Theorem 10.5
Let A be self-adjoint, $\lambda \in \rho(A)$. Then,

$$\|R_A(\lambda)\| \leq [dist(\lambda, \sigma(A))]^{-1}.$$

DEFINITION 10.0.11. *The Riesz projection for an operator A is given by the contour integral*

$$P_{\lambda_0} = \frac{1}{2\pi i} \oint_{\Gamma_{\lambda_0}} R_A(\lambda) d\lambda,$$

where $\lambda_0 \in \sigma_d(A)$ and Γ_{λ_0} is a contour which does not intersect $\sigma(A)$ anywhere except λ_0. Note, P_{λ_0} is a projection onto the eigenspace at λ_0. Hence, if we allow Γ to encircle all of $\sigma(A)$, we can say

$$f(A) = \frac{1}{2\pi i} \oint_{\Gamma} f(\lambda) R_A(\lambda) d\lambda,$$

for sufficiently regular f.

Theorem 10.6
Let A be a self-adjoint operator and B be a closed operator such that:
(a) For some (and hence all) $z \in \rho(B) \cap \rho(A)$, $(A-z)^{-1} - (B-z)^{-1}$ is compact,
and either
(b1) $\sigma(A) \neq \mathbb{R}$ and $\rho(B) \neq \emptyset$
or
(b2) There are points of $\rho(B)$ in both the upper and lower half-planes.
Then, $\sigma_{ess}(A) = \sigma_{ess}(B)$.

Theorem 10.7 (Weyl's Theorem)
Let A be a self-adjoint operator and let C be a relatively compact perturbation of A. Then $\sigma_{ess}(A) = \sigma_{ess}(B)$.

Note that from here, for V with sufficient decay, it is clear that $\sigma_{ess}(-\Delta + V) = \sigma_{ess}(-\Delta) = [0, \infty)$.

We begin with some preliminaries from the notes of Tataru [**Tat00**] and [**Tay97b**].

10.0.2. Pseudodifferential Operators. In this section, we discuss properties of Pseudodifferential Operators (PDO's). In $\mathbb{R}^n$, we define the Weyl quantization of an operator $a(x, hD)$ where $a \in \mathscr{S}(\mathbb{R}^{2n})$, $a = a(x, \xi)$ by:

$$a(x, D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y, \xi \rangle} a(x, \xi) u(y) dy d\xi,$$

for $u \in \mathscr{S}$ and a symbol class by:

$$S^k_\delta(m) = \{a \in C^\infty(\mathbb{R}^{2n}) \| |\partial^\alpha a| \leq C_\alpha |\xi|^{-\delta|\alpha|-k} m \text{ for all multi-indices } \alpha\},$$

where $m : \mathbb{R}^{2n} \to (0, \infty)$ is is an order function, i.e. there exist constants C, N such that

$$m(z) \leq C\langle z - w \rangle^N m(w).$$

We also define

$$S^{-\infty}_\delta(m) = \bigcap_{k=-\infty}^{\infty} S^k_\delta(m).$$

For $k, \delta = 0$ we write simply $S(m)$.

Look at $P(x,D)u = f$. One possible approach to such problems involves approximate solutions or parametrices:

$$K \Rightarrow P(x,D)K = I + \text{error}.$$

If the error is small, iterative methods give an exact solution. If the error is smooth, the problem is solved modulo smooth functions.
As seen in [**Str03**], let R = error from above. Then, to solve the equation locally, we need to find g such that $(PK) * g = g + (K * g)$ in some set U. Thus, we have changed the problem to solving the integral equation $g + K * g = f$ on U. Choose a cut-off function ψ which is one on U and zero off V where $U \subset V$. Choose ϕ a cut-off function on V. Note that $\phi\psi = \psi$. Hence, we need only solve the system

$$g + \psi(K * (\phi g)) = \psi f.$$

We define

$$\tilde{K}g = \int \psi(x)K(x-y)\phi(y)\, dy,$$

where $\tilde{K}g$ is supported on V. Then, we have

$$g = \sum_{k=0}^{\infty} (-1)^k (\tilde{K})^k (\psi f),$$

which converges for for U, V small enough to make $\|\tilde{K}\| < 1$.

Let us approach the problem more formally. Consider $u \to P(x,D)u$, and suppose that $P(x,D)u = c(x)D_x^\alpha u$ which implies

$$\begin{aligned} c(x)D^\alpha u &= c(x)\mathscr{F}^{-1}\mathscr{F}D^\alpha u \\ &= c(x)\mathscr{F}^{-1}\xi^\alpha \hat{u} \\ &= \int c(x)\xi^\alpha e^{ix\xi}\hat{u}d\xi. \end{aligned}$$

Thus for $P(x,D) = \sum c_\alpha(x)D^\alpha$, we have $p(x,\xi) := \sum c_\alpha \xi^\alpha$, which we call the symbol of the operator. That is,

$$P(x,D)u = \int p(x,\xi)e^{ix\xi}\hat{u}(\xi)d\xi.$$

Suppose now that $p(x,\xi)$ is a smooth function on $\mathbb{R}^{2n}$. Define $P(x,D)u$ by

p(x,d) (10.5)
$$P(x,D)u = \int p(x,\xi)e^{ix\xi}\hat{u}(\xi)d\xi.$$

Since $u \in \mathscr{S} \Rightarrow \hat{u} \in \mathscr{S}$, if

$$|\partial_x^\beta \partial_\xi^\alpha p(x,\xi)| \le C_\alpha(1 + |\xi| + |x|)^{N_{\alpha,\beta}},$$

then

$$P(x,D) : \mathscr{S} \to \mathscr{S}.$$

We want to use the Fourier transform to get an idea of what the adjoints will look like. Let $Pu(x) = \int p(x,\xi)e^{i\xi(x-y)}u(y)dyd\xi$.

$$\begin{aligned}\langle P(x,D)u,v\rangle &= \int\int p(x,\xi)\hat{u}(\xi)e^{ix\xi}\bar{v}(x)d\xi dx\\ &= \int \hat{u}\int\overline{\bar{p}(x,\xi)e^{-ix\xi}v(x)}dxd\xi\\ &= \int u(y)\overline{\mathscr{F}^{-1}\int(\bar{p}(x,\xi)e^{-ix\xi}v(x))dxdy}\\ \Rightarrow P^*v(y) &= \int\int\bar{p}(x,\xi)e^{-i(x-y)\xi}v(x)dxd\xi.\end{aligned}$$

Thus if $P(x,D)=\sum c_\alpha(x)D^\alpha$, $(P(x,D))^*=\sum D^\alpha\overline{c_\alpha(x)}$.

REMARK 10.0.12. *We say* $(P(x,D))^*=\bar{P}(D,x)$.

10.0.3. Weyl Calculus. We make a slight variation in the definition of $P(x,D)$ in (10.5). Set

p-weyl

$$p^w(x,D)(u)=\int p\Big(\frac{x+y}{2},\xi\Big)e^{i\xi(x-y)}u(y)dyd\xi. \tag{10.6}$$

The Weyl calculus is extremely useful since $(p^w(x,D))^*=\bar{p}^w(x,D)$. We will see more on functional calculi in section 9.

REMARK 10.0.13. *Note that Pseudodifferential Operators do not necessarily commute.*

EXAMPLE 10.0.14. $[x,D]=xD-Dx=i$.

Thus, we need a functional calculus. That is, we need to determine how pseudodifferential operators act under compositions. First, we prove the uncertainty principle.

10.0.4. The Uncertainty Principle. Recall that functions of the form $e^{ix\xi_0}$ are purely oscillatory, whereas δ_{x_0} are purely spatial. Suppose we see a concentration around $(0,0)$ in phase space. Note that $x\delta_0=0$. Given a function u, $\|xu\|_{L^2}$ measures the concentration around $x=0$, and $\|\xi\hat{u}\|_{L^2}$ measures the concentration around $\xi=0$.

LEMMA 10.0.15 (Uncertainty Principle).

$$\|xu\|_{L^2}\|Du\|_{L^2}\geq\frac{n}{2}\|u\|_{L^2}^2$$

in $\mathbb{R}^n$.

PROOF.

$$\begin{aligned}\left|\int_{\mathbb{R}^n}n|u|^2dx\right| &= \left|\int_{\mathbb{R}^n}x\cdot\partial_x|u|^2dx\right|\\ &= \left|\int_{\mathbb{R}^n}x(u\cdot\partial_x\bar{u}+\bar{u}\cdot\partial u)dx\right|\\ &\leq 2\|xu\|_{L^2}\|Du\|_{L^2}.\end{aligned}$$

□

Note that equality holds if u is a Gaussian.

10.0.5. L^2 boundedness of Pseudodifferential Operators.

t:6 **Theorem 10.8** *Suppose $m(x,\xi)$ is supported on $B(0,1)\times\mathbb{R}^n$ and that $|\partial_x^\alpha m(x,\xi)| \le C_\alpha$, then $m(x,D): L^2 \to L^2$.*

PROOF. We know $B(0,1) \subset Q(0,1)$ where $Q(a,r) = \{x \,:\, |x_i - a_i| \le r,\ \forall\ 1 \le i \le n\}$, so we use Fourier series in $L^2(Q(0,1))$. Let ϕ_j be a basis which has the property that each entry is bounded in L^2 and L^∞. Then, $m(x,\xi) = \sum_j \phi_j(x) m_j(\xi)$, where $m_j(\xi) = \int m(x,\xi)\bar{\phi}_j(x)dx$. Using integration by parts and our assumptions on m, $|m_j(\xi)| \le Cj^{-N}, \forall N$. Therefore,

$$m(x,D) = \sum_{j=1}^{\infty} \phi_j(x) m_j(D)$$

and

$$\|m(x,D)\|_{L^2\to L^2} \le \sum_{j=1}^{\infty} \|\phi_j\|_{L^\infty}\|m_j\|_{L^\infty} < \infty.$$

□

t:7 **Theorem 10.9** *Suppose $m(x,\xi)$ is:*

- *bounded*
- *smooth in ξ*
- *homogeneous of order 0 in ξ.*

Then, $m(x,D): L^2 \to L^2$.

PROOF. Since m is homogeneous in ξ, we have $|\partial_\xi^\alpha m| \le c_\alpha(\frac{1}{|\xi|})^\alpha$, $m(x,\xi) = m(x,\frac{\xi}{|\xi|})$. We choose ϕ_j to be an L^2 basis of eigenfunctions for $-\Delta_{S^{n-1}}$. The ϕ_j are restrictions of homogeneous harmonic polynomials of degree j.

$$\begin{aligned} \|\phi_j\|_{L^\infty} &\le j^{-cN}; \\ m(x,\frac{\xi}{|\xi|}) &= \sum \phi_j(\frac{\xi}{|\xi|}) m_j(x), \end{aligned}$$

where $m_j(x) = \int m(x,\frac{\xi}{|\xi|})\bar{\phi}_j(\frac{\xi}{|\xi|})dS^{n-1}$. Thus we have rapid decay $|m_j(x)| \le j^{-N}$.

$$\begin{aligned} m(x,D) &= \sum m_j(x)\phi_j(\frac{D}{|D|}) \Rightarrow \\ \|m(x,D)\|_{L^2\to L^2} &\le \sum \|m_j\|_{L^\infty}\|\phi_j\|_{L^\infty} < \infty. \end{aligned}$$

□

10.0.6. Symbol Classes.

DEFINITION 10.0.16 (Symbol Classes). *We say a symbol $a(x,\xi)$ is the the symbol class $S^m_{\rho,\delta}$ if and only if $a \in \mathscr{C}^\infty(\mathbb{R}^{2n})$ and*

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le c_{\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|},$$

where m is called the order of a, and $0 \le \delta \le \rho \le 1$.

EXAMPLE 10.0.17. *If $|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \le c_{\alpha,\beta}$, then $p \in S^0_{0,0}$.*

t:10 **Theorem 10.10** *If $a \in S^0_{\rho,\rho}$, $0 \le \rho \le 1$, then $a(x,D)$ is L^2 bounded.*

PROOF. Let us use a partition of unity: $\mathbb{R}^n = \cup_j A_j$, where

$$A_j = \{2^{j-1} \le |\xi| \le 2^{j+1}\},\ A_0 = B(0,1).$$

Then,

$$1 = \sum_{j=0}^{\infty} \phi_j(\xi),\ \text{supp}\ (\phi_j) = A_j.$$

$|\partial^\alpha \phi_j| \le c_\alpha 2^{-|\alpha| j} \approx c_\alpha (1+|\xi|)^{-|\alpha|}$ since ϕ_j moves from 0 to 1 in a region the size of 2^j. Write $a(x,\xi) = \sum_j a_j(x,\xi)$, where $a_j(x,\xi) = a(x,\xi)\phi_j(x,\xi)$. Then

$$\left|\partial^\alpha_\xi \partial^\beta_x a_j(x,\xi)\right| \le c_{\alpha,\beta} 2^{j\rho(|\beta|-|\alpha|)}.$$

Step 1: Boundedness of $a_j(x,D)$.

Let us introduce a change of variables $x = \lambda y$, $\partial_x = \frac{1}{\lambda}\partial_y$ so that $a_j(x,D) \to a_j(\lambda y, \frac{D}{\lambda})$. In other words, we dilate space by λ and contract the frequency by $\frac{1}{\lambda}$. Then we have

$$\partial^\alpha_y \partial^\beta_\nu a_j((\lambda y, \frac{\nu}{\lambda})| \le (\lambda 2^{j\rho})^{(|\beta|-|\alpha|)}.$$

If $\lambda = 2^{-j\rho}$, then $a_j \in S^0_{0,0} \Rightarrow a_j$ is L^2 bounded.

Step 2: Orthogonality of the a_j's (Summing the a_j's).

We will need the Cotlar-Stein Lemma.

l:c-s LEMMA 10.0.18 (Cotlar-Stein). *Let $A = \int_M A_x dx$, where $A_x : H \to H$. If*

$$\int \|A^*_x A_y\|^{\frac{1}{2}}_{H\to H} dx \le C$$

and

$$\int \|A_x A^*_y\|^{\frac{1}{2}}_{H\to H} dx \le C,$$

then $\|A\|_{H\to H} \le C$.

PROOF OF LEMMA 10.0.18. We will use the following facts from functional analysis:

Fact 1: $\|A\|^2 = \|AA^*\|$

Fact 2: If B is a self-adjoint operator, then $\|B\|^n = \|B^n\|$.

Thus, we have $\|A\|^{2n} = \|A^*A\|^n = \|(A^*A)^n\|$, where

$$(A^*A)^n = \int_{M^{2n}} A^*_{x_1} A_{x_2} A^*_{x_3} A_{x_4} \cdots A^*_{x_{2n-1}} A_{x_{2n}} dx.$$

But we also have

$$\|A^*_{x_1} A_{x_2} A^*_{x_3} A_{x_4} \cdots A^*_{x_{2n-1}} A_{x_{2n}}\| \le \|A^*_{x_1} A_{x_2}\| \cdots \|A^*_{x_{2n-1}} A_{x_{2n}}\|$$

and

$$\begin{aligned} &\|A^*_{x_1} A_{x_2} A^*_{x_3} A_{x_4} \cdots A^*_{x_{2n-1}} A_{x_{2n}}\| \le \\ &\le \quad \|A^*_{x_1}\| \|A_{x_2} A^*_{x_3}\| \cdots \|A_{x_{2n-2}} A^*_{x_{2n-1}}\| \|A_{x_{2n}}\|. \end{aligned}$$

Thus, we can say

$$\|A^*_{x_1} A_{x_2} A^*_{x_3} A_{x_4} \cdots A^*_{x_{2n-1}} A_{x_{2n}}\| \le \|A^*_{x_1}\|^{\frac{1}{2}} \|A^*_{x_1} A_{x_2}\|^{\frac{1}{2}} \cdots \|A_{x_{2n}}\|^{\frac{1}{2}}.$$

Now, let us make some simplifying assumptions:

1. M has a finite measure m.

2. $\|A_x\|$ is uniformly bounded by N.
Then,

$$\begin{aligned}\|(A^*A)^n\| &\leq \int_{M^{2n}} \|A^*_{x_1}\|^{\frac{1}{2}} \|A^*_{x_1}A_{x_2}\|^{\frac{1}{2}} \cdots \|\|A_{x_{2n}}\|^{\frac{1}{2}} dx \leq NC^{2n-1}m \\ &\Rightarrow \|A\| \leq (mN)^{\frac{1}{2n}} C^{1-\frac{1}{2n}} \overset{n\to\infty}{\to} C.\end{aligned}$$

For complete details, see [**Ste93a**]. □

Now, continuing with our proof of the theorem, let $A = a(x,D)$, $a = \sum A_j$, $A_j = a_j(x,D)$. Thus, $A_j^* = a_j(\bar{D},x)$, and

$$\begin{aligned}(A_jA_k^*u)(x) &= a_j(x,D)a_k(\bar{D},x)u(x) \\ &= \int a_j(x,\xi)e^{i\xi x}e^{-i\xi y}a_k(\bar{y},\xi)u(y)dyd\xi \\ &= 0\end{aligned}$$

when $|j-k|>1$ as a_j, a_k have different regions of support. Thus, we get $\sum_{j=k-1}^{k+1}\|A_jA_k^*\|^{\frac{1}{2}} \leq C$.

LEMMA 10.0.19 (Rapid Decay of Mixed Terms). $|j-k|>1 \Rightarrow \|A_j^*A_k\| \leq (2^j+2^k)^{-N}$.

PROOF. Suppose $j<k-1$. We have $A_j^*A_k = A_j^*S_k(D)A_k + A_j^*(1-S_k(D))A_k$ where S_k selects a ball inside the support of A_{k-2}. Now, A_j^* is L^2 bounded, while $S_k(D)$ and A_k have disjoint supports. Similarly, A_k is L^2 bounded, while A_j^* and $(1-S_k(D))$ have disjoint supports. Thus,

$$\begin{aligned}\sum_{j=0}^{\infty}\|A_j^*A_k\|^{\frac{1}{2}} &\leq \sum_{|j-k|>1}(2^j+2^k)^{-N} \\ &\leq C,\end{aligned}$$

if we can properly bound $S_k(D)a_k(x,D)$ and $a_j(\bar{D},x)(1-S_k(D))$. To do this, let us use the Bargmann Transform and scaling:

$$S_k\left(\frac{D}{\lambda}\right)a_k\left(\lambda x,\frac{D}{\lambda}\right) = T^*TS_k\left(\frac{D}{\lambda}\right)T^*Ta_k\left(\lambda x,\frac{D}{\lambda}\right)T^*T.$$

We have the following two kernels to work with,

$$\begin{aligned}K_1(\xi_1,x_1,\xi_2,x_2) &= (Ta_k(\lambda x,D/\lambda)T^*) \\ K_2(\xi_2,x_2,\xi_3,x_3) &= (TS_k(D/\lambda)T^*).\end{aligned}$$

$$\begin{aligned}|K_1(\xi_1,x_1,\xi_2,x_2)| &\leq \left[1+|\xi_1-\xi_2|+|x_1-x_2|+\text{dist}\left((x_2,\xi_2),\text{supp } a_k\right)\right]^{-N}. \\ |K_2(\xi_2,x_2,\xi_3,x_3)| &\leq [1+|\xi_2-\xi_3|+|x_2-x_3|+\text{dist}\left((x_2,\xi_2),\text{supp } S_k\right)]^{-N}.\end{aligned}$$

Thus,

$$\begin{aligned}&|K(\xi_1,x_1,\xi_3,x_3)| \leq \\ &\leq \int [1+|\xi_1-\xi_2|+|x_1-x_2|+\text{dist}\left((x_2,\xi_2),\text{supp } a_k\right)]^{-N} \\ &\qquad \cdot[1+|\xi_2-\xi_3|+|x_2-x_3|+\text{dist}\left((x_2,\xi_2),\text{supp } S_k\right)]^{-N}d\xi_2dx_2 \\ &\leq r^{-\frac{N}{2}}(1+|(\xi_1,x_1)-(\xi_3,x_3)|)^{-N},\end{aligned}$$

where $r = 2^{k(1-\rho)}$. □

□

10.0.7. Adjoints and Compositions of Pseudodifferential Operators. Adjoints.

Formally, using Taylor series, we see

$$\begin{aligned}[A(x,D)]^* u(x) &= \int a(\bar{y},\xi) e^{i\xi(x-y)} u(y)\,dy\,d\xi \\ &= \int \sum_\alpha \frac{\partial_x^\alpha a(\bar{x},\xi)(y-x)^\alpha}{\alpha!} e^{i\xi(x-y)} u(y)\,dy\,d\xi \\ &= \int \sum_\alpha \frac{(-i)^\alpha \partial_x^\alpha \partial_\xi^\alpha a(\bar{x},\xi)}{\alpha!} e^{i\xi(x-y)} u(y)\,dy\,d\xi.\end{aligned}$$

So, we would like to show $A(x,D)^* = B(x,D)$ where

$$b(x,\xi) \sim \frac{(-1)^\alpha \partial_x^\alpha \partial_\xi^\alpha a(\bar{x},\xi)}{\alpha!}.$$

But does this converge? We need $a(x,\xi)$ nice as $|\alpha| \to \infty$. Let $a(x,\xi) \in S^m_{\rho,\delta}$, then $b(x,\xi) \in \sum S^{m-|\alpha|(\rho-\delta)}_{\rho,\delta}$. Note that this works fine if $\rho > \delta$ but is meaningless if $\rho \le \delta$.

Products.

Again, formally using Taylor series, we have

$$\begin{aligned}(A(x,D)B(x,D)u)(x) &= \int a(x,\xi) e^{i\xi(x-y)} b(y,v) e^{iv(y-z)} u(z)\,dz\,dv\,dy\,d\xi \\ &= \int \sum_\alpha \frac{\partial_\xi^\alpha a(x,\xi)\partial_x^\alpha b(x,v)}{i^{|\alpha|}\alpha!} e^{i\xi(x-y)} u(y)\,dz\,dv\,dy\,d\xi.\end{aligned}$$

Then, by taking the Fourier transform with respect to y, then integrating over v, we have

$$\int \sum_\alpha \frac{\partial_\xi^\alpha a(x,\xi)\partial_x^\alpha b(x,\xi)}{i^{|\alpha|}\alpha!} e^{i\xi(x-z)} u(y)\,dz\,d\xi.$$

So, formally,

$$c(x,\xi) \sim \frac{\partial_\xi^\alpha a(x,\xi)\partial_x^\alpha b(x,\xi)}{i^{|\alpha|}\alpha!},$$

and again we need $\rho > \delta$ for this to make sense.

DEFINITION 10.0.20. *Let $a_j \in S^{m_j}_{\rho,\delta}$, $m_j \searrow -\infty$, then we say $a \sim \sum_{j=1}^\infty a_j$ if*

$$a - \sum_{j=1}^N a_j \in S^{m_{N-1}}_{\rho,\delta}.$$

t:16 **Theorem 10.11 (Formal Series)** *(i) Let $a(x,\xi) \in S^m_{\rho,\delta}$ for $0 \le \delta < \rho \le 1$. Then,*

t:16.1.a (10.7) $$A(x,D)^* = B(x,D)$$

where $b(x,\xi) \in S^m_{\rho,\delta}$, and

t:16.1.b (10.8) $$b(x,\xi) \sim \frac{(-i)^\alpha \partial_x^\alpha \partial_\xi^\alpha \overline{a(x,\xi)}}{\alpha!}.$$

(ii) Let $a_j \in S^{m_j}_{\rho,\delta}$, $j = 1,2$ for $0 \le \delta < \rho \le 1$. The

t:16.2.a (10.9) $$A_1(x,D)A_2(x,D) = C(x,D)$$

for $c(x,\xi) \in S^{m_1+m_2}_{\rho,\delta}$ where

t:16.2.b (10.10) $$c(x,\xi) \sim \frac{\partial^\alpha_\xi a_1(x,\xi)\partial^\alpha_x a_2(x,\xi)}{i^{|\alpha|}\alpha!}.$$

We need to prove a few propositions before proving Theorem 10.11.

p:3 PROPOSITION 10.0.21. *Let $a_j \in S^{m_j}_{\rho,\delta}$, $m_j \searrow -\infty$, then there exist $a \in S^{m_1}_{\rho,\delta}$ such that*

$$a \sim \sum_{j=1}^{\infty} a_j.$$

p:4 PROPOSITION 10.0.22. *Let $f_j \in \mathbb{R}$, then there exists a function $f \in C^\infty$ such that $f^{(j)}(0) = f_j$.*

p:5 PROPOSITION 10.0.23. *Suppose $a \in S^m_{\rho,\rho}$, $a_j \in S^{m_j}_{\rho,\rho}$, $m_j \searrow -\infty$, $(a - \sum_{j=1}^m a_j) \in S^{n_j}_{\rho,\rho}$, $n_j \searrow -\infty$, then $a \sim \sum_{j=1}^\infty a_j$.*

We will first prove Proposition 10.0.22.

PROOF OF PROPOSITION 10.0.22. Define the function

$$\chi = \{1 \text{ in } [-1,1],\ 0 \text{ in } [-2,2]^c\},$$

where χ is smooth. Let

$$f(x) = \sum_{j=0}^{\infty} \frac{f_j}{j!} x^j \chi(\frac{x}{\epsilon_j})$$

for $\epsilon_j \to 0$ as $j \to \infty$ to be chosen. Then

$$|f| \le \sum_{j=0}^{\infty} \frac{f_j}{j!}(2\epsilon_j)^j.$$

We wish to choose our ϵ_j's such that the sum converges.

$$|\partial_x f| \le C_1 \sum_{j=0}^{\infty} \frac{f_j}{j!}(2\epsilon_j)^{j-i} + c_1 \frac{f_0 \epsilon_0^{-1}}{0!}.$$

Similarly,

$$|\partial^k f| \le C_k \sum_{j=k}^{\infty} \frac{f_j}{(j-k)!}(2\epsilon_j)^{j-k} + c_k \sum_{j=0}^{k-1} \frac{f_j \epsilon_j^{j-k}}{j!}.$$

So, we need $|\frac{f_j}{(j-k)!}(2\epsilon_j)^{j-k}| \le \frac{1}{2^j}$ for $k \le j$. Let $h = j-k$. Then, $(2\epsilon_j)^h \frac{f_j}{h!} \le 2^{-j}$. Choose ϵ_j such that this occurs for $h = 0, 1, ..., j$. □

PROOF OF PROPOSITION 10.0.21. Let $\chi = \{1 \in B_2^c,\ 0 \in B_1\}$,

$$a(x,\xi) = \sum_{j=0}^{\infty} a_j(x,\xi)\chi\left(\frac{\xi}{R_j}\right),$$

where $R_j \to \infty$. Choose the R_j's such that this sum converges, or

$$|\partial^\alpha_\xi \partial^\beta_x a_j(x,\xi)\chi\left(\frac{\xi}{R_j}\right)| \le \frac{1}{2^j}(1+|\xi|)^{m_{j-1}-|\alpha|\rho+|\beta|\delta}$$

for $|\alpha|+|\beta| \leq j$, $m_{j-1} > m_j$. We have different cases for the different derivatives:
Case 1. All derivatives fall on $a_j \Rightarrow$

(1) A bad constant,

(2) An extra $(1+|\xi|)^{m_j - m_{j-1}}$.

We must choose R_j large enough such that 2 controls 1.

Case 2. Some derivatives fall on $\chi(\frac{\xi}{R_j}) \Rightarrow$ a factor of $\frac{1}{R_j}$ where $\xi \approx R_j$.

Show convergence:

$$\begin{aligned}\sum_{j=0}^{\infty} |\partial_\xi^\alpha \partial_x^\beta a_j(x,\xi)\chi\left(\frac{\xi}{R_j}\right)| &\leq C \sum_{j=|\alpha|+|\beta|}^{\infty} \frac{1}{2^j}(1+|\xi|)^{m_j - |\alpha|\rho+|\beta|\delta} + c \sum_{j=i}^{|\alpha|+|\beta|} \dots \\ &< \infty.\end{aligned}$$

□

REMARK 10.0.24. *A note on remainder estimates:*

$$a - \sum_{j=1}^{m} = -\sum_{j=1}^{m} a_j(x,\xi)\left(1-\chi\left(\frac{\xi}{R_j}\right)\right) + \sum_{j=m+1}^{\infty} a_j \chi\left(\frac{\xi}{R_j}\right).$$

We have:

$$\begin{aligned}&\sum_{j=m+1}^{\infty} \left|\partial_\xi^\alpha \partial_x^\beta \left[a_j)(x,\xi)\chi\left(\frac{\xi}{R_j}\right)\right]\right| \leq \\ &\leq \sum_{j=m+1}^{\max\{m+1,|\alpha|+|\beta|\}} \cdots + \sum_{j=|\alpha|+|\beta|}^{\infty} \frac{1}{2^j}(1+|\xi|)^{m_{j-1} - |\alpha|\rho+|\beta|\delta}.\end{aligned}$$

PROOF OF PROPOSITION 10.0.23. We want to show

$$\left|\partial_\xi^\alpha \partial_x^\beta (a(x,\xi) - \sum_{j=1}^{m} a_j(x,\xi))\right| \leq c_{\alpha,\beta}(1+|\xi|)^{m+1-\rho|\alpha|+\delta|\beta|}.$$

But,

$$\begin{aligned}&|\partial_\xi^\alpha \partial_x^\beta (a(x,\xi) - \sum_{j=1}^{m} a_j(x,\xi))| \leq \\ &\leq |\partial_\xi^\alpha \partial_x^\beta [a(x,\xi) - \sum_{j=1}^{\tilde{m}} a_j(x,\xi)]| + |\partial_\xi^\alpha \partial_x^\beta \sum_{j=m+1}^{\tilde{m}} a_j(x,\xi)| \\ &\leq (1+|\xi|)^{\tilde{m}-\rho|\alpha|+\delta|\beta|} + C.\end{aligned}$$

So,

$$a \sim \sum_{j=1}^{\infty} a_j.$$

□

DEFINITION 10.0.25.

$$S^{-\infty} = \bigcap_{m=0}^{-\infty} S^m_{1,0}.$$

This is the symbol class with all derivatives rapidly decaying.

We are now in a position ro prove Theorem 10.11

PROOF OF THEOREM 10.11. Use a Littlewood-Paley Decomposition to get

$$a(x,D)=\sum_{j=0}^{\infty}a_j(x,D)$$

where $\operatorname{supp}(a_j)\subset\{|\xi|\sim 2^j\}$.

Given a_j, we want b_j s.t.

$$b_j\approx\sum\frac{\partial_\xi^\alpha\partial_x^\beta\overline{a_j(x,\xi)}}{i^{|\alpha|}\alpha!}.$$

We know that:

$$|\partial_\xi^\alpha\partial_x^\beta a_j(x,\xi)|\le c_{\alpha,\beta}2^{j(m-\rho|\alpha|+\delta|\beta|)}.$$

Rescale using $x\to\lambda x$ and $\xi\to\lambda^{-1}\xi$. Then, we have

$$|\partial_\xi^\alpha\partial_x^\beta a_j(\lambda x,\lambda^{-1}\xi)|\le c_{\alpha,\beta}2^{j(m-\rho|\alpha|+\delta|\beta|)}\lambda^{-|\alpha|+|\beta|}.$$

We want to choose λ so that $2^{-j\rho}\lambda^{-1}=2^{j\delta}\lambda$, or $\lambda^2=2^{j(\delta-\rho)}$.

Let $\tilde{a}_j=a_j(\lambda x,\lambda^{-1}\xi)$, then

$$|\partial_\xi^\alpha\partial_x^\beta\tilde{a}_j|\le c_{\alpha,\beta}2^{jm}\mu^{|\alpha|+|\beta|},$$

where $\mu=\lambda^{-1}2^{-j\rho}=2^{j\delta}\lambda\Rightarrow\mu=2^{\frac{j}{2}(\delta-\rho)}\le 1$. Thus,

$$\begin{aligned}a_j(y,\xi)&=\\&=\sum_{|\alpha|=0}^{N}\frac{\partial_x^\alpha a_j(x,\xi)}{\alpha!}(x-y)^\alpha+(x-y)^{N+1}\int_0^1\frac{h^N\partial_x^{N+1}a_j((1-h)x+hy,\xi)}{(N+1)!}dh\end{aligned}$$

Then,

$$\begin{aligned}a_j(x,D)^*u&=\\&=\int\overline{a_j(y,\xi)}e^{i\xi(x-y)}u(y)dyd\xi\\&=\int\sum_{|\alpha|}^{N}\frac{\partial_x^\alpha\overline{a_j(x,\xi)}(y-x)^\alpha}{\alpha!}e^{i\xi(x-y)}u(y)dyd\xi\\&\quad+\int\frac{(x+y)^{N+1}}{(N+1)!}h^N\partial_x^{N+1}\overline{a_j((1-h)x+hy,\xi)}dhdyd\xi,\end{aligned}$$

where $|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\le c_{\alpha,\beta}\mu^{|\alpha|+|\beta|}$. We need to prove $a(x,D)^*-b_{N-1}(x,\xi)\in\mu^{-2N}\cdot S^0_{0,0}$. To do this, let us restate the problem in a stronger form.

Assuming $|\partial_x^{\alpha+\alpha_1}\partial_\xi^{\alpha+\beta_1}a(x,\xi)|\le c_{\alpha,\alpha_1,\beta_1}$, $|\alpha|=N$, then

$$a(x,D)^*-b_{N-1}(x,\xi)\in S^0_{0,0}.$$

This is the same as above, we have simply moved the factors of μ. The proof of this statement goes as follows:

$$\begin{aligned}
&a(x,D)^* u = \\
&= \int \overline{a(y,\xi)} e^{i\xi(x-y)} u(y)\,dy\,d\xi \\
&= \int \sum_{|\alpha|}^{N} \frac{\partial_x^\alpha \overline{a(x,\xi)}(y-x)^\alpha}{\alpha!} e^{i\xi(x-y)} u(y)\,dy\,d\xi \\
&\qquad + \int \left((y-x)\partial_x\right)^{N+1} h^N \overline{a((1-h)x+hy,\xi)}\,dh\,dy\,d\xi \\
&= \int b_N e^{i\xi(x-y)} u(y)\,dy \\
&\qquad + \int\int_0^1 \partial_x^N \partial_\xi^N h^N e^{i\xi(x-y)} a((1-h)x+hy,\xi) u(y)\,dy\,d\xi\,dh,
\end{aligned}$$

where

$$b_N = \sum_{|\alpha|\le N} \frac{\partial_x^\alpha \partial_\xi \overline{a(x,\xi)}}{i^{|\alpha|}\alpha!}.$$

Is the remainder in $S^0_{0,0}$?

CLAIM 10.0.26. *The function $q(x,y,\xi) = \partial_x^N \partial_\xi^N a((1-h)x+hy,\xi)$ is bounded with bounded derivatives.*

PROOF OF CLAIM. Let

$$Qu(x) = \int q(x,y,\xi) e^{i\xi(x-y)} u(y)\,dy\,d\xi.$$

We need the kernel of TQT^*. So, we have:

$$\begin{aligned}
&K(\xi_2,x_2,\xi_1,x_1) = \\
&= e^{-\frac{1}{2}(y-x_1)^2} e^{i\xi_1(y-x_1)} q(x,y,\xi) e^{i\xi(x-y)} e^{-\frac{1}{2}(x-x_2)^2} e^{i\xi_2(x_2-x)}\,dx\,dy\,d\xi \\
&= \int e^{-\frac{1}{2}(y-x_1)^2} e^{i\xi_1(y-x_1)} q(x-x_2,y,\xi) e^{i\xi(x-y)+i\xi_2(x_2-x)}\,dx\,dy\,d\xi \\
&= \int e^{-\frac{1}{2}(x_1-y)^2} e^{i\xi_1(y-x_1)} q(\xi-\xi_2,y,\xi) e^{i\xi(x_2-y)}\,dy\,d\xi,
\end{aligned}$$

as $(x-y) = (x-x_2)+(x_2-y)$.

Repeat this argument to get proper bounds on $K(\xi_2,x_2,\xi_1,x_1)$ and thus $q \in S^0_{0,0}$. This will prove (10.7-10.8). □

Now we proceed to prove (10.9-10.10).

$$\begin{aligned}
|\partial_\xi^{\alpha+\alpha_1}\partial_x^\beta a_1(x,\xi)| &\le c_{\alpha,\alpha_1,\beta},\ |\alpha| = N \\
|\partial_\xi^{\alpha+\alpha_1}\partial_x^\beta a_2(x,\xi)| &\le c_{\alpha,\alpha_1,\beta},\ |\alpha| = N.
\end{aligned}$$

We need to prove $a_1(x,D)a_2(x,D) - b_{N-1}(x,D) \in S^0_{0,0}$.

$$\begin{aligned}
&(a_1(x,D)a_2(x,D)u)(x) = \\
&= \int a_1(x,\xi)e^{i\xi(x-z)}a_2(z,\nu)e^{i\nu(z-y)}u(y)dydzd\xi d\nu \\
&= \int \sum_{|\alpha|<N} a_1(x,\xi)e^{i\xi(x-z)}a_2(z,\nu)e^{i\nu(z-y)}u(y)dydzd\xi d\nu \\
&= \int \sum_{|\alpha|<N} a_1(x,\xi)e^{i\xi(x-z)}\frac{\partial_x^\alpha a_2(x,\nu)}{\alpha!}(z-x)^\alpha e^{i\nu(z-y)}u(y)dydzd\xi d\nu \\
&\quad + \int\int_0^1 a_1(x,\xi)e^{i\xi(x-z)}\partial_x^N a_2((1-h)x+hy,\nu) \\
&\quad\quad \cdot(z-x)^N e^{i\nu(z-y)}u(y)dydzd\xi d\nu dh \\
&= \int b_{N-1}e^{i\xi(x-z)}e^{i\nu(z-y)}u(y)dydzd\xi d\nu.
\end{aligned}$$

Note that $q \in \mathscr{C}^\infty$ with bounded derivatives. Let

$$Qu(x) = \int q(x,\xi,y,\nu)e^{i\xi(x-z)}e^{i\nu(z-y)}u(y)dyd\xi dzd\nu.$$

We need derivatives on the Bargman Kernel of Q and TQT^*.

$$\begin{aligned}
&K(x_2,\xi_2,x_1,\xi_1) = \\
&= \int e^{-\frac{1}{2}(x-x_2)^2}e^{i\xi_2(x_2-x)}q(x,\xi,z,\nu,y)e^{i\xi(x-z)}e^{i\nu(z-y)} \\
&\quad\quad \cdot e^{-\frac{1}{2}(x_1-y)^2}e^{i\xi_1(y-x_1)}dxdydzd\xi d\nu \\
&= \int q(x-x_2,\xi,z,\nu,y-x_1)e^{i\xi_2(x_2-x)+i\xi(x-z)}e^{i\nu(z-y)+i\xi_1(y-x_1)}dxdydzd\xi d\nu \\
&= \int q(x-x_2,z,\nu,y-x_1)e^{i\phi}dxdydzd\xi d\nu,
\end{aligned}$$

where $\phi = (\xi_2-\xi)(x_2-x)+\xi(x_2-x)+\nu(z-y)+\xi_1(y-x_1)$.

Integrate in x to get:

$$= \int q(\xi-\xi_2,z,\nu,y-x_1)e^{i(\xi(x_2-z)+\nu(z-y)+\xi_1(y-x_1))}dyd\xi dzd\nu.$$

Repeat this process to get the desired bounds on $K(\xi_2,x_2,\xi_1,x_1)$. □

10.0.8. Properties of the Calculus of P.D.O.'s. Left Calculus: $a(x,D)$
Right Calculus: $a(D,y)$
Weyl Calculus: $a^w(\frac{x+y}{2},D)$.

$$a\Big(\frac{x+y}{2},D\Big)b\Big(\frac{y+z}{2},D\Big) = c\Big(\frac{x+z}{2},D\Big), \text{ where}$$

$$c(x,\xi) = \sum \frac{(-1)^\alpha \partial_x^\alpha \partial_\xi^\beta a(x,\xi)\partial_x^\beta \partial_\xi^\alpha b(y,\xi)}{(2i)^{|\alpha|+|\beta|}\alpha!\beta!}$$

Using Taylor Series, take y into z from a, y into x from b. In this series, we get

$$c = ab + \frac{1}{2i}\{a,b\} + \ldots$$

What is the symbol of a P.D.O.?

Normally, we must talk about a Calculus and a symbol. We have:

$$a(x,D) \to a(x,\xi)$$
$$a(D,x) \to b(x,D) \to b(x,\xi) = a(x,\xi) + \frac{1}{i}a_{x\xi}(x,\xi) + ...$$

The Principal Symbol of an Operator.

If $a \in \mathrm{O}P\mathscr{S}^m_{\rho,\delta}$ with error in $OPS^{m+\delta-\rho}_{\rho,\delta}$, then the principle symbol is contained in

$$S^m_{\rho,\delta} \Big/ S^{m+\delta-\rho}_{\rho,\delta},$$

and it is independent of the Calculus.

EXAMPLE 10.0.27. *The principal symbol of $a(x,D)^*$ is $\overline{a(x,\xi)}$.*

EXAMPLE 10.0.28. *The principal symbol for the product $a_1(x,D)a_2(x,D)$ is*

$$(\textit{princ. symb.}(a_1))(\textit{princ. symb.}(a_2)).$$

An important fact is that the the principal symbol of a commutator is well-defined. A commutator of a P.D.O. is given by:

$$[a_1(x,D), a_2(x,D)] = a_1(x,D)a_2(x,D) - a_2(x,D)a_1(x,D),$$

where $a_1 \in S^{m_1}_{\rho,\delta}$, $a_2 \in S^{m_2}_{\rho,\delta}$. In the expansion of the commutator, we see the first term is simply 0, while the second term is $\frac{1}{i}\{a_1, a_2\}$. Hence, the principal symbol of a commutator is the essentially the Poisson Bracket of the two elements.

10.0.9. L^p bounds for P.D.O.'s. For multipliers, $|\partial^\alpha m(\xi)| \le (1+|\xi|)^{-|\alpha|} \Longrightarrow m(D) : L^p \to L^p$.

Theorem 10.12
If $a \in S^0_{1,\delta}$, $\delta < 1$, then $a(x,D) : L^p \to L^p$.

PROOF. For multipliers, we have

$$m(D)u(x) = \int K(x-y)u(y)dy,$$

where $K = \check{m}$, $|K(x)| \le |x|^{-n}$, and $|\partial_x K(x)| \le |x|^{-n-1}$. In this case, we have

$$K(x,y) = \int a(x,\xi)e^{i\xi(x-y)}d\xi.$$

Using a Littlewood-Paley decomposition, we can set $a = \sum a_j$ and similarly $K = \sum K_j$. Then,

$$K_j(x,y) = K_j(x, x-y)$$

Let $z = x - y$. We have:

$$|K_j(x,z)| \le 2^{jm}(1+2^j|z|)^{-N}$$

and

$$|\partial_z K_j(x,z)| \le 2^{j(n+1)}(1+2^j|z|)^{-N}.$$

Thus, by summing over j, we see:

$$|K(x, x-y)| \le |x-y|^{-n}$$

and

$$|\partial_y K(x, x-y)| \le |x-y|^{-n-1}.$$

As $\partial_x a(x,\xi) \in S^{\delta}_{1,\delta}$ and $\partial_x a_j(x,\xi) \approx 2^{j\delta}$,

$$|\partial_x K_j(x, x-y)| \le 2^{j\delta} 2^{jn} (1+2^j|z|)^{-N} + 2^{j(n+1))} (1+2^j|z|)^{-N}$$
$$\Longrightarrow |\partial_x K(x,y)| \le |x-y|^{-n-1}.$$

□

REMARK 10.0.29. *We only need $\delta < 1$ for L^2 boundedness. The kernel bounds are not integrable. In general, we have $a(x,D) : L^1 \not\to L^1$. However, we do have:*

$$a(x,D) : H_1 \to H_1$$
$$a(x,D) : BMO \to BMO.$$

10.0.10. Sobolev Spaces. For P.D.O's, we have $OP\mathscr{S}^0_{\rho,\delta} : L^2 \to L^2$.

PROPOSITION 10.0.30. $OP\mathscr{S}^m_{\rho,\delta} : H^s \to H^{s-m}$, $s \in \mathbb{R}$.

PROOF. We know that $(1+|D|^2)^{\frac{s}{2}} : H^s \to L^2$ is an isometry simply by Fourier transform. Hence,

$$a(x,D) : H^s \to H^{s-m} \Leftrightarrow (1+|D|^2)^{\frac{s-m}{2}} a(x,D)(1+|D|^2)^{\frac{-s}{2}} : L^2 \to L^2$$

□

PROPOSITION 10.0.31. $OPS^m_{1,\delta} : W^{s,p} \to W^{s-m,p}$ *for* $1 < p < \infty$.

PROOF. $W^{s,p} = (1+|D|^2)^{\frac{-s}{2}} L^p$, for $1 < p < \infty$. □

Note that derivatives of L^∞ functions are Lipschitz functions. Half derivatives are Holder functions. Roughly, $L^\infty \to C^\alpha \to Lip$ where α is not an integer. We have that

$$C^\alpha \neq (1+|D|^2)^{\frac{-\alpha}{2}} C^\infty$$

but that

$$C^\alpha = (1+|D|^2)^{\frac{-\alpha}{2}} BMO.$$

10.0.11. Invertibility of P.D.O.'s. Let $a^{ij}(x) D_i D_j u = f$, $a^{ij} \in C^\infty$, positive definite.

PROPOSITION 10.0.32. $f \in H^2$, $u \in H^r$ *implies that* $u \in H^{s+2}$ *where* u, f *are as above.*

We have here an expression of the form $p(x,D)u = f$ where $p(x,D) = a^{ij}\xi_i\xi_j$, so $p(x,\xi) \in S^2_{1,0}$ and $|p(x,\xi)| \ge |\xi|^2$ for large $|\xi|$.

PROOF. Let $q(x,\xi) \in S^{-2}_{1,0}$. Act $q(x,D)$ on both sides of $p(x,D)u = f$. Thus,

$$\begin{aligned} p(x,D)u = f \quad &\Rightarrow \quad q(x,D)p(x,D) - I = r(x,D) \\ &\Rightarrow \quad u + r(x,D)u = q(x,D)f, \end{aligned}$$

where $r \in S^{-\infty}$, $q \in S^{-2}_{1,0}$. □

Theorem 10.13
Suppose $a \in S^m_{\rho,\delta}$, $\rho > \delta$, $a(x,\xi) \ge |\xi|^m$ for large $|\xi|$, then there is a $b \in S^{-m}_{\rho,\delta}$ s.t. $a(x,D)b(x,D) - I \in S^{-\infty}$.

REMARK 10.0.33. $S^{-\infty}: \mathscr{S}' \to C^\infty$. *The Kernel of an operator in* $S^{-\infty}$ *satisfies:*

$$|\partial_x^\alpha \partial_\xi^\beta K(x,y)| \leq c_{\alpha,\beta}(1+|x-y|)^{-N}.$$

Note that inverses are unique up to $S^{-\infty}$.

$$b_l(x,D) = b_l(x,D)a(x,D)b_r(x,D) + \mathrm{OP}\mathscr{S}^{-\infty} = b_r(x,D) + \mathrm{OP}\mathscr{S}^{-\infty}.$$

PROOF. We seek

$$b(x,\xi) \sim \sum_{j=0}^{\infty} b_j(x,\xi),$$

where $b_j \in \mathrm{OP}\mathscr{S}^{m-j(\rho-\delta)}_{\rho,\delta}$ s.t. $b(x,D)a(x,D) - I \in S^{-\infty}$. We will choose the b_j's inductively such that:

$$(b_0 + ... + b_N)(x,D)a(x,D) - I \in OPS^{-(N+1)(\rho-\delta)}_{\rho,\delta}.$$

For $N = 0$, seek $b_0 \in OPS^{-m}_{\rho,\delta}$ s.t. $I - b_0(x,D)a(x,D) \in OPS^{-(\rho-\delta)}_{\rho,\delta}$. Take $b_0(x,\xi) = a(x,\xi)^{-1}$ for large $|\xi|$ and use symbol calculus.

For the induction step, assume that $I - (b_0 + ... + b_{N-1})(x,D)a(x,D) = r_N(x,D) \in S^{-N(\rho-\delta)}_{\rho,\delta}$. Choose

$$b_N(x,\xi) = \frac{r_N(x,\xi)}{a(x,\xi)}$$

for large $|\xi|$. Note that we do not worry about small $|\xi|$ values as we can control them by adding $S^{-\infty}$ symbols. Then,

$$b_N(x,\xi) \in OPS^{-N(\rho-\delta)}_{\rho,\delta}.$$

Thus, by the symbol calculus, we get

$$\begin{aligned} b_N(x,D)a(x,D) &= r_N(x,D) + r_{N+1}(x,D) \\ &\Rightarrow r_{N+1} \in S^{-(N+1)(\rho-\delta)}_{\rho,\delta}. \end{aligned}$$

□

Idea: Look at the differential operator $-\triangle$. We have

$$\langle -\triangle u, u\rangle = \|\nabla u\|^2_{L^2}.$$

DEFINITION 10.0.34. *We define* $a(x,\xi) \in S^m_{1,0}$ *to be positive definite if*

$$\langle a(x,D)u, u\rangle \geq c\|u\|^2_{H^{\frac{m}{2}}} - D\|u\|^2_{H^{\frac{m-1}{2}}}.$$

That is, the operator $a(x,D)$ *is positive definite in the usual sense, modulo an error of the next lowest order.*

EXAMPLE 10.0.35. $(1+|D|^2)^{\frac{m}{2}}$ *is positive definite.*

We also have the notion of a non-negative definite symbol:

$$\begin{aligned} 0 \quad &\leq \quad \|a(x,D)u\|^2 = \langle a(x,D)u, a(x,D)u\rangle = \langle a(x,D)^* a(x,D)u, u\rangle \\ &\Rightarrow \quad a(x,D)^* a(x,D) \text{ is non-negative definite.} \end{aligned}$$

Note that $(1+|D|^2)^{\frac{m}{2}} + a(x,D)^* a(x,D)$ is positive definite. $a \in S^{\frac{m}{2}}$ and the principal symbol is $(1+|\xi|^2)^{\frac{m}{2}} + |a(x,\xi)|^2$.

We present here the proof that a general bounded operator is in fact a psuedodifferential operator for some symbol b as present in [**EZ06**]. In particular, this allows us

to say that our smooth, decaying error is in fact a PDO in the discussion from Section 13.6. This proof is done here from the semiclassical analysis point of view with $h = 1$, while the original microlocal result is presented in [**Bea77**]. However, as the proofs are similar, we present the more concise version from [**EZ06**].

thm:beals1 **Theorem 10.14** *There exist constants $C, M > 0$ such that*

$$\|b\|_{L^\infty} \leq C \sum_{|\alpha|\leq M} \|Op(\partial^\alpha b)\|_{L^2\to L^2},$$

for all $b \in \mathscr{S}'(\mathbb{R}^{2d})$.

PROOF. We have

$$b(x,D)u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} b(x,\xi)e^{i\langle x,\xi\rangle}\hat{u}(\xi)d\xi,$$

where the Fourier transform is taken formally here in $\mathscr{S}'$.

Set $\phi = \phi(x)$, $\psi = \psi(x) \in \mathscr{S}$. Then, we can take $\mathscr{F}(b\phi\hat{\psi}e^{i\langle x,\xi\rangle})$ as a function of dual variables $(x^*,\xi^*) \in \mathbb{R}^{2d}$. We have

$$\begin{aligned}\frac{1}{(2\pi)^d}|\mathscr{F}(b\phi\hat{\psi}e^{i\langle x,\xi\rangle})(0,0)| &= \frac{1}{(2\pi)^d}\left|\int\int b(x,\xi)\phi(x)\hat{\psi}(\xi)e^{i\langle x,\xi\rangle}dxd\xi\right| \\ &= |\langle b(x,D)\psi,\phi\rangle| \leq \|b\|_{L^2\to L^2}\|\phi\|_{L^2}\|\psi\|_{L^2}.\end{aligned}$$

Fix $(x^*,\xi^*) \in \mathbb{R}^{2d}$ and rewrite the inequality with $\phi(x)e^{i\langle x^*,x\rangle}$ replacing $\phi(x)$ and $\hat{\psi}e^{-i\langle\xi^*,\xi\rangle}$ replacing $\hat{\psi}(\xi)$, which does not alter the L^2 norm. Then,

$$\frac{1}{(2\pi)^d}|\mathscr{F}(b\phi\hat{\psi}e^{i\langle x,\xi\rangle})(x^*,\xi^*)| \leq \|b\|_{L^2\to L^2}\|\phi\|_{L^2}\|\psi\|_{L^2}.$$

Take $\chi \in C_c^\infty(\mathbb{R}^{2d})$. Take $\phi, \psi \in \mathscr{S}$ so that

$$\begin{cases}\phi(x) = 1 \text{ if } (x,\xi) \in \operatorname{supp}\chi \\ \hat{\psi}(\xi) = 1 \text{ if } (x,\xi) \in \operatorname{supp}\chi.\end{cases}$$

Write

$$\phi_1 = \chi e^{-i\langle x,\xi\rangle}.$$

Then,

$$\chi(x,\xi) = \phi_1(x,\xi)\phi(x)\hat{\psi}(\xi)e^{i\langle x,\xi\rangle}.$$

Using standard estimates on the Fourier transform, we have

$$\|\mathscr{F}\chi\|_{L^1} \leq C\sum_{|\alpha|\leq 2d+1}\|\partial^\alpha\chi\|_{L^1},$$

so

$$\|\mathscr{F}\phi\|_{L^1} \leq C\sum_{|\alpha|\leq 2d+1}\|\partial^\alpha\chi\|_{L^1}.$$

Thus, for any $(x^*,\xi^*)\in\mathbb{R}^{2d}$, we have

$$\begin{aligned}|\mathscr{F}(\chi b)(x^*,\xi^*)| &\leq \|\mathscr{F}(\phi_1 b\phi\hat{\psi}e^{i\langle x,\xi\rangle})\|_{L^\infty}\\ &= \frac{1}{(2\pi)^d}\|\mathscr{F}(\phi_1)*\mathscr{F}(b\phi\hat{\psi}e^{i\langle x,\xi\rangle})\|_{L^\infty}\\ &\leq \frac{1}{(2\pi)^d}\|\mathscr{F}(\phi_1)\|_{L^1}\|\mathscr{F}(b\phi\hat{\psi}e^{i\langle x,\xi\rangle})\|_{L^\infty}\\ &\leq C\|b\|_{L^2\to L^2}\end{aligned}$$

where C depends only on ϕ, ψ and χ, not (x^*,ξ^*). Hence,

eqn:infbd (10.11) $$\|\mathscr{F}(\chi b)\|_{L^\infty}\leq C\|b\|_{L^2\to L^2}.$$

CLAIM 10.0.36.

$$|\mathscr{F}(\chi b)(x^*,\xi^*)|\leq C\langle(x^*,\xi^*)\rangle^{-2d-1}\sum_{|\alpha|\leq 2d+1}\|Op(\partial^\alpha b)\|_{L^2\to L^2}.$$

PROOF. We have

$$\begin{aligned}(x^*)^\alpha(\xi^*)^\beta\mathscr{F}(\chi b)(x^*,\xi^*) &= \int_{\mathbb{R}^{2d}}(x^*)^\alpha(\xi^*)^\beta e^{-i(\langle x^*,x\rangle+\langle\xi^*,\xi\rangle)}\chi b(x,\xi)dxd\xi\\ &= \int\int_{\mathbb{R}^{2d}}e^{-i(\langle x^*,x\rangle+\langle\xi^*,\xi\rangle)}D_x^\alpha D_\xi^\beta(\chi b(x,\xi))dxd\xi.\end{aligned}$$

Summing over (α,β) with $|\alpha|+|\beta|\leq 2d+1$ and using (10.11), we have

$$\begin{aligned}\|\langle(x^*,\xi^*)\rangle^{2d+1}\mathscr{F}(\chi b)\|_{L^\infty} &\leq C_1\sum_{|\alpha|+|\beta|\leq 2d+1}\|\mathscr{F}(D_x^\alpha D_\xi^\beta(\chi b))\|_{L^\infty}\\ &\leq C_2\sum_{|\alpha|\leq 2d+1}\|Op(\partial^\alpha b)\|_{L^2\to L^2}.\end{aligned}$$

□

Hence,

$$\|\chi b\|_{L^\infty}\leq C\|\mathscr{F}(\chi b)\|_{L^1}\leq C\sum_{|\alpha|\leq 2d+1}\|Op(\partial^\alpha b)\|_{L^2\to L^2}.$$

Note that this can be applied in any quantization by standard arguments. □

DEFINITION 10.0.37. *Write*

$$ad_B A=[B,A],$$

where ad is called the adjoint action.

DEFINITION 10.0.38. *A pair $(x^*,\xi^*)\in\mathbb{R}^{2d}$ is identified with a linear operator $l(x,\xi)=\langle x^*,x\rangle+\langle\xi^*,\xi\rangle$.*

Theorem 10.15 (Beals's Theorem)
Let $A:\mathscr{S}\to\mathscr{S}'$ be a continuous linear operator. Then $A=Op(a)$ for a symbol $a\in\mathscr{S}$ if and only if for all $N=0,1,2,\ldots$ and all linear functions $l_1,\ldots,l_N$, we have

eqn:beal (10.12) $$\|ad_{l_1(x,D)}\circ\cdots\circ ad_{l_N(x,D)}A\|_{L^2\to L^2}=O(1).$$

PROOF. If $a \in \mathscr{S}$, then (10.12) holds via standard microlocal techniques. Namely, since $\|A\|_{L^2 \to L^2} = O(1)$ and commuting with $l_j(x,D)$ yields a term of order $O(1)$.

Since A is a continuous linear operator, there is a Schwartz kernel

$$Au(x) = \int_{\mathbb{R}^d} K_A(x,y)u(y)dy$$

where $K_A \in \mathscr{S}'(\mathbb{R}^{2d})$. Denote K_A the kernel of A.

If $a \in \mathscr{S}'(\mathbb{R}^{2d})$ is defined by

eqn:opa

$$a(x,\xi) = \int_{\mathbb{R}^d} e^{-i\langle w,\xi\rangle} K_A(a + \frac{w}{2}, x - \frac{w}{2})dw, \tag{10.13}$$

then

$$K_A(x,y) = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d} a(\frac{x+y}{2},\xi) e^{i\langle x-y,\xi\rangle} d\xi,$$

where the integrals are shorthand for the formal Fourier transform on $\mathscr{S}'$. Indeed,

$$\begin{aligned} K_A(x,y) &= \frac{1}{(2\pi)^d} \int\int e^{i\langle x-y-w,\xi\rangle} K_A\Big(\frac{x+y}{2} + \frac{w}{2}, \frac{x+y}{2} - \frac{w}{2}\Big) dw d\xi \\ &= \frac{1}{(2\pi)^d} \int a(\frac{x+y}{2},\xi) e^{i\langle x-y,\xi\rangle} d\xi \end{aligned}$$

since

$$\frac{1}{(2\pi)^d} \int e^{i\langle x-y-w,\xi\rangle} d\xi = \delta_{x-y}(w)$$

in $\mathscr{S}'$. Hence, $A = Op(a)$ for a as defined in (10.13).

Now, we would like to know that $a \in \mathscr{S}$, or

$$\sup_{\mathbb{R}^{2d}} |\partial^\alpha a| \le C_\alpha$$

for each multi-index α.

To do so, we use (10.12). Define l_j by the pair (x_j, ξ_j). Compute

$$\begin{aligned} Op(D_{\xi_j} a)u(x) &= \frac{1}{(2\pi)^d} \int D_{\xi_j} \Big(a\Big(\frac{x+y}{2},\xi\Big)\Big) e^{i\langle x-y,\xi\rangle} u(y) d\xi dy \\ &= -\frac{1}{(2\pi)^d} \int a\Big(\frac{x+y}{2},\xi\Big) D_{\xi_j}\Big(e^{i\langle x-y,\xi\rangle}\Big) u(y) d\xi dy \\ &= -\frac{1}{(2\pi)^d} \int a\Big(\frac{x+y}{2},\xi\Big) e^{i\langle x-y,\xi\rangle} (x_j - y_j) u(y) d\xi dy \\ &= -[x_j, A]u = -ad_{x_j} Au(x). \end{aligned}$$

Similarly,

$$Op(D_{x_j} a)u(x) = ad_{D_{x_j}} Au(x).$$

In other words, for $j = 1, \dots, n$,

$$\begin{cases} ad_{x_j} A = -Op(D_{\xi_j} a) \\ ad_{D_{x_j}} A = Op(D_{x_j} a). \end{cases}$$

Hence, from our hypothesis, we have

$$\|Op(\partial^\beta a)\| \lesssim 1$$

for all multi-indices β. From Theorem 10.14, we have

$$\|\partial^\alpha a\|_{L^\infty} \leq C \sum_{|\beta|\leq d+1} \|Op(\partial^{\alpha+\beta} a)\|_{L^2\to L^2} \leq C_\alpha.$$

Hence, $a \in \mathscr{S}$ and we have the result. □

In this note, we derive the integral kernel in $\mathbb{R}^3$ for the inverse of the differential operator

$$P_\mu = -\Delta - \mu^2.$$

Specifically, given $u, f : \mathbb{R}^3 \to \mathbb{R}$, we find $K_\mu(x,y)$ such that if

$$P_\mu u = f,$$

then

$$u = \int_{\mathbb{R}^3} K_\mu(x,y) f(y) dy.$$

To begin, we Fourier transform the equation to see

$$(\xi^2 - \mu^2)\hat{u} = \hat{f},$$

hence

$$u = \mathscr{F}^{-1}[(\xi^2-\mu^2)^{-1}] * f.$$

So,

$$K_\mu(x,y) = \mathscr{F}^{-1}[(\xi^2-\mu^2)^{-1}](x-y),$$

if we can define

$$G(x) = \mathscr{F}^{-1}[(\xi^2-\mu^2)^{-1}](x)$$

in a meaningful sense.

Without loss of generality, set $\mu > 0$. Initially, asssume that $x \neq 0$, though this will be easily seen as a limiting case in the end. We have

$$\begin{aligned} G(x) &= \int_{\mathbb{R}^3} \frac{e^{ix\cdot\xi}}{(\xi^2-\mu^2)} d\xi \\ &= \int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{e^{i|x|rcos(\theta)}}{(r-\mu)(r+\mu)} r^2 \sin(\theta) d\theta d\phi dr \\ &= \frac{4\pi}{|x|} \int_0^\infty \frac{r\sin(|x|r)}{(r-\mu)(r+\mu)} dr \end{aligned}$$

by first making a rotational change of variables where $\xi_3 \to \frac{x}{|x|}$, then using polar coordinates.

Now, we are set up to use contour integration to find $G(x)$. See Figure 1 for the contour over which we integrate. We call this contour $\Gamma_{R,\epsilon}$.

Then, we have from residue theory

$$\begin{aligned} \int_{\Gamma_{R,\epsilon}} \frac{ze^{iz|x|}}{(z-\mu)(z+\mu)} dz &= 2\pi i \left[\frac{\mu e^{i|x|\mu}}{2\mu}\right] \\ &= \pi i e^{i|x|\mu}. \end{aligned}$$

However, breaking Γ down, we also have

$$\int_{\Gamma_{R,\epsilon}} \frac{ze^{iz|x|}}{(z-\mu)(z+\mu)} dz = 2i \int_0^R \frac{r\sin(|x|r)}{(r-\mu)(r+\mu)} dr + \frac{\pi e^{i|x|\mu}}{2} - \frac{\pi e^{-i|x|\mu}}{2}.$$

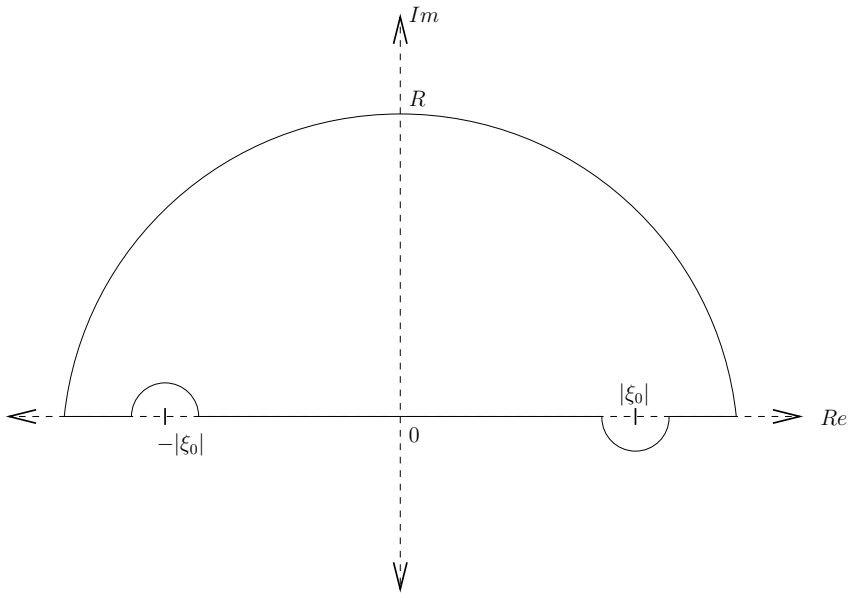

FIGURE 1. The contour for computing the behavior of K_μ fig:fund1

Combining terms and taking $R \to \infty$, we have

$$G(x) = \frac{4\pi^2 \cos(\mu|x|)}{|x|}.$$

This is valid for all x since the integral diverges as $x \to 0$.

Notice that in this analysis we have used the distributional convention

$$f(\lambda) = \frac{1}{2}\left[f(\lambda + i0) + f(\lambda - i0)\right].$$

However, it is not hard to see that taking the convention

$$f(\lambda) = f(\lambda \pm i0)$$

results in the

$$G(x) = \frac{4\pi^2 e^{\pm i|x|\mu}}{|x|}.$$

In this note we discuss the principles of stationary and nonstationary phase used to prove decay in time for the linearized Hamiltonian. The following come from [**Ste93a**] and [**EZ06**].

lem:nsp **LEMMA 10.0.39 (Non-stationary phase)** *Suppose ψ is smooth with sufficient decay and that ϕ is a smooth function with no critical points in the support of ψ. Then*

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\phi(x)}\psi(x)\,dx = O(\lambda^{-N}),$$

as $\lambda \to \infty$, for all $N \geq 0$. In particular,

$$|I(\lambda)| \lesssim \lambda^{-N} \sum_{|\alpha| \leq N} \sup_{\mathbb{R}^d} |\partial^\alpha \psi|.$$

PROOF. Define

$$L = \frac{1}{\lambda i}\frac{1}{|\partial\phi|^2}\langle\partial\phi,\partial\rangle$$

for $x \in \text{supp}(\psi)$. Note that

$$L^N(e^{i\lambda\phi(x)}) = e^{i\lambda\phi(x)},$$

hence

$$|I(\lambda)| = \left|\int_{\mathbb{R}^d} L^N(e^{i\lambda\phi(x)})\psi(x)dx\right| = \left|\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}(L^*)^N\psi(x)dx\right| \lesssim \lambda^{-N}.$$

□

DEFINITION 10.0.40. *A critical point x_0 of ϕ is called nondegenerate if the symmetric matrix*

$$\left[\frac{\partial^2\phi}{\partial_i\partial_j}\right](x_0)$$

is invertible.

LEMMA 10.0.41 (Stationary phase)
Suppose $\phi(x_0) = 0$, ϕ has a nondegenerate critical point at x_0. If ψ is supported in a sufficiently small neighborhood of x_0, then

$$\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}\psi(x)dx \sim \lambda^{-\frac{d}{2}}\sum_{j=0}^{\infty} a_j\lambda^{-j},$$

where the a_j's depend only on finitely many derivatives of ϕ and ψ.

PROOF. Let us assume that $\phi(x) = \sum_1^m x_j^2 - \sum_{m+1}^d x_j^2$. Let $l = (l_1,\dots,l_d)$ be a multi-index. Then, note that

$$\prod_{j=1}^d\int_{-\infty}^{\infty} e^{\pm i\lambda x_j^2}e^{-x_j^2}x_j^{l_j}dx_j = \prod_{j=1}^d\int_{-\infty}^{\infty} e^{-x^2}x^{l_j}dx(1\mp i\lambda)^{-\frac{1}{2}-\frac{l_j}{2}}$$

by deforming the contour of integration using the decay of e^{-x^2}. Factor out

$$\prod_{j=1}^d \lambda^{-(l_j+1)/2} = \lambda^{-(d+|l|)/2},$$

then expand

$$\prod_{j=1}^d(\lambda^{-1}\mp i)^{-\frac{1}{2}-\frac{l_j}{2}}$$

into a power series in λ^{-1} for large λ. Then, we see that

$$\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}e^{-x^2}x^l dx \sim \lambda^{-(d+|l|)/2}\sum_{j=0}^{\infty} c_j(m,l)\lambda^{-j}.$$

Let $\eta(x) \in C_0^\infty(\mathbb{R}^d)$. We need that

$$\left|\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}x^l\eta(x)dx\right| \lesssim \lambda^{-(d+|l|)/2}.$$

To prove this, let

$$\Gamma_j = \{x : |x_j|^2 \geq \frac{x^2}{2d}\},$$

and

$$\Gamma_j = \{x : |x_j|^2 \geq \frac{x^2}{d}\}.$$

Since we have

$$\bigcup_{j=1}^{d} \Gamma_j^0 = \mathbb{R}^d,$$

find functions Ω_j, $j = 1, \ldots, d$ such that $\text{supp}(\Omega_j) \subset \Gamma_j$, Ω_j is homogeneous of degree 0 for all j, smooth away from the origin and

$$\sum_{j=1}^{d} \Omega_j(x) = 1$$

for all $x \neq 0$. Then, we have

$$\int_{\mathbb{R}^d} e^{i\lambda\phi(x)} x^l \eta(x) dx = \sum_j \int_{\mathbb{R}^d} e^{i\lambda\phi(x)} x^l \eta(x) \Omega_j(x) dx.$$

In each cone, use integration by parts with the operator

$$D_j f(x) = (\pm 2i\lambda x_j)^{-1} \frac{\partial f}{\partial x_j},$$

such that

$$D_j e^{i\lambda\phi(x)} = e^{i\lambda\phi(x)}.$$

Using the bound $|x_j| \geq (2d)^{-1/2}|x|$ in Γ_j and

$$|(D_j^*)^N \Omega_j(x)| \leq C_N \lambda^{-N} |x|^{-2N}.$$

By splitting the integral into $\int_{B(0,\epsilon)}$ and $\int_{\mathbb{R}^d \setminus B(0,\epsilon)}$, then choosing $\epsilon = \lambda^{-d/2}$, we are done with the proof. By the principal of nonstationary phase, when $\eta \in \mathscr{S}$ and η vanishes near the origin, then

$$\int e^{i\lambda\phi(x)} \eta(x) dx = O(\lambda^{-N})$$

for every $N \geq 0$. Then, using a Taylor series and the previous estimates, we have the result. □

10.1. General Distorted Fourier Basis Theory

lin:gdfb

We present here the combined results of [**Agm75b**] and [**Hör05**], Chapter 14. Both presentations are valid for operators of the form

$$(P(D) + V(x, D))u = 0,$$

where $P(D)$ is a self-adjoint, constant coefficient differential operator and $V(x, D)$ is a *short range*, symmetric differential operator. The perturbation $V(x, D)$ is defined to be *short range* in order to say that

$$\lim_{z \to \lambda, \pm \Im \mathrm{m} z > 0} R(z) = R^{\pm}(z)$$

exists in the uniform operator topology of $B(L^{2,s}, \mathscr{H}_{2,-s})$, where

$$L^{2,s}(\mathbb{R}^d) = \{u(x) | (1+|x|^2)^{\frac{s}{2}} u(x) \in L^2\}$$

and

$$\mathscr{H}_{m,s} = \{u(x) | D^\alpha u \in L^{2,s},\ 0 \le |\alpha| \le m\}.$$

Also, for any $f \in L^{2,s}$,

$$R^\pm(\lambda) f = R_0^\pm(\lambda) f - R_0^\pm(\lambda) V R^\pm(\lambda) f,$$

where R_0 is the resolvent for the constant coefficient operator. As the notion of short range deals with compactness of the operator $Z(u) = R(Vu)$, being short range requires sufficient decay assumptions at ∞. Heuristically, it is required that the coefficients of V decrease as fast as an integrable function in $|x|$ and for each fixed x_0, we have

$$\frac{V(x_0, \xi)}{P_0(x, \xi)} \to 0 \text{ as } \xi \to \infty.$$

The reasons why these heuristics hold true are explored in this chapter, hence we forego this analysis here and move on with the fact that $V(x, D)$ is a short range perturbation as an assumption. Note that in the case explored below, V is Schwartz in x and is dominated by $P(\xi)$ as $|\xi| \to \infty$. It is also important to note that while our contour integration works out nicely in $3d$, the results presented here hold in any dimension where R_0^+ and R_0^- are arrived at through a limiting procedure.

The Agmon approach to the distorted Fourier transform is equivalent to the approach taken by the author. Namely, define

$$\phi_\pm(x, \xi) = e^{ix\xi} - R^\mp(|\xi|)[V e^{ix\cdot\xi}](x).$$

Then, we define the distorted Fourier transform to be $\mathscr{F}_\pm : L^2 \to L^2$ such that
(i) $Ker(\mathscr{F}_\pm) = L^2_d$. The restriction of $\mathscr{F}_\pm$ is a unitary operator from L^2_c onto L^2,
(ii) for any $f \in L^2$

$$(\mathscr{F}_\pm f)(\xi) = (2\pi)^{-\frac{d}{2}} \lim_{N\to\infty} \int_{|x|<N} f(x) \overline{\phi_\pm(x, \xi)} dx \text{ in } L^2_\xi,$$

and

$$({\mathscr{F}^*}_\pm f)(x) = (2\pi)^{-\frac{d}{2}} \lim_{j\to\infty} \int_{K_j} f(\xi) \phi_\pm(x, \xi) d\xi \text{ in } L^2_x$$

where K_j is an increasing sequence of compact sets such that $\cup_j K_j = \mathbb{R}^d \setminus \mathscr{N}$ for

$$\mathscr{N}(H) = \{\xi \in \mathbb{R}^d | |\xi|^2 \text{ is an eigenvalue for } H\} \cup 0,$$

and
(iii) If P_c is the projection of L^2 onto L^2_c, then

$$(P_c H) f = (\mathscr{F}_\pm^* M_{P(\xi)} \mathscr{F}_\pm) f$$

for any $f \in D(H)$ where $M_{P(\xi)}$ denotes multiplication by $P(\xi)$.

In addition to all of this, we have $\|P_c f\|_{L^2} = \|\mathscr{F}_\pm f\|_{L^2}$. In other words, we have a Plancherel theorem for our distorted Fourier basis.

Now, [**Hör05**], Chapter 14 arrives at the same conclusions using

$$(\mathscr{F}_\pm f)(\xi) = \mathscr{F}(I + V R_0^\pm)^{-1} f(\xi).$$

However, using the resolvent identity

$$R(z) = R_0(z)(I + V R_0(z))^{-1},$$

we see that a formal iteration argument shows equivalence between these definitions for ξ large. It is precisely this iteration we use to get uniform bounds in ξ.

We derive the integral kernel in $\mathbb{R}^3$ for the inverse of the differential operator

$$P_\mu = -\Delta - \mu^2.$$

Specifically, given $u, f : \mathbb{R}^3 \to \mathbb{R}$, we find $K_\mu(x,y)$ such that if

$$P_\mu u = f,$$

then

$$u = \int_{\mathbb{R}^3} K_\mu(x,y) f(y) dy.$$

To begin, we Fourier transform the equation to see

$$(\xi^2 - \mu^2)\hat{u} = \hat{f},$$

hence

$$u = \mathscr{F}^{-1}[(\xi^2 - \mu^2)^{-1}] * f.$$

So,

$$K_\mu(x,y) = \mathscr{F}^{-1}[(\xi^2 - \mu^2)^{-1}](x-y),$$

if we can define

$$G(x) = \mathscr{F}^{-1}[(\xi^2 - \mu^2)^{-1}](x)$$

in a meaningful sense.

Without loss of generality, set $\mu > 0$. Initially, asssume that $x \neq 0$, though this will be easily seen as a limiting case in the end. We have

$$\begin{aligned} G(x) &= \int_{\mathbb{R}^3} \frac{e^{ix\cdot\xi}}{(\xi^2 - \mu^2)} d\xi \\ &= \int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{e^{i|x| r cos(\theta)}}{(r-\mu)(r+\mu)} r^2 \sin(\theta) d\theta d\phi dr \\ &= \frac{4\pi}{|x|} \int_0^\infty \frac{r \sin(|x| r)}{(r-\mu)(r+\mu)} dr \end{aligned}$$

by first making a rotational change of variables where $\xi_3 \to \frac{x}{|x|}$, then using polar coordinates.

Now, we are again set up to use contour integration to find $G(x)$. See Figure 1 for the contour over which we integrate. We call this contour $\Gamma_{R,\epsilon}$.

Then, we have from residue theory

$$\begin{aligned} \int_{\Gamma_{R,\epsilon}} \frac{z e^{iz|x|}}{(z-\mu)(z+\mu)} dz &= 2\pi i \left[\frac{\mu e^{i|x|\mu}}{2\mu} \right] \\ &= \pi i e^{i|x|\mu}. \end{aligned}$$

However, breaking Γ down, we also have

$$\int_{\Gamma_{R,\epsilon}} \frac{z e^{iz|x|}}{(z-\mu)(z+\mu)} dz = 2i \int_0^R \frac{r \sin(|x| r)}{(r-\mu)(r+\mu)} dr + \frac{\pi e^{i|x|\mu}}{2} - \frac{\pi e^{-i|x|\mu}}{2}.$$

Combining terms and taking $R \to \infty$, we have

$$G(x) = \frac{4\pi^2 \cos(\mu|x|)}{|x|}.$$

This is valid for all x since the integral diverges as $x \to 0$.

Notice that in this analysis we have used the distributional convention

$$f(\lambda) = \frac{1}{2}\left[f(\lambda + i0) + f(\lambda - i0)\right].$$

However, it is not hard to see that taking the convention

$$f(\lambda) = f(\lambda \pm i0)$$

results in the

$$G(x) = \frac{4\pi^2 e^{\pm i|x|\mu}}{|x|}.$$

10.2. Strichartz Estimates

From the above time decay, we can also prove the standard space-time Strichartz estimates for $e^{i\mathscr{H}t}\phi$ where $\phi \in P_c\mathscr{H}$. We review the standard methods here as seen in [**SS99**]. From henceforward, let us assume that we work on the subspace of functions contained in $P_c\mathscr{H}$.

thm:strich1 **Theorem 10.16** *For p and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$, with $2 \le p \le \infty$, and $t \neq 0$, the transformation $e^{i\mathscr{H}t}$ maps continuously $L^{p'}(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ and*

eqn:strich1 (10.14)
$$\|e^{i\mathscr{H}t}\phi\|_{L^p} \lesssim \frac{1}{|t|^{d(\frac{1}{2}-\frac{1}{p})}}\|\phi\|_{L^{p'}}.$$

PROOF. This result follows from interpolation, see REFERENCE. □

DEFINITION 10.2.1. *The pair (q, r) of real numbers is called admissible if $\frac{2}{q} = \frac{d}{2} - \frac{d}{r}$ with $2 \le r < \frac{2d}{d-2}$ when $d > 2$, or $2 \le r \le \infty$ when $d = 1$ or $d = 2$.*

One uses a duality argument when the operator $e^{i\mathscr{H}t}$ is unitary. Namely,

$$|\langle e^{i\mathscr{H}t}\phi, G\rangle_{L^2(\mathbb{R}^{d+1})}| \lesssim \|\phi\|_{L^2}\|G\|_{L^{q'}L^{r'}}.$$

To this end, write

$$\begin{aligned}
|\int_{-\infty}^{\infty}\langle e^{i\mathscr{H}t}\phi, G\rangle_{L^2(\mathbb{R}^d)}ds| &= |\left\langle \phi, \int_{-\infty}^{\infty} e^{i\mathscr{H}-s}G(s)ds\right\rangle_{L^2(\mathbb{R}^d)}| \\
&\le \|\phi\|_{L^2(\mathbb{R}^d)}\left\|\int_{-\infty}^{\infty} e^{i\mathscr{H}-s}G(s)ds\right\|_{L^2(\mathbb{R}^d)},
\end{aligned}$$

where

$$\begin{aligned}
\left\|\int_{-\infty}^{\infty} e^{i\mathscr{H}-s}G(s)ds\right\|^2_{L^2(\mathbb{R}^d)} &= \left\langle \int_{-\infty}^{\infty} e^{i\mathscr{H}-s}G(s)ds, \int_{-\infty}^{\infty} e^{i\mathscr{H}-t}G(t)dt\right\rangle_{L^2(\mathbb{R}^d)} \\
&= \left\langle \int_{-\infty}^{\infty} G(t)dt, \int_{-\infty}^{\infty} e^{i\mathscr{H}t-s}G(s)ds\right\rangle_{L^2(\mathbb{R}^d)} \\
&\le \|G\|_{L^{q'}L^{r'}}\left\|\int_{-\infty}^{\infty} e^{i\mathscr{H}\cdot-s}G(s)ds\right\|_{L^qL^r}.
\end{aligned}$$

Using Equation 12.147, we have

$$\begin{aligned}\left\|\int_{-\infty}^{\infty} e^{i\mathscr{H}t-s}G(s)ds\right\|_{L^r} &\leq \int_{-\infty}^{\infty}\left\|e^{i\mathscr{H}t-s}G(s)\right\|_{L^r}ds \\ &\leq \int_{-\infty}^{\infty}\frac{1}{|t-s|^{d(\frac{1}{2}-\frac{1}{r})}}\|G(s)\|_{L^{r'}}ds \\ &\leq \int_{-\infty}^{\infty}\frac{1}{|t-s|^{\frac{2}{q}}}\|G(s)\|_{L^{r'}}ds.\end{aligned}$$

Hence, using the Hardy-Littlewood-Sobolev Theorem with $\gamma = -\frac{2}{q}$,

$$\left\|\int_{-\infty}^{\infty} e^{i\mathscr{H}t-s}G(s)ds\right\|_{L^qL^r} \lesssim \|G\|_{L^{q'}L^{r'}}.$$

10.3. Exercises

CHAPTER 11

Examples and Applications - Dirac δ potentials

sec:nonapp

We review here work done with Justin Holmer, Maciej Zworski, Vincent Duchêne and Michael Weinstein.

11.1. Karlsruhe Lecture 2 - Scattering Theory for One δ function

11.2. Wave Operators

Wave operators provide a means for converting operator bounds for a "free" dynamics generated by a constant coefficient Hamiltonian, $H_0 = -\Delta$ to analogous operator bounds about "interacting" dynamics associated with a variable coefficient Hamiltonian, $H = -\Delta + V$, on its continuous spectral subspace. Indeed let $W_\pm$ and $W_\pm^*$ denote wave operators associated with the free and interacting Hamiltonians H_0 and H (defined by (11.8) and (11.9)). Then we have

ww-star
$$W_\pm W_\pm^* = P_c, \quad W_\pm^* W_\pm = Id \tag{11.1}$$

wfwstar
$$f(H)P_c = W_\pm f(H_0) W_\pm^*, \; f(H_0) = W_\pm^* f(H) W_\pm, \quad f \text{ Borel on } \mathbb{R}\,. \tag{11.2}$$

It follows that bounds on $f(H)P_c$ acting between $W^{k_1,p_1}(\mathbb{R}^d)$ and $W^{k_2,p_2}(\mathbb{R}^d)$ can be derived from bounds on $f(H_0)$ between these spaces if the wave operators $W_\pm$ are bounded between $W^{k_1,p_1}(\mathbb{R}^d)$ and $W^{k_2,p_2}(\mathbb{R}^d)$ for $k_j \geq 0$ and $p \geq 1$. Here, $W^{k,p}(\mathbb{R}^d)$, $k \geq 1$, $p \geq 1$ denotes the Sobolev space of functions having derivatives up to order k in $L^p(\mathbb{R}^d)$.

Boundedness of wave operators in $W^{k,p}(\mathbb{R}^d)$, under smoothness and decay assumptions on $V(x)$ was proved by Yajima [**Yaj95**] in dimensions $d \geq 2$. Weder [**Wed99**] proved boundedness in dimension one; see also [**DF06**]. In [**Wed99**] it is assumed that $V \in L^1_\gamma(\mathbb{R})$, the space of all complex-valued measurable functions ϕ defined on $\mathbb{R}$ such that

wfwstar
$$\|\phi\|_{L^1_\gamma} = \int |\phi(x)|(1+|x|)^\gamma dx < \infty. \tag{11.3}$$

For V falling into a class of generic potentials, the assumption is $\gamma > 3/2$, otherwise it is assumed $\gamma > 5/2$. Wave operator bounds can be used to establish dispersive estimates, namely

$$\left\| e^{-iHt} P_c(H) f \right\|_{L^p(\mathbb{R}^d)} = \left\| W_\pm e^{-iH_0 t} W_\pm^* f \right\|_{L^p(\mathbb{R}^d)} \leq C\,|t|^{-\frac{d}{2}-\frac{d}{p}}\; \|f\|_{L^q(\mathbb{R}^d)}\,, \tag{11.4}$$
$$p^{-1} + q^{-1} = 1, \;\; p \geq 1.$$

Applications of dispersive estimates and wave operator bounds for singular potentials have appeared in the work of two separate works by two different combinations of the authors [**MW10**, **DW11**], which motivated this work. In general, Schrödinger operators with singular potentials arise in several mathematical models, which have recently been extensively investigated. See, for example, [**GHW04**, **JW04**, **HMZ07b**, **HMZ07a**, **HZ07**, **FOO08**, **LCFF+08**, **FJ08**, **MW10**], where Dirac delta function potentials are considered. Boundedness of wave operators for singular potentials is used implicitly in

references [**JW04**] and [**GHW04**], but this boundedness appears not to have been addressed previously. This gap in the literature is addressed by the current work. Another motivation for the present work is the study of scattering for highly oscillatory structures in the homogenization limit [**DW11**], where bounds on $(m^2+H)^{-1}P_c(H)(m^2-\partial_x^2)$, where $H=-\partial_x^2+V(x)$ is a Schrödinger operator with a singular (distribution) part to the potential $V(x)$, are required; see section 11.

This article is devoted to an extension of the one-dimensional results [**Wed99**] to the case of singular potentials. In particular, our results apply to Hamiltonians of the form

$$H \;=\; -\partial_x^2 \;+\; V(x),$$

where $V(x)$ satisfies:

Hypotheses (V)

Vdecomp
$$V(x) \;=\; V_{sing}(x) \;+\; V_{reg}(x), \tag{11.5}$$

Vsing
$$V_{sing}(x) \;=\; \sum_{j=0}^{N-1} q_j\,\delta(x-y_j),\;\; q_j, y_j \in \mathbb{R},\;\; y_j < y_{j+1},\;\; q_j \neq 0, \tag{11.6}$$

Vreg
$$\|V_{reg}\|_{L^1_{\frac{3}{2}+}(\mathbb{R})} \;\equiv\; \int_{\mathbb{R}} (1+|s|)^{\frac{3}{2}+}|V_{reg}(s)|\,ds \;<\; \infty. \tag{11.7}$$

sec:main

11.3. Main results

We first define and review properties of the wave operators. For basic results on wave operators see, for example, [**Agm75a**, **RS79**, **Sch02**].

Introduce the self-adjoint operators $H_0=-\Delta$ and $H=-\Delta+V$. Here, V is a real-valued potential, satisfying assumptions given below; see Section 11.6. Let $P_c=P_c(H)$ denote the continuous spectral projection associated with H. The wave operators, $W_\pm$ and their adjoints $W_\pm^*$ are defined by

eqn:wpm
$$W_\pm \equiv s-\lim_{t\to\pm\infty} e^{itH}e^{-itH_0} \tag{11.8}$$

eqn:wpmstar
$$W_\pm^* \equiv s-\lim_{t\to\pm\infty} e^{itH_0}e^{-itH}P_c. \tag{11.9}$$

The wave operators satisfy the properties (11.1) and (11.2). The notion of wave operators is intimately related to the idea of distorted Fourier bases, which are discussed in detail in [**Agm75b**], [**Hör05**], [**RS78b**]. In one dimension, this is directly related to the Jost solutions. These objects are studied in general in [**RS78b**] and generalized to even a certain class of non-self-adjoint operators in [**KS06a**].

The central Theorem 11.3, combined with the calculations of section 11.8, implies the following:

thm:main
Theorem 11.1 *Consider the Schrödinger operator with a potential, $V(x)$, satisfying* **Hypotheses (V)**. *Then $W_\pm$ and $W_\pm^*$ originally defined on $W^{1,p}\cap L^2$, $1\le p\le\infty$, have extensions to bounded operators on $W^{1,p}$, $1<p<\infty$. Moreover, there are constants C_p such*

that:

$$\|W_\pm f\|_{W^{1,p}(\mathbb{R})} \le C_p \|f\|_{W^{1,p}(\mathbb{R})},\ \|W_\pm^* f\|_{W^{1,p}(\mathbb{R})} \le C_p \|f\|_{W^{1,p}(\mathbb{R})}, \tag{11.10}$$

$$f \in W^{1,p}(\mathbb{R}),\ 1 < p < \infty. \tag{11.11}$$

REMARK 11.3.1. *In general, the wave operators are not bounded on L^1. The constraint $p > 1$ is due to the Hilbert transform, $\mathscr{H}$ not being bounded on L^1; see [**Wed99**].*

11.4. Strategy of Proof

sec:strategy

We use the approach for wave operators on $\mathbb{R}$ initiated by Weder in [**Wed99**]. The heart of the matter concerns the detailed low and high frequency behavior of Jost solutions, worked out by Deift and Trubowitz [**DT79**], or a consequence of their methods. The idea is to split the wave operators into high and low frequency components:

$$W_\pm = W_{\pm,high} + W_{\pm,low}.$$

For the high frequency component we prove for $\phi \in \mathscr{S}$,

$$W_{\pm,high}\phi = \sum_j S_{A_j}\phi, \text{ where } S_A\phi \equiv \int_{-\infty}^{\infty} A(x,y)\phi(y)\,dy.$$

For each $A = A_j$, we use the criterion (Young's inequality [**Fol95**]) for L^p, $1 \le p \le \infty$ boundedness:

$$\begin{aligned} C_A \equiv \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} |A(x,y)|\,dy + \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} |A(x,y)|\,dx < \infty \\ \Longrightarrow \left\|S_A\phi\right\|_{L^p} \le C_A \left\|\phi\right\|_{L^p}. \end{aligned}$$

to prove

Whigh

$$\left\|W_{\pm,high}\phi\right\|_{W^{k,p}} \le C_p \left\|\phi\right\|_{W^{k,p}},\ 1 < p < \infty,\ k \ge 0. \tag{11.12}$$

For the low frequency components, we have

$$W_{\pm,low} \sim \mathscr{H} + \sum_j S_{A_j},$$

where S_{A_j} is as above and $\mathscr{H}$ denotes the Hilbert Transform

HTdef

$$(\mathscr{H}\phi)(x) = \frac{1}{\pi}\,\text{P.V.}\int \frac{\phi(x-y)}{y}dy = \int_{-\infty}^{\infty} e^{ikx}\left(-i\,\text{sgn}(k)\right)\hat{\phi}(k)dk \tag{11.13}$$

Here, $\mathscr{F}$ and $\mathscr{F}^{-1}$ denote the Fourier Transform on $\mathbb{R}$ and its inverse, defined by

$$\hat{\phi}(k) \equiv \mathscr{F}\phi(k) = \frac{1}{2\pi}\int e^{-ikx}\phi(x)dx,\quad \check{\Phi}(x) \equiv \mathscr{F}^{-1}\Phi(x) = \int e^{ikx}\Phi(k)dk. \tag{11.14}$$

Thus, for low frequencies, boundedness

Wlow

$$\left\|W_{\pm,low}\phi\right\|_{W^{k,p}} \le C_p \left\|\phi\right\|_{W^{k,p}},\ 1 < p < \infty,\ k \ge 0 \tag{11.15}$$

reduces to the boundedness properties of the Hilbert transform [**Ste93a**]:

thm:CZest

Theorem 11.2 *$\mathscr{H}: W^{k,p} \to W^{k,p}$, for $1 < p < \infty$ and $k \ge 0$, with $\left\|\mathscr{H}\phi\right\|_{W^{k,p}(\mathbb{R})} \le K_p \|\phi\|_{W^{k,p}(\mathbb{R})}$.*

Estimates (11.12) and (11.15) then imply the theorem. The proof of (11.12) and (11.15) is given in section 11.7. We now develop some background for implementing the strategy.

11.5. Background spectral theory of $H = -\partial_x^2 + V$

sec:back

11.5.1. Distorted plane waves, $e_\pm(x;k)$. Consider the operator $H = -\partial_x^2 + V(x)$, defined as a self-adjoint operator on $L^2(\mathbb{R})$. Denote by P_d and P_c the discrete and continuous spectrum projections. P_d and P_c are orthogonal projections with $P_c = Id - P_d$.

Denote by R_0 the outgoing "free" resolvent operator $R_0(k) = (-\partial_x^2 - k^2)^{-1}$ with kernel

$$R_0(k)(x,y) = -(2ik)^{-1} e^{(ik|x-y|)}$$

and finally introduce the *distorted plane waves*, $e_\pm(x;k)$:

DEFINITION 11.5.1. *The functions $u = e_\pm(x;k)$ are the unique solutions to $(H-k^2)u = 0$ satisfying*

outgoing (11.16) $$e_\pm(x;k) \;=\; e^{\pm ikx} \;+\; \text{outgoing}(x),$$

where a function U is said to be outgoing as $|x| \to \infty$ if

$$(\,\partial_x \mp ik\,)\,U \to 0,\ x \to \pm\infty.$$

Thus, $e_\pm(x;k)$ is given by the integral equation:

eqn:epm1 (11.17) $$e_\pm(x;k) = e^{\pm ikx} - R_0(k)Ve_\pm(x;k)$$

or equivalently

eqn:epm2 (11.18) $$e_\pm(x;k) = e^{\pm ikx} - R_V(k)Ve^{\pm ikx}.$$

The continuous spectral projection, P_c, is given by

Pcf (11.19) $$P_c f(x) = \frac{1}{2\pi}\int\int_0^\infty \left(e_+(x,k)\,\overline{e_+(y,k)} + e_-(x,k)\,\overline{e_-(y,k)}\right) f(y)\,dk\,dy.$$

see, for example, [**TZ**].

We write

$$P_c f \;\equiv\; F_+^*\,F_+\,f, \text{ where it follows from (11.19) that}$$

F+def (11.20) $$F_+ f \;\equiv\; \int_{\mathbb{R}} \overline{\Psi_+(y,k)}\,f(y)\,dy, \quad F_+^* f \;\equiv\; \int_{\mathbb{R}} \Psi_+(y,k)\,f(k)\,dk \text{ and}$$

Psi+def (11.21) $$\Psi_+(y,k) = \frac{1}{\sqrt{2\pi}}\begin{cases} e_+(x;k) & k \ge 0, \\ e_-(x;-k) & k < 0. \end{cases}$$

We also define $\Psi_-(x,k) = \overline{\Psi_+(x,-k)}$.

11.5.2. Jost solutions. To make direct use of the arguments in [**Wed99**] and [**DT79**], we express the results of the preceding subsection in terms of *Jost solutions*, commonly introduced for one-dimensional Schrödinger operators.

Given the Schrödinger equation

eqn:statsch (11.22) $$-\frac{d^2}{dx^2}u + Vu = k^2 u,\ k \in \mathbb{C},$$

we define the Jost solutions, $f_j(x,k)$, $j = 1,2$, $\text{Im}k \ge 0$, to be the unique solutions of (11.22) satisfying the conditions:

Jostdef (11.23) $$f_1(x,k) \;-\; e^{ikx} \to 0, \quad x \to \infty, \text{ and}$$
$$f_2(x,k) \;-\; e^{-ikx} \to 0, \quad x \to -\infty.$$

The Jost solutions are linearly independent solutions of (11.22) for $k \neq 0$. Therefore, there are unique functions $T(k)$, $R_j(k)$, $j = 1,2$ such that for $k \in \mathbb{R} \setminus 0$

eqn:f1f2 (11.24) $$f_2(x,k) = \frac{R_1(k)}{T(k)} f_1(x,k) + \frac{1}{T(k)} f_1(x,-k),$$

(11.25) $$f_1(x,k) = \frac{R_2(k)}{T(k)} f_2(x,k) + \frac{1}{T(k)} f_2(x,-k)$$

For a potential, V, with compact support within $(-r,r)$, $R_j(k)$ and $T(k)$ are defined via the solutions:

u1exp (11.26) $$e_+(x;k) = t(k) f_1(x;k) = \begin{cases} e^{ikx} + R_2(k) e^{-ikx}, & x < -r, \\ T(k) e^{ikx}, & x > r \end{cases}$$

u2exp (11.27) $$e_-(x;k) = t(k) f_2(x;k) = \begin{cases} T(k) e^{-ikx}, & x < -r, \\ e^{-ikx} + R_1(k) e^{ikx}, & x > r \end{cases}$$

Generically,

(11.28) $$T(k) = \alpha k + o(k), \quad 1 + R_j(k) = \alpha_j k + o(k), \quad j = 1,2, \quad k \to 0.$$

$T(k)$ is called the transmission coefficient associated with H. $R_1(k)$ is the right to left reflection coefficient, and $R_2(k)$ the left to right reflection coefficient.

Finally, it is convenient to denote by $m_j(x,k)$, $j = 1,2$

def:mj (11.29) $$m_1(x,k) = e^{-ikx} f_1(x,k), \qquad \text{and} \qquad m_2(x,k) = e^{ikx} f_2(x,k).$$

It follows from (11.16), (11.23) and (11.24) that

Psi+T (11.30) $$\Psi_+(x,k) = \frac{1}{\sqrt{2\pi}} \begin{cases} T(k)\, e^{ikx}\, m_1(x,k) & k \geq 0, \\ T(-k)\, e^{ikx}\, m_2(x,-k) & k < 0, \end{cases}$$

where $m_1(x,k) - 1 \to 0$ as $x \to \infty$ and $m_2(x,k) - 1 \to 0$ as $x \to -\infty$. The detailed smoothness and decay properties, in x and k, of $m_j(x;k) - 1$ are required in estimates. These are given in section 11.8.

sec:state

11.6. Statement of the Central Theorem

Our central result, from which Theorem 11.1 follows, is:

thm:central **Theorem 11.3** *Let $H = -\partial_x^2 + V(x)$ be self-adjoint on $L^2(\mathbb{R})$ for which the transmission and reflection coefficients (see* (11.24)*) satisfy the bounds:*

RT-assume (11.31) $$|R_1(k)| + |R_2(k)| + |T(k) - 1| \leq \frac{C}{\langle k \rangle},$$

dkRT-assume (11.32) $$|\partial_k R_1(k)| + |\partial_k R_2(k)| + |\partial_k T(k)| = \mathcal{O}\left(\frac{1}{|k|}\right) (|k| \to \infty).$$

Assume further that S_1 is bounded on $W^{1,p}(\mathbb{R}_+)$ and S_2 is bounded on $W^{1,p}(\mathbb{R}_-)$ for $1 < p < \infty$, with

Sjdef (11.33) $$\big(S_j \Phi\big)(x) \equiv \int_{\mathbb{R}} R_j(x,y)\, \Phi(y)\, dy,$$

Rjdef (11.34) $$R_j(x,y) \equiv \int_{\mathbb{R}} e^{ikx} \big(m_j(x,k) - 1\big) e^{-iky}\, dk.$$

Then $W_\pm$ and $W_\pm^*$ originally defined on $W^{1,p}\cap L^2$, $1\le p\le\infty$, extend to bounded operators on $W^{1,p}$, $1<p<\infty$. Furthermore, there are constants C_p such that:

$$\|W_\pm f\|_{W^{1,p}}\le C_p\|f\|_{W^{1,p}},\ \|W_\pm^* f\|_{W^{1,p}}\le C_p\|f\|_{W^{1,p}}, \tag{11.35}$$

$$f\in W^{1,p}\cap L^2,\ 1<p<\infty. \tag{11.36}$$

REMARK 11.6.1. *Deift and Trubowitz [**DT79**] establish the hypotheses of the Theorem for any potential $V(x)$, for which $(1+|x|)\,|V(x)|\in L^1(\mathbb{R})$. We show in section 11.8 that their proof also applies to a potential of the type in Hypothesis* **(V)**, *$V=V_{sing}+V_{reg}$, where V_{sing} has a finite set of Dirac masses within an interval $(-A,A)$, and such that $(1+|x|)\left|V_{reg}(x)\right|\in L^{\frac{3}{2}+}(\mathbb{R})$.*

REMARK 11.6.2. *In fact, less restrictive bounds on V_{reg} as developed in [**DF06**] would suffice. However, for simplicity we will follow the work of [**Wed99**] as it makes some computations more explicit.*

sec:proof-of-Whighlow

11.7. Proof of Central Theorem 11.3

We follow the strategy described in section 11.4. Theorem 11.1 will follow from 11.3 by verifying the hypotheses of Theorem 11.3 for $V=V_{sing}+V_{reg}$. This verification is computed in section 11.8.

Let $\chi(x\ge 1)\in C^\infty(\mathbb{R})$ denote non-decreasing cut-off functions such that

$$\chi(x\ge 1)=\begin{cases}0 & x\le\frac{1}{2},\\ 1 & x\ge 1\,.\end{cases} \tag{11.37}$$

To localize in frequency space, introduce $\psi(|k|\le k_0)\in C_0^\infty(\mathbb{R})$ be a compactly supported cut-off function, depending on a parameter, k_0, to be chosen, such that

$$\psi(|k|\le k_0)=\begin{cases}1 & |k|\le\ k_0,\\ 0 & |k|\ge\ 2k_0\ .\end{cases} \tag{11.38}$$

We decompose any $\phi\in L^2(\mathbb{R})$ into its low and high frequency parts:

low+high
$$\phi(x)\ =\ \phi_{low}(x)\ +\ \phi_{high}(x),\quad \text{where using}\quad D\equiv -i\partial_x, \tag{11.39}$$

phi-low
$$\phi_{low}(x)\ \equiv\ \psi(|D|\le k_0)\phi(x)\ \equiv\ \int_{\mathbb{R}} e^{ikx}\psi(|k|\le k_0)\,\hat\phi(k)\,dk, \tag{11.40}$$

phi-high
$$\phi_{high}(x)\ \equiv\ \big(\,1\ -\ \psi(|D|\le k_0)\big)\,\phi(x)\ \equiv\ \int_{\mathbb{R}} e^{ikx}\big(\,1-\psi(|k|\le k_0)\,\big)\hat\phi(k)\,dk\,. \tag{11.41}$$

11.7.1. Bounds on $W_+\phi_{low}$. For $x\ge 0$, we can express $W_+\phi_{low}(x)$, in terms of $m_1(x,k)$, and for $x\le 0$, we can express $W_+\phi_{low}(x)$, in terms of $m_2(x,k)$. Since the cases $x\ge 0$ and $x\le 0$ are very similar, we only carry this calculation out in detail for $x\ge 0$. We have,

using the notation $Pf(x) = f(-x)$,

$$\begin{aligned}
& W_+\phi_{low} \;=\; F_+^*\mathscr{F}\,\psi(|D|\le k_0)\phi \\
= \;& \int_0^\infty e^{ikx}\;T(k)\;m_1(x,k)\;\psi\left(|k|\le k_0\right)\;\hat{\phi}(k)\;dk + \int_{-\infty}^0 e^{ikx}\;T(-k)\;m_2(x,-k)\;\psi\left(|k|\le k_0\right)\;\hat{\phi}(k)\;dk \\
= \;& \int_0^\infty e^{ikx}\;T(k)\;m_1(x,k)\;\psi\left(|k|\le k_0\right)\;\hat{\phi}(k)\;dk \\
& + \int_{-\infty}^0 e^{ikx}\;[R_1(-k)e^{-2ikx}\;m_1(x,-k) + m_1(x,k)]\;\psi\left(|k|\le k_0\right)\hat{\phi}(k)\;dk \\
= \;& \int_0^\infty e^{ikx}\;m_1(x,k)\;[T(k)+R_1(k)P]\;\psi\left(|k|\le k_0\right)\;\hat{\phi}(k)\;dk + \int_{-\infty}^0 e^{ikx}\;m_1(x,k)\;\hat{\phi}(k)\;dk, \quad x\ge 0,
\end{aligned}$$

where we have applied (11.20) and (11.30).

We continue by using that $\int_0^\infty\;[\dots]\;dk = \frac{1}{2}\int_{-\infty}^\infty(1+\mathrm{sgn}(k))\;[\dots]\;dk$, we have

W+phi (11.42)
$$\begin{aligned}
W_+\phi_{low} \;=\; & \frac{1}{2}\int_{-\infty}^\infty (1+\mathrm{sgn}(k))\;e^{ikx}(m_1(x,k)-1)T(k)\;\psi\left(|k|\le k_0\right)\hat{\phi}(k)dk \\
& + \frac{1}{2}\int_{-\infty}^\infty (1+\mathrm{sgn}(k))\;e^{ikx}(m_1(x,k)-1)R_1(k)P\;\psi\left(|k|\le k_0\right)\hat{\phi}(k)dk \\
& + \frac{1}{2}\int_{-\infty}^\infty (1-\mathrm{sgn}(k))\;e^{ikx}(m_1(x,k)-1)\;\psi\left(|k|\le k_0\right)\hat{\phi}(k)dk \\
& + \frac{1}{2}\int_{-\infty}^\infty (1+\mathrm{sgn}(k))\;e^{ikx}T(k)\;\psi\left(|k|\le k_0\right)\hat{\phi}(k)dk \\
& + \frac{1}{2}\int_{-\infty}^\infty (1+\mathrm{sgn}(k))\;e^{ikx}R_1(k)P\;\psi\left(|k|\le k_0\right)\hat{\phi}(k)dk \\
& + \frac{1}{2}\int_{-\infty}^\infty (1-\mathrm{sgn}(k))\;e^{ikx}\;\psi\left(|k|\le k_0\right)\hat{\phi}(k)dk, \quad x\ge 0.
\end{aligned}$$

For $x\le 0$ an analogous representation holds with $m_1(x,k)$ replaced by $m_2(x,k)$.

We now show that $W_{+,1,low}$ is a bounded operator on $W^{k,p}(\mathbb{R}_+)$. Each term in the first three lines of (11.42) is of the form:

first3 (11.43)
$$\phi \;\mapsto\; S_j \circ (I \pm i\,\mathscr{H}) \circ \Psi(D)\,\phi\,,$$

and each term in the last three lines is of the form

last3 (11.44)
$$\phi \;\mapsto\; (I \pm i\,\mathscr{H}) \circ \Psi(D)\,\phi\,,$$

where S_j, $j=1,2$ is defined in (11.33)-(11.34), $\mathscr{H}$ denotes the Hilbert transform (11.13), and

$$\begin{aligned}
\Psi(D) = \;& \mathscr{F}^{-1}\,\hat{\Psi}(k)\,\mathscr{F} \quad \text{and} \\
\hat{\Psi}(k) \;=\; & T(k)\,\psi(|k|\le k_0) \text{ or } R_1(k)\;P\,\psi(|k|\le k_0) \text{ or } \psi(|k|\le k_0).
\end{aligned}$$

By hypotheses on $T(k)$ and $R(k)$, $\hat{\Psi}(k)$ is a multiplier on $W^{k,p}(\mathbb{R})$ for $1<p<\infty$ [**Ste93a**]. The Hilbert transform is bounded (Theorem 11.2), so that the boundedness of the operators in (11.43) and (11.44) on $W^{k,p}$ for $1<p<\infty$ follows from boundedness of S_j, that holds by hypothesis. Therefore, one has

low-bound (11.45)
$$\|W_+\phi_{low}\|_{W^{1,p}(\mathbb{R})} \;\le\; C\,\|\phi\|_{W^{1,p}(\mathbb{R})}\,,$$

and this completes the low frequency analysis.

11.7.2. High Frequencies. We have, using (11.24) and the notation $Pf(x) = f(-x)$,

$$\begin{aligned}
W_+\phi_{high} &= F_+^*\mathscr{F}\left(1-\psi(|D|\le k_0)\right)\phi \\
&= \int_0^\infty T(k)e^{ikx}m_1(x,k)(1-\psi(|k|\le k_0))\hat{\phi}(k)dk \\
&\quad + \int_{-\infty}^0 T(-k)e^{ikx}m_2(x,-k)(1-\psi(|k|\le k_0))\hat{\phi}(k)dk \\
&= \int_0^\infty T(k)e^{ikx}m_1(x,k)(1-\psi(|k|\le k_0))\hat{\phi}(k)dk \\
&+ \int_{-\infty}^0 e^{ikx}[R_1(-k)e^{-2ikx}m_1(x,-k)+m_1(x,k)](1-\psi(|k|\le k_0))\hat{\phi}(k)dk \\
&= \int_0^\infty e^{ikx}m_1(x,k)[T(k)+R_1(k)P](1-\psi(|k|\le k_0))\hat{\phi}(k)dk + \int_{-\infty}^0 e^{ikx}m_1(x,k)\hat{\phi}(k)dk.
\end{aligned}$$

For $x \ge 0$ we rewrite this expression as

$$\begin{aligned}
W_+\phi_{high} &= \frac{1}{2}\int_{-\infty}^\infty e^{ikx}\,(1+\mathrm{sgn}(k))\,(m_1(x,k)-1)T(k)(1-\psi(|k|\le k_0))\hat{\phi}(k)dk \\
&\quad + \frac{1}{2}\int_{-\infty}^\infty e^{ikx}\,(1+\mathrm{sgn}(k))(m_1(x,k)-1)R_1(k)P(1-\psi(|k|\le k_0))\hat{\phi}(k)dk \\
&\quad + \frac{1}{2}\int_{-\infty}^\infty e^{ikx}\,(1-\mathrm{sgn}(k))(m_1(x,k)-1)(1-\psi(|k|\le k_0))\hat{\phi}(k)dk \\
&\quad + \frac{1}{2}\int_{-\infty}^\infty e^{ikx}\,(1+\mathrm{sgn}(k))T(k)(1-\psi(|k|\le k_0))\hat{\phi}(k)dk \\
&\quad + \frac{1}{2}\int_{-\infty}^\infty e^{ikx}\,(1+\mathrm{sgn}(k))R_1(k)P(1-\psi(|k|\le k_0))\hat{\phi}(k)dk \\
&\quad + \frac{1}{2}\int_{-\infty}^\infty e^{ikx}\,(1-\mathrm{sgn}(k))(1-\psi(|k|\le k_0))\hat{\phi}(k)dk, \quad x\ge 0.
\end{aligned}$$

An analogous expression, with $m_1(x,k)$ replaced by $m_2(x,k)$, is used for $x \le 0$. We proceed now to show that each term is bounded on $W^{1,p}(\mathbb{R}_+)$, $p \ge 1$.

Each summand in this decomposition of $W_+\phi_{high}$ is of the form:

model-high (11.46) $$\phi \mapsto S_j \circ \rho(D)\,\phi, \quad \text{or} \quad \phi \mapsto \rho(D)\phi.$$

where $\rho(D) = \mathscr{F}^{-1}\hat{\rho}(k)\mathscr{F}$. Here, S_j, $j = 1,2$, defined in (11.33) and (11.34), is bounded on $W^{1,p}(\mathbb{R}_+)$ for $1 < p < \infty$ by hypothesis. Moreover, $\rho(k)$ is a multiplier on $W^{1,p}(\mathbb{R})$ for $1 < p < \infty$ due to hypotheses on $R(k)$, $T(k)-1, \partial_k R(k)$ and $\partial_k T(k)$, and the fact that $1-\psi(|k|\le k_0)$ is smooth, asymptotically constant as $k \to \infty$ and vanishing in a neighborhood of 0. It follows that

W+highbound (11.47) $$\|W_+\phi_{high}\|_{W^{1,p}(\mathbb{R}_+)} \le C\|V\|_{L^1_{\frac{3}{2}+}(\mathbb{R})}\,\|\phi\|_{W^{1,p}(\mathbb{R}_+)}.$$

An estimate analogous to (11.47), similarly proved using a representation of $W_+\phi_{high}(x)$ for $x \le 0$, in terms of S_2, also holds. Thus,

high-bound (11.48) $$\|W_+\phi_{high}\|_{W^{1,p}(\mathbb{R})} \le C\|V\|_{L^1_{\frac{3}{2}+}(\mathbb{R})}\,\|\phi\|_{W^{1,p}(\mathbb{R})}\,.$$

The decomposition (11.39) and the bounds (11.45) and (11.48) imply the result. This completes the proof of the central result, Theorem 11.3.

sec:completion

11.8. Completion of the proof of Theorem 11.1

This section is devoted to the completion of the proof of Theorem 11.1, as a consequence of Theorem 11.3. The hypotheses of Theorem 11.3 are satisfied for potentials $V \in L^1_{\frac{3}{2}+}(\mathbb{R})$, by results in [**DT79**]. We briefly recall the argument below, and then we generalize it to potentials of the form (11.5) in Section 11.8.2.

11.8.1. The case of regular potentials. It follows from definition, $m_1(x,k) = e^{-ikx} f_1(x,k)$, that $m_1(x,k)$ is the unique solution of

defm1 (11.49) $$\frac{d^2}{dx^2} m_1 + 2ik \frac{d}{dx} m_1 = V\, m_1,\ k \in \mathbb{C},$$

with $m_1(x;k) \to 1$ as $x \to \infty$. Consequently, we have

propm1 (11.50) $$\begin{aligned} m_1(x,k) &= 1 + \int_x^\infty D_k(y-x) V(y) m_1(y,k)\, dy, \\ D_k(x) &= \int_0^x e^{2iky}\, dy. \end{aligned}$$

Indeed, for $V \in L^1_{\frac{3}{2}+}(\mathbb{R})$, the iterates of the Volterra integral are bounded by $\frac{\gamma(x)^n}{n!}$, with

$$\gamma(x) \equiv \int_x^\infty (t-x)|V(t)|\, dt,$$

and therefore converge. It follows that (11.50) defines equivalently $m_1(x,k)$, and that

$$m_1(x,k) \le e^{\gamma(x)} \gamma(x).$$

Now, by a careful analysis when $|x| \to -\infty$, one can improve the estimate with

eqn:uniform-bounds (11.51) $$m_1(x,k) \le C(1 + \max(-x,0)) \int_x^\infty (1+|t|)|V(t)|\, dt.$$

As a consequence, the function $m_1(x,k) - 1$ is in the Hardy space, and therefore there exists $B_1 \in L^2(\mathbb{R} \times \mathbb{R}_+)$ such that

defB1 (11.52) $$m_1(x,k) = 1 + \int_0^\infty B_1(x,y) e^{2iky}\, dy.$$

Now, the function $B_1(x,y)$ is equivalently defined with

defB-1 (11.53) $$B_1(x,y) \equiv \int_{x+y}^\infty V(t)\, dt + \int_0^y \int_{x+y-z}^\infty V(t) B_1(t,z)\, dt\, dz$$

defB-2 (11.54) $$= \sum_{n=0}^\infty K_n(x,y),$$

where K_n is defined by induction with

$$K_0(x,y) = \int_{x+y}^\infty V(t) dt\,, \qquad K_{n+1}(x,y) = \int_0^y \int_{x+y-z}^\infty V(t) K_n(t,z) dt dz.$$

It is then easy to prove by induction that

$$|K_n(x,y)| \le \frac{\gamma^n(x)}{n!} \eta(x+y), \quad \text{with } \eta(x) \equiv \int_x^\infty |V(t)|\, dt.$$

This allows us to confirm that the sum in (11.54) is well-defined and satisfies (11.53), plus the estimates

eqn:b1bd (11.55) $$|B_1(x,y)| \le e^{\gamma(x)} \eta(x+y), \quad \|B_1(x,\cdot)\|_{L^1} \le e^{\gamma(x)} \gamma(x),$$

from which we can deduce, taking the $x-$derivative of (11.53)

eqn:b1primebd

$$\left|\partial_x B_1(x,y)\right| \le C\, e^{\gamma_1(x)} \left(V(x+y) + \int_{x+y}^{\infty} |V(t)|dt \right),\ x\in\mathbb{R},\ y>0. \tag{11.56}$$

Now, the construction above, and (11.52) with the estimates (11.55) and (11.56), are sufficient to prove

lem:Sjbound

LEMMA 11.8.1. *S_1 is bounded on $W^{1,p}(\mathbb{R}_+)$ and S_2 is bounded on $W^{1,p}(\mathbb{R}_-)$ for $1<p<\infty$.*

PROOF. We focus on the bound for S_1 on $W^{k,p}(\mathbb{R}_+)$. The bound for S_2 is bounded on $W^{k,p}(\mathbb{R}_-)$ is similar. To prove boundedness of S_1 and ∂S_1 on L^p of the operator we use that the operator

$$S_R\Phi(x) = \int_{\mathbb{R}} R(x,y)\,\Phi(y)\,dy,$$

is bounded on L^p with estimate

younginеq

$$\|S_R\Phi\|_{L^p} \le C_R\,\|\Phi\|_{L^p},\ 1\le p\le\infty \tag{11.57}$$

if

youngprime

$$C_R \equiv \sup_{x\ge 0}\int_{\mathbb{R}} |R(x,y)|\,dy + \sup_{y\ge 0}\int_{\mathbb{R}} |R(x,y)|\,dx < \infty. \tag{11.58}$$

Using the representation formula (11.52) we have

$$R_j(x,y) \equiv \int_{\mathbb{R}} e^{ik(x-y)}\int_0^{\infty} e^{2ikz}\,B_1(x,z)\,dk\,dz = B_1\left(x,\frac{y-x}{2}\right).$$

Thus, the operator S_1 simplifies to

$$(S_1\Phi)(x) = \int_x^{\infty} B_1\left(x,\frac{y-x}{2}\right)\Phi(y)dy = \int_0^{\infty} B_1\left(x,\frac{\zeta}{2}\right)\Phi(\zeta-x)\,d\zeta,\quad x\ge 0.$$

Since we must estimate S_1 on $W^{1,p}$ we also compute

$$\begin{aligned}\partial_x(S_1\Phi)(x) &= \int_0^{\infty} B_1\left(x,\frac{\zeta}{2}\right)(-\partial_\zeta)\Phi(\zeta-x)\,d\zeta + \int_0^{\infty}\partial_x B_1\left(x,\frac{\zeta}{2}\right)\Phi(\zeta-x)\,d\zeta\\ &= \int_x^{\infty} B_1\left(x,\frac{y-x}{2}\right)(-\partial_y)\Phi(y)\,dy + \int_x^{\infty}\partial_x B_1\left(x,\frac{y-x}{2}\right)\Phi(y)\,dy,\quad x\ge 0.\end{aligned}$$

Note that by (11.55) and (11.56) we have for large enough x that

b1-bounds

$$|B_1(x,z)| \lesssim \int_{x+z}^{\infty}|V(s)|ds \text{ and } |\partial_x B_1(x,z)| \lesssim |V(x)| + \int_{x+z}^{\infty}|V(s)|ds. \tag{11.59}$$

Therefore,

$$\begin{aligned}&\sup_{x\ge 0}\int \mathbf{1}_{y\ge x}\left|B_1\left(x,\frac{y-x}{2}\right)\right|\,dy + \sup_{y\ge 0}\int \mathbf{1}_{y\ge x}\left|B_1\left(x,\frac{y-x}{2}\right)\right|\,dx\\ &\quad\le 2\sup_{x\ge 0}\int_0^{\infty}\int_{\frac{x+y}{2}}^{\infty}|V(s)|ds\,dy\\ &\quad\le 2\int_0^{\infty}\left(1+\frac{x+y}{2}\right)^{-\frac{3}{2}-}\int_{\frac{x+y}{2}}^{\infty}(1+s)^{\frac{3}{2}+}|V(s)|ds\\ &\quad\le const\times\|V\|_{L^1_{\frac{3}{2}+}(\mathbb{R})}.\end{aligned}$$

A similar bound applies to the kernel $\mathbf{1}_{x\ge y}\partial_x B_1\left(x,\frac{y-x}{2}\right)$. Thus, we have

$$\|S_1\Phi\|_{W^{1,p}(\mathbb{R}_+)} \equiv \|S_1\Phi\|_{L^p(\mathbb{R}_+)} + \|\partial_x(S_1\Phi)\|_{L^p(\mathbb{R}_+)} \le C\|V\|_{L^1_{\frac{3}{2}+}(\mathbb{R})}\,\|\Phi\|_{W^{1,p}(\mathbb{R}_+)}.$$

Applying similar arguments with S_1 replaced by S_2 for $x \le 0$ yields boundedness of S_2 on $W^{1,p}$. □

lem:RTbound

LEMMA 11.8.2.

$$|R_j(k)|,\ |T(k)-1| \le \frac{C}{\langle k\rangle} \quad \forall k \in \mathbb{R}, \qquad |\partial_k T(k)|,\ |\partial_k R_1(k)|,\ |\partial_k R_2(k)| \le \frac{C}{|k|} \quad as\ |k| \to \infty.$$

PROOF. We follow again the method of [**DT79**]. From (11.50), one has

$$\begin{aligned} m_1(x,k) &= 1 + \frac{1}{2ik}\int_0^\infty (e^{2ik(y-x)} - 1)V(y)m_1(y,k)\,dy \\ &= e^{-2ikx}\left(\frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2iky} m_1(y,k)V(y)\,dy\right) \\ &\quad + \left(1 - \frac{1}{2ik}\int_{-\infty}^{+\infty} m_1(y,k)V(y)\,dy\right) + o(1)(x \to -\infty). \end{aligned}$$

Moreover, one has from (11.23) and (11.24)

$$m_1(x,k) \;=\; e^{-2ikx}\frac{R_2(k)}{T(k)} \;+\; \frac{1}{T(k)} \;+\; o(1)(x \to -\infty).$$

This, and the same study on $m_2(x,k)$, leads to the following integral representations:

eqn:ir-T (11.60)
$$\frac{1}{T(k)} \;=\; 1 - \frac{1}{2ik}\int_{-\infty}^{+\infty} m_1(y,k)V(y)\,dy,$$

eqn:ir-R2 (11.61)
$$\frac{R_2(k)}{T(k)} \;=\; \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2iky} m_1(y,k)V(y)\,dy,$$

eqn:ir-R1 (11.62)
$$\frac{R_1(k)}{T(k)} \;=\; \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2iky} m_2(y,k)V(y)\,dy.$$

These integral representations, together with the $L^{1,3/2+}$ decay assumption on the potential and the uniform bounds (11.51), lead immediately to

$$|R_j(k)|,\ |T(k)-1| \le \frac{C}{\langle k\rangle}.$$

Now, differentiating (11.52) with respect to k leads to the uniform estimate

$$\partial_k m_1(x,k) \;\le\; C\langle k\rangle\langle x\rangle,$$

so that (11.60) yields

$$|\partial_k T(k)| \le \frac{C}{|k|}, \qquad \text{as} \qquad |k| \to \infty.$$

The equivalent estimates for $R_1(k), R_2(k)$ follow similarly from (11.61) and (11.62), and Lemma 11.8.2 holds. □

sec:completion-Vsing

11.8.2. The case of potentials with a singular component. We prove that one can generalize the construction above for generalized potentials, satisfying Hypothesis **(V)**, with equivalent estimates, so that Lemma 11.8.1 and 11.8.2 hold. As a consequence, the hypotheses of Theorem 11.3 are satisfied, and Theorem 11.1 is proved.

Proof of Lemma 11.8.1. We prove the desired estimates for $m_1(x,k)$, $x \geq 0$, and similar results apply to $m_2(x,k)$, $x \leq 0$. Let us define the function $B_1(x,y)$ with

$$B_1(x,y) \equiv \int_{x+y}^{\infty} V_{reg}(t)\, dt + \sum_{l=0}^{N-1} c_l \mathbf{1}(x_l - (x+y)) + \int_0^y \int_{x+y-z}^{\infty} V_{reg}(t) B_1(t,z)\, dt\, dz$$

defB1-1-sing (11.63) $$+ \int_0^y \sum_{l=0}^{N-1} c_l B_1(x_l,z) \mathbf{1}(x_l - (x+y-z))\, dz$$

defB1-2-sing (11.64) $$= \sum_{n=0}^{\infty} K_n(x,y),$$

with $\mathbf{1}$ the classical symmetric Heaviside function defined such that

$$\mathbf{1}(x) = \begin{cases} 1\,, & x > 0, \\ \frac{1}{2}\,, & x = 0, \\ 0\,, & x < 0, \end{cases}$$

and K_n defined by induction, with

$$K_0(x,y) = \int_{x+y}^{\infty} V_{reg}(t) dt + \sum_{l=0}^{N-1} c_l \mathbf{1}(x_l - (x+y)),$$
$$K_{n+1}(x,y) = \int_0^y \int_{x+y-z}^{\infty} V_{reg}(t) K_n(t,z) dt dz + \sum_{l=0}^{N-1} c_l \int_0^y K_n(x_l,z) \mathbf{1}(x_l - (x+y-z))\, dz.$$

Following the proof of Lemma 3, [**DT79**], it is easy to show by induction the following pointwise bound

$$|K_n(x,y)| \leq \frac{\gamma_1^n(x)}{n!} \eta_1(x+y),$$

with γ_1 and η_1 defined as

$$\gamma_1(x) \equiv \int_x^{\infty} (t-x)|V_{reg}(t)|\, dt + \sum_{l=0}^{N-1} |c_l| (x_l - x) \mathbf{1}(x_l - x),$$
$$\eta_1(x) \equiv \int_x^{\infty} |V_{reg}(t)|\, dt + \sum_{l=0}^{N-1} |c_l| \mathbf{1}(x_l - x).$$

This allows us to confirm that the sum in (11.64) is well-defined and satisfies (11.63), plus the estimates

eqn:b1bd-sing (11.65) $$|B_1(x,y)| \leq e^{\gamma_1(x)} \eta_1(x+y), \quad \|B_1(x,\cdot)\|_{L^1} \leq e^{\gamma(x)} \gamma(x)$$

and, differentiating with respect to x,

eqn:b1primebd-sing (11.66) $$\left|\partial_x B_1(x,y)\right| \leq C\, e^{\gamma_1(x)} \left(V(x+y) + \int_{x+y}^{\infty} |V(t)| dt \right),\ x \in \mathbb{R},\ y > 0.$$

Then, when we define by abusing the notation (which we justify in the following)

defm-sing (11.67) $$m_1(x,k) \equiv 1 + \int_0^{\infty} B_1(x,y) e^{2iky}\, dy,$$

it is easy to deduce from (11.63) that

$$\partial_x m_1(x,k) = \int_0^\infty \big(\partial_x B_1(x,y) - \partial_y B_1(x,y)\big) e^{2iky}\, dy + \int_0^\infty \partial_y B_1(x,y) e^{2iky}\, dy$$
$$= -\int_0^\infty \int_x^\infty V_{reg}(t) B_1(t,y)\, dt\, e^{2iky}\, dy - \int_0^\infty \sum_{l=0}^{N-1} c_l B_1(x_l,y) \mathbf{1}(x_l - x) e^{2iky}\, dy$$

dxm1 (11.68) $$- \int_0^\infty 2ik B_1(x,y) e^{2iky}\, dy - B_1(x,0).$$

$$\partial_x^2 m(x,k) = \int_0^\infty V_{reg}(x) B_1(x,y) e^{2iky}\, dy + \int_0^\infty \sum_{l=0}^{N-1} c_l B_1(x_l,y) \delta(x_l - x) e^{2iky}\, dy$$

(11.69) $$- \int_0^\infty 2ik \partial_x B_1(x,y) e^{2iky}\, dy + V_{reg}(x) + \sum_{l=0}^{N-1} c_l \delta(x_l - x).$$

Therefore, $m_1(x,k)$ is the unique function satisfying

$$\frac{d^2}{dx^2} m_1 + 2ik \frac{d}{dx} m_1 = \sum_l c_l \delta(x - x_l) + V_{reg}\, m_1,\ k \in \mathbb{C},$$

with $m_1(x;k) \to 1$ as $x \to \infty$, so that the abuse of notation is genuine.

Finally, Lemma 11.8.1 follows from (11.67), with the estimates (11.65) and (11.66).

trbounds *Proof of Lemma 11.8.2.* We follow again the method of [**DT79**]. The generalization of (11.50) to potentials satisfying Hypotheses **(V)** is

$$m_1(x,k) = 1 + \frac{1}{2ik} \int_0^\infty (e^{2ik(y-x)} - 1) V_{reg}(y) m_1(y,k)\, dy + \frac{1}{2ik} \sum_{l=0}^{N-1} c_l (e^{2ik(x_l - x)} - 1) m_1(x_l,k) \mathbf{1}(x_l - x)$$
$$= \frac{e^{-2ikx}}{2ik} \left(\int_{-\infty}^{+\infty} e^{2iky} m_1(y,k) V_{reg}(y)\, dy + \sum_{l=0}^{N-1} c_l e^{2ikx_l} m_1(x_l,k) \right)$$
$$+ 1 - \frac{1}{2ik} \left(\int_{-\infty}^{+\infty} m_1(y,k) V_{reg}(y)\, dy + \sum_{l=0}^{N-1} c_l m_1(x_l,k) \right) + o(1)(x \to -\infty).$$

Moreover, one has from (11.23) and (11.24)

$$m_1(x,k) = e^{-2ikx} \frac{R_2(k)}{T(k)} + \frac{1}{T(k)} + o(1)(x \to -\infty).$$

This, and the same study on $m_2(x,k)$, leads to the following integral representations:

eqn:ir-T-sing (11.70) $$\frac{1}{T(k)} = 1 - \frac{1}{2ik} \int_{-\infty}^{+\infty} m_1(y,k) V_{reg}(y)\, dy - \frac{1}{2ik} \sum_{l=0}^{N-1} c_l m_1(x_l,k),$$

eqn:ir-R2-sing (11.71) $$\frac{R_2(k)}{T(k)} = \frac{1}{2ik} \int_{-\infty}^{+\infty} e^{2iky} m_1(y,k) V_{reg}(y)\, dy + \frac{1}{2ik} \sum_{l=0}^{N-1} c_l e^{2ikx_l} m_1(x_l,k),$$

eqn:ir-R1-sing (11.72) $$\frac{R_1(k)}{T(k)} = \frac{1}{2ik} \int_{-\infty}^{+\infty} e^{2iky} m_2(y,k) V_{reg}(y)\, dy + \frac{1}{2ik} \sum_{l=0}^{N-1} c_l e^{2ikx_l} m_2(x_l,k).$$

The identity (11.67), with the estimates (11.65), guarantees the uniform bounds

$$|m_1(x,k) \le C\langle x \rangle, \qquad |\partial_k m_1(x,k)| \le C\langle k \rangle \langle x \rangle.$$

Therefore the $L^{1,3/2+}$ decay assumption on the potential V_{reg} leads immediately to

$$|R_j(k)|,\ |T(k) - 1| \le \frac{C}{\langle k \rangle}.$$

Now, differentiating (11.70) with respect to k yields

$$|\partial_k T(k)| \leq \frac{C}{|k|}, \qquad \text{as} \qquad |k| \to \infty.$$

The equivalent estimates for $R_1(k), R_2(k)$ follow similarly from (11.71) and (11.72), and Lemma 11.8.2 holds.

sec:singpot

11.8.3. $V(x) =$ **a sum of Dirac delta masses.** In this section we verify directly the hypotheses of Theorem 11.3 for the case of a potential, which is the sum of Dirac delta functions, thereby establishing the applicability of our main results to this case.

We follow the analysis from [**HMZ07b**] and [**TZ**], see also [**GS01**], [**GT92**] for specific examples. Seek solutions of the form

$$\left(H_{\vec{q},\vec{y}} - \frac{1}{2}k^2 \right) e_\pm(x,k) = 0, \tag{11.73}$$

where $H_{\vec{q},\vec{y}} = \sum_{j=0}^{N-1} q_j \delta(x - y_j)$ when $\vec{q} = (q_0, \cdots, q_{N-1})$, $\vec{y} = (y_0, \cdots, y_{N-1})$, and where $e_\pm(x,k)$ represent the distorted Fourier basis functions as defined (11.16). Thus,

$$e_+(x,k) = \begin{cases} e^{ikx} + B_0 e^{-ikx} \text{ for } x < y_0, \\ A_1 e^{ikx} + B_1 e^{-ikx} \text{ for } y_0 < x < y_1, \\ \vdots \\ A_N e^{ikx} \text{ for } x > y_{N-1}, \end{cases} \tag{11.74}$$

where we have taken $A_0 = 1$ and $B_N = 0$. With this choice of notation, we have, referring to (11.26) and (11.27), $A_N = T$ the transmission coefficient and $B_0 = R_1$ the reflection coefficient for the "incoming" plane wave e^{ikx} from $-\infty$. Then, we have the following system of equations implied by continuity and jump conditions at the points $\{y_j\}$ for $j = 0, \ldots, N-1$:

$$\begin{aligned} e^{iky_0} + B_0 e^{-iky_0} &= A_1 e^{ikx_0} + B_1 e^{-iky_0} \\ ik\left[A_1 e^{iky_0} - B_1 e^{-iky_0} - e^{iky_0} + B_0 e^{-iky_0} \right] &= 2q_0 \left[A_1 e^{iky_0} + B_1 e^{-iky_0} \right] \\ &\vdots \\ A_{N-1} e^{iky_{N-1}} + B_{N-1} e^{-iky_{N-1}} &= A_N e^{iky_{N-1}} \\ ik\left[A_N e^{iky_{N-1}} - A_{N-1} e^{iky_0} + B_{N-1} e^{-iky_0} \right] &= 2q_{N-1} \left[A_N e^{iky_{N-1}} \right]. \end{aligned}$$

Note, the above system guarantees unitarity, or that

$$|B_0|^2 + |A_N|^2 = 1. \tag{11.75}$$

We can define similarly

$$e_-(x,k) = \begin{cases} D_0 e^{-ikx} \text{ for } x < y_0, \\ C_1 e^{ikx} + D_1 e^{-ikx} \text{ for } y_0 < x < y_1, \\ \vdots \\ C_N e^{ikx} + e^{-ikx} \text{ for } x > y_{N-1}, \end{cases} \tag{11.76}$$

where now the incoming wave is e^{-ikx} from ∞ and the scattering matrix is determined by the transmission coefficients $D_0 = T$ and the reflection coefficient $C_N = R_2$ for the "incoming" plane wave e^{-ikx} from ∞.

11.8.3.1. *Bounds on m_1, m_2:* In addition, for general singular potentials with compact support, we have

$$m_1(x,k) = e^{-ikx} f_1(x,k) = \begin{cases} e^{-ikx}\frac{e_+(x,k)}{T(k)} \text{ for } x < y_{N-1}, \\ 1, \text{ for } x > y_{N-1}, \end{cases}$$

$$m_2(x,k) = e^{ikx} f_2(x,k) = \begin{cases} e^{ikx}\frac{e_-(x,k)}{T(k)} \text{ for } x > y_0, \\ 1, \text{ for } x < y_0. \end{cases}$$

Hence, there exists constants $C^1_\alpha(y_{N-1})$ and $C^2_\alpha(y_0)$ such that

eqn:regm1 (11.77) $$|\partial^\alpha_k m_1(x,k)| \leq C^1_\alpha(y_{N-1}) \text{ for } y_{N-1} > x \geq 0,$$

eqn:regm2 (11.78) $$|\partial^\alpha_k m_2(x,k)| \leq C^2_\alpha(y_0) \text{ for } y_0 < x \leq 0.$$

As a result, we see that an arbitrary collection of δ functions satisfies the required estimate for the proof of Lemma 11.8.1.

We conclude this subsection with explicit computations of the transmission and reflection coefficients for single and double δ well potentials:

11.8.3.2. **Single δ potential ($H_q = -q\delta(x)$):** Setting up the appropriate equations, we have

eqn:refco-d (11.79) $$R_1 = r_q = \frac{q}{ik-q},$$

eqn:tranco-d (11.80) $$T = t_q = \frac{ik}{ik-q},$$

where r_q, t_q are the reflection and transmission coefficients for H_q respectively. We must show the bounds from (11.31) hold, however such bounds follow clearly for (11.80), (11.79).

11.8.3.3. **Double δ potential ($H_{q,L} = -q(\delta(x+L)+\delta(x-L))$):** Setting up the appropriate equations, we have

eqn:refco-dd (11.81) $$R_1 = r_{q,L} = \left(\frac{q(ik-q)e^{2ikL} + q(ik+q)e^{-2ikL}}{q^2e^{2ikL} - (ik+q)^2e^{-2ikL}}\right)e^{-2ikL},$$

eqn:tranco-dd (11.82) $$T = t_{q,L} = \left(\frac{k^2}{q^2e^{2ikL} - (ik+q)^2e^{-2ikL}}\right)e^{-2ikL},$$

where $r_{q,L}$, $t_{q,L}$ are the reflection and transmission coefficients for $H_{q,L}$ respectively.

Again, we must verify Lemma (11.8.2), hence we must prove for instance

$$\dot{t}_{q,L}(k) \leq C(1+|k|)^{-1},$$

provided $qL \neq 1/2$. Indeed, we have

$$\dot{t}_{q,L}(k) = \frac{2k(k^2 - 2ikq + q^2(e^{4ikL}-1)) - 2ik^2(2Lq^2e^{4ikL} - (ik+q))}{(k^2 - 2ikq + q^2(e^{4ikL}-1))^2},$$

which satisfies

$$|\dot{t}_{q,L}(k)| \sim \mathcal{O}(|k|^{-1})$$

as $k \to \infty$ and

$$|\dot{t}_{q,L}(k)| \sim \mathcal{O}(\frac{1}{4q^2L - 2q})$$

as $k \to 0$. A similar computation holds for $r_{q,L}$.

CHAPTER 12

Nonlinear Dynamics

In this chapter, we discuss nonlinear scattering theory and finite dimensional dynamics.

In this note we study the dynamics of specific solutions to the focusing nonlinear Schrödinger equation

eqn:nls

$$\begin{cases} i u_t + \Delta u + \beta(|u|^2)u = 0,\ u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C} \\ u(0,x) = u_0(x),\ u_0 : \mathbb{R}^d \to \mathbb{C}. \end{cases} \tag{12.1}$$

Specifically, we will set $u_0 = \alpha\phi_\lambda(x)$ where $0 < \alpha \le 1$ and ϕ_λ is the unique positive, radial, exponentially decaying soliton solution of the elliptic equation

eqn:sol

$$-\Delta\phi_\lambda + \lambda^2\phi_\lambda - \beta(x,\phi_\lambda^2)\phi_\lambda. \tag{12.2}$$

The above elliptic equation results from using the ansatz $u(t,x) = e^{it\lambda^2}\phi_\lambda(x)$ in the Eq. (12.1). Hence, this constitutes a steady state, nondispersive solution. For a general study of the existence and properties of such solutions, see [**?**]. Also, great deal of work has gone into studying the stability of soliton solutions, see [**BW98**, **?**, **?**, **?**, **RSS03b**, **Sch09**, **?**, **Sha83**, **Sha83**, **?**, **Wei85b**, **Wei86**, **?**, **?**, **?**]. In general, there are several types of stability studied:

def1

DEFINITION 12.0.3. *A soliton ϕ is orbitally stable if for any $\epsilon > 0$, there exists $\delta > 0$ such that if the initial condition u_0 is such that*

$$\inf_{\theta,y} \| u_0(x) - e^{i\theta}\phi(x+y)\|_{H^1(x)} < \delta,$$

then the solution $u(x,t)$ of NLS satisfies

$$\inf_{\theta,y} \| u(x,t) - e^{i\theta}\phi(x+y)\|_{H^1(x)} < \epsilon.$$

def2

DEFINITION 12.0.4. *A soliton ϕ is said to be linearly stable if we can control the growth in some sense of the solution to the problem linearized around the soliton. This linearization will be explored later, but for now, let $\mathcal{H}$ be the Hamiltonian resulting from linearizing the equation about a soliton. Then, the new solution space is $L^2 \times L^2 = \mathcal{N} + \perp \mathcal{N}^*$ for*

$$\mathcal{N} = \bigcup_{l\ge 1} ker\mathcal{H}^l,$$

and $\mathcal{N}^*$ defined similarly for $\mathcal{H}^*$. If P is the projection onto $\perp \mathcal{N}^*$, then we have linear stability if

$$\sup_t \| e^{i\mathcal{H}t} P\psi\|_{L^2} < \infty,$$

where $\psi = e^{i\theta}\left(\phi_{\lambda^2}(x-vt) + R(t,x-vt)\right)$. This notion is closely related to that of orbital stability in most cases.

def3

DEFINITION 12.0.5. *Let $u_0 = \phi_\lambda(x) + R(x)$. A solution to NLS is said to be asymptotically stable if there exists $\epsilon > 0$ such that $\|R\| < \epsilon$ in some norm implies that there exists $w(x)$ such that*

$$\| u(x,t) - \phi_{\lambda_\infty} - e^{i\frac{t}{2}\Delta} w\|_{L^2(x)} \to 0. \tag{12.3}$$

In other words, a small perturbation of a soliton eventually converges to a soliton solution (of possibly different soliton parameter) plus dispersion. This is closely related to the notion of scattering for dispersive equations and soliton resolution.

In the case of orbitally stable solitons, we are particularly interested in understanding when one actually has asymptotic stability. For the cubic NLS equation in $\mathbb{R}$

$$i u_t + \Delta u + |u|^2 u = 0, \tag{12.4}$$

the works of [**Wei85b**, **Wei86**, **?**, **?**, **Sha83**, **Sha85**] show that we have orbital stability, while the results of [**?**] uses inverse scattering techniques to prove that asymptotic stability actually occurs. The methods require that the equation be integrable, which is in general not satisfied for other nonlinearities orin higher dimensions. In the sequel, we study how this remarkable phenomenon actually depends upon the regularity of the function $\beta(s)$.

12.1. Conserved Quantities

We assume that $u_0 \in H^1$ and $|x|u_0 \in L^2$. We refer to this property by saying that u_0 has finite variance. For this initial data, from the spatial and phase invariance of NLS, we have the following conserved quantities. Conservation of Mass:

$$Q(u) = \frac{1}{2}\int_{\mathbb{R}^d} |u|^2 dx = \frac{1}{2}\int_{\mathbb{R}^d} |u_0|^2 dx.$$

Conservation of Energy:

$$E(u) = \int_{\mathbb{R}^d} |\nabla u|^2 dx - \int_{\mathbb{R}^d} G(|u|^2) dx = \int_{\mathbb{R}^d} |\nabla u_0|^2 dx - \int_{\mathbb{R}^d} G(|u_0|^2) dx,$$

where

$$G(t) = \int_0^t \beta(s) ds.$$

We also have the so-called pseudoconformal conservation law:

$$\|(x + 2it\nabla)u\|_{L^2}^2 - 4t^2 \int_{\mathbb{R}^d} G(|u|^2) dx = \|x\phi\|_{L^2}^2 - \int_0^t \theta(s) ds, \tag{12.5}$$

where

$$\theta(s) = \int_{\mathbb{R}^d} (4(d+2)G(|u|^2) - 4d\beta(|u|^2)|u|^2).$$

12.2. Variational Techniques and Stability Results

Once we have the existence of soliton solutions, we would like to know if those solutions are stable. There are several forms of stability that defined above. All of these are tied very closely to the spectral properties of the linearized equation. From analysis due in [**Wei85b**] and [**GSS90**], these spectral properties depend heavily on the function $\delta(\lambda)$ described above. From a Lagrange multiplier approach, we see that

$$\frac{d}{d\lambda} E(\phi_\lambda) = -\lambda \frac{d}{d\lambda} Q(\phi_\lambda). \tag{12.6}$$

Hence, the condition for orbital stability and instability becomes

$$\frac{d}{d\lambda}Q(\phi_\lambda) > 0$$

and

$$\frac{d}{d\lambda}Q(\phi_\lambda) < 0,$$

respectively. For monomial nonlinearities, a simple scaling argument shows that L^2 subcritical powers give stable solitons and supercritical powers give unstable solitons.

Now, we will also study nonlinearities for which no such scaling occurs. Hence, we must use the differentiability of $Q(\phi_\lambda)$ proved in [**Sha83**, **Sha85**] and analyze at this curve numerically. Rodnianski, Soffer and Schlag [**RSS03b**] prove that this condition holds for a slightly different class of nonlinearities that depends on a small parameter θ and are small perturbations of stable power nonlinearities for any λ provided θ small enough. Numerics show that for the nonlinearity above, the graph of $Q(\phi_\lambda)$ with respect to λ for $0 < \lambda < \infty$ is a map which is ∞ at $\lambda = 0, \infty$ and has a global minimum at some $\lambda_0 > 0$ depending upon p, q. Hence, we have orbital instability for $0 < \lambda < \lambda_0$ and orbital stability for $\lambda_0 < \lambda$. At λ_0, we should expect to see instability as some perturbations can lead to orbitally unstable solitons and some perturbations can lead to energies for which soliton solutions are not allowed (below $Q(\phi_{\lambda_0})$).

A recent result of Comech-Pelinovsky [**?**] prove using spectral analysis that in fact the minimal mass ground state is unstable. Also, the recent results of Schlag [**Sch09**] and Krieger-Schlag [**?**] prove that though solitons for L^2 supercritical NLS are unstable, there is in fact a submanifold of perturbations along which there is in fact asymptotic stability in three and one dimensions respectively. To do this, they project onto on of the eigenvalues of the linearized Hamiltonian that does not cause instability. As shown in [**?**] and discussed later, we linearize around a minimal energy ground state, all the eigenvalues collapse to degree 4 generalized null space at 0. Hence, we hope that by projecting onto some subset of the kernel for this Hamiltonian, we might see a similar result at λ_0. We also hope to show that any perturbation which reduces the L^2 energy leads to linearized decay. These results will be discussed in more detail in later sections.

12.3. Linearization

Before we begin, we comment that all asymptotic stability results apart from inverse scattering rely heavily upon making the assumption $u(t,x) = e^{it}(\phi(x) + R(t,x))$, where $\lambda = 1$ here for simplicity. By linearizing about the soliton and studying the resulting equation on the perturbation R, one can study the long time dynamics under assumptions on the initial perturbation $R(0,x)$, see [**?**, **?**, **Sch09**, **?**]. However, this linearization scheme relies heavily upon the error in the scheme actually being quadratic in R, which will not occur if the regularity of β is too low.

To begin, let us write down the form of the linearized equation. First of all, let us assume we are looking for a solution $\psi = e^{i\lambda^2 t}(\phi_\lambda + R(x,t))$. For simplicity, set $\phi = \phi_\lambda$. Inserting this into the equation we know that since ϕ is a soliton solution we have

$$i(R)_t + \Delta(R) = -\beta(|\phi|^2)R - 2\beta'(|\phi|^2)\phi \mathrm{Re}(R) + O(R^2), \tag{12.7}$$

by splitting R up into its real and imaginary parts, then doing a Taylor Expansion. Hence, if $R = u + iv$, we get

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \mathscr{H} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{12.8}$$

where

$$\mathscr{H} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}, \tag{12.9}$$

where

$$L_- = -\Delta + \lambda^2 - \beta(\phi_\lambda^2)$$

and

$$L_+ = -\Delta + \lambda^2 - \beta(\phi_\lambda^2) - 2\beta'(\phi_\lambda^2).$$

For a detailed look at the spectrum of such operators for general nonlinearities, see [**?**, **?**]. For linearized Hamiltonians resulting from perturbations of stable solitons, we have the following:

- L_+, L_- are self-adjoint operators.
- $L_- \geq 0$.
- There exists one simple, negative eigenvalue for L_+.
- The continuous spectrum of L_-, L_+ is (λ^2, ∞).
- The null space of L_- is spanned by ϕ_λ.
- The null space of L_+ is spanned by $\partial_j \phi_\lambda$ for $j = 1, ..., d$.
- The generalized null space is of order 2 and spanned by $2d+2$ eigenfunctions. To see this, look at $\partial_\lambda \phi_\lambda$ and $x_j \phi_\lambda$ for $j = 1, ..., d$. We have $L_+ \partial_\lambda \phi_\lambda = 2\phi_\lambda$. Hence, $L_- L_+ \partial_\lambda \phi_\lambda = 0$. A similar analysis shows $L_+ L_- x_j \phi_\lambda = 0$.
- An embedded resonance is in fact an embedded eigenvalue.
- The continuous spectrum of $\mathscr{H}$ is $(-\infty, -\lambda^2) \cup (\lambda^2, \infty)$.
- There are no large embedded eigenvalues for $\mathscr{H}$.
- There are a finite number of allowed spherical harmonics in an expansion for an embedded eigenvalue.

12.4. Applicable Nonlinearities

In this section, we develop the nonlinearities that do not have the proper regularity described to make the above linearization rigorous. To begin, we study sub-critical monomial nonlinearities

$$\beta(s) = s^{\frac{q}{2}},$$

where $q < \frac{4}{d}$, $q \neq 2$. In other words, all subcritical exponents in all dimensions except for the cubic when $d = 1$. Specifically, we will focus on $d = 1$ and the nonlinearities known as the quadratic

$$\beta(s) = s^{\frac{1}{2}}$$

and the quartic

$$\beta(s) = s^{\frac{3}{2}}.$$

Also, we study what are called saturated nonlinearities given by

$$\beta(s) = s^{\frac{q}{2}} \frac{s^{\frac{p-q}{2}}}{1 + s^{\frac{p-q}{2}}},$$

where $\infty > p > \frac{4}{d} > q > 0$ for $d \geq 3$ and $\infty > p > \frac{d}{2} > q > 0$ for $d < 3$. In this case, for $|u|$ large, the behavior is L^2 subcritical and for $|u|$ small, the behavior is L^2 supercritical. Note that

$$\begin{aligned}
\beta(x) &= \frac{x^{\frac{p}{2}}}{1+x^{\frac{p-q}{2}}},\\
\beta'(x) &= \frac{x^{\frac{p}{2}-1}(\frac{p}{2}+\frac{q}{2}x^{\frac{p-q}{2}})}{(1+x^{\frac{p-q}{2}})^2},\\
\beta''(x) &= \frac{x^{\frac{p}{2}-2}(\frac{p}{2}(\frac{p}{2}-1)+(pq-\frac{q^2}{4}-\frac{q}{2}-\frac{p^2}{2})x^{\frac{p-q}{2}}+(\frac{q^2}{4}-\frac{q}{2})x^{p-q})}{(1+x^{\frac{p-q}{2}})^3},
\end{aligned}$$

hence there will be regularity issues for $p < 4$ (possible for $d \geq 2$).

12.5. Effective Hamiltonian Approach

For now, let us assume we are working with cubic NLS. We assume that the oscillation observed in orbital stability occurs through an exchange of mass between solitons. Then, we assume the manifold of solutions is of the form

$$M = \{e^{i\gamma 1}\mu_1\eta(\mu_1 x) + e^{i\gamma_2}\mu_2\eta(\mu_2 x)\},$$

with symplectic form $\omega(u,v) = \mathfrak{Im}\int u\bar{v}dx$. Then, we have

$$\begin{aligned}
\partial_{\gamma_1} &\to ie^{i\gamma_1}\mu_1\eta(\mu_1 x),\\
\partial_{\mu_1} &\to e^{i\gamma_1}\eta(\mu_1 x) + e^{i\gamma_1}\mu_1 x\eta'(\mu_1 x)\\
\partial_{\gamma_2} &\to ie^{i\gamma_2}\mu_2\eta(\mu_2 x)\\
\partial_{\mu_2} &\to e^{i\gamma_2}\eta(\mu_2 x) + e^{i\gamma_2}\mu_2 x\eta'(\mu_2 x).
\end{aligned}$$

Then, to find the 2-form on the manifold M, we look at

$$\begin{aligned}
\omega(\partial_{\gamma_1},\partial_{\mu_1}) &= \frac{1}{2}\int \eta^2(x)dx,\\
\omega(\partial_{\gamma_1},\partial_{\gamma_2}) &= \sin(\gamma_1-\gamma_2)\mu_1\mu_2\int \eta(\mu_1 x)\eta(\mu_2 x)dx = \alpha_1(\mu_1,\mu_2,\gamma_1,\gamma_2)\\
\omega(\partial_{\gamma_1},\partial_{\mu_2}) &= \cos(\gamma_1-\gamma_2)\mu_1\int [\eta(\mu_1 x)\eta(\mu_2 x) + x\mu_2\eta'(\mu_2 x)\eta(\mu_1 x)]dx = \alpha_2(\mu_1,\mu_2,\gamma_1,\gamma_2)\\
\omega(\partial_{\mu_1},\partial_{\gamma_2}) &= -\cos(\gamma_1-\gamma_2)\mu_2\int [\eta(\mu_1 x)\eta(\mu_2 x) + x\mu_1\eta'(\mu_1 x)\eta(\mu_2 x)]dx = \alpha_3(\mu_1,\mu_2,\gamma_1,\gamma_2)\\
\omega(\partial_{\mu_1},\partial_{\mu_2}) &= \sin(\gamma_1-\gamma_2)\int [\eta(\mu_1 x) + x\mu_1\eta'(\mu_1 x)][\eta(\mu_2 x) + x\mu_2\eta'(\mu_2 x)]dx = \alpha_4(\mu_1,\mu_2,\gamma_1,\gamma_2)\\
\omega(\partial_{\gamma_2},\partial_{\mu_2}) &= \frac{1}{2}\int \eta^2(x)dx.
\end{aligned}$$

For later simplicity, we set

$$\begin{aligned}
g_1(\mu_1,\mu_2) &= \int [\eta(\mu_1 x)\eta(\mu_2 x)]dx\\
g_2(\mu_1,\mu_2) &= \int [x\eta'(\mu_1 x)\eta(\mu_2 x)]dx\\
g_3(\mu_1,\mu_2) &= \int [x\eta(\mu_1 x)\eta'(\mu_2 x)]dx\\
g_4(\mu_1,\mu_2) &= \int [x^2\eta'(\mu_1 x)\eta'(\mu_2 x)]dx.
\end{aligned}$$

Since $\omega(\cdot,\Sigma_f) = f_{\gamma_1}d\gamma_1 + f_{\mu_1}d\mu_1 + f_{\gamma_2}d\gamma_2 + f_{\mu_2}d\mu_2$ and $\Sigma f = \alpha f_{\gamma_1}\partial_{\gamma_1} + \beta\partial_{\gamma_2} + \sigma\partial_{\mu_1} + \rho\partial_{\mu_2}$. In order to find α, β, σ and ρ, we use

$$\begin{aligned}
\omega(\cdot,\partial_{\mu_1}) &\rightarrow d\gamma_1 + \alpha_3 d\gamma_2 - \alpha_4 d\mu_2,\\
\omega(\cdot,\partial_{\mu_2}) &\rightarrow d\gamma_2 + \alpha_2 d\gamma_1 + \alpha_4 d\mu_1,\\
\omega(\cdot,\partial_{\gamma_1}) &\rightarrow d\mu_1 + \alpha_1 d\gamma_2 + \alpha_2 d\mu_2,\\
\omega(\cdot,\partial_{\gamma_2}) &\rightarrow d\mu_2 - \alpha_1 d\gamma_1 + \alpha_3 d\mu_1.
\end{aligned}$$

Then, we get

$$A\begin{bmatrix}\alpha\\ \beta\\ \sigma\\ \rho\end{bmatrix} = \begin{bmatrix}-1 & -\alpha_3 & 0 & \alpha_4\\ -\alpha_2 & -1 & \alpha_4 & 0\\ 0 & \alpha_1 & 1 & \alpha_2\\ -\alpha_1 & 0 & \alpha_3 & 1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \sigma\\ \rho\end{bmatrix} = \begin{bmatrix}f_{\mu_1}\\ f_{\mu_2}\\ f_{\gamma_1}\\ f_{\gamma_2}\end{bmatrix}.$$

Then,

$$A^{-1} = \begin{bmatrix}
\frac{1}{(\alpha_1\alpha_4+\alpha_2\alpha_3-1)} & \frac{\alpha_3}{(\alpha_1\alpha_4-\alpha_2\alpha_3+1)} & \frac{2\alpha_3\alpha_4}{(\alpha_1^2\alpha_4^2-\alpha_2^2\alpha_3^2+2\alpha_2\alpha_3-1)} & \frac{-\alpha_4(1+\alpha_1\alpha_4+\alpha_2\alpha_3)}{(\alpha_1^2\alpha_4^2-\alpha_2^2\alpha_3^2+2\alpha_2\alpha_3-1)}\\
\frac{-\alpha_2}{(\alpha_1\alpha_4+\alpha_2\alpha_3-1)} & \frac{-1}{(\alpha_1\alpha_4-\alpha_2\alpha_3+1)} & \frac{\alpha_4(\alpha_1\alpha_4-\alpha_2\alpha_3-1)}{(\alpha_1^2\alpha_4^2-\alpha_2^2\alpha_3^2+2\alpha_2\alpha_3-1)} & \frac{2\alpha_2\alpha_4}{(\alpha_1^2\alpha_4^2-\alpha_2^2\alpha_3^2+2\alpha_2\alpha_3-1)}\\
0 & \frac{\alpha_1}{(\alpha_1\alpha_4-\alpha_2\alpha_3+1)} & \frac{1}{(\alpha_1\alpha_4-\alpha_2\alpha_3+1)} & \frac{-\alpha_2}{(\alpha_1\alpha_4-\alpha_2\alpha_3+1)}\\
\frac{\alpha_1}{(\alpha_1\alpha_4+\alpha_2\alpha_3-1)} & 0 & \frac{\alpha_3}{(\alpha_1\alpha_4+\alpha_2\alpha_3-1)} & \frac{-1}{(\alpha_1\alpha_4+\alpha_2\alpha_3-1)}
\end{bmatrix}.$$

Hence,

$$\begin{aligned}
\dot\gamma_1 &= \left[\frac{f_{\mu_1}}{(\alpha_1\alpha_4+\alpha_2\alpha_3-1)} + \frac{\alpha_3 f_{\mu_2}}{(\alpha_1\alpha_4-\alpha_2\alpha_3+1)} + \frac{2\alpha_3\alpha_4 f_{\gamma_1}}{(\alpha_1^2\alpha_4^2-\alpha_2^2\alpha_3^2+2\alpha_2\alpha_3-1)}\right.\\
&- \left.\frac{\alpha_4(1+\alpha_1\alpha_4+\alpha_2\alpha_3)f_{\gamma_2}}{(\alpha_1^2\alpha_4^2-\alpha_2^2\alpha_3^2+2\alpha_2\alpha_3-1)}\right],\\
\dot\gamma_2 &= \left[\frac{-\alpha_2 f_{\mu_1}}{(\alpha_1\alpha_4+\alpha_2\alpha_3-1)} - \frac{f_{\mu_2}}{(\alpha_1\alpha_4-\alpha_2\alpha_3+1)} + \frac{\alpha_4(\alpha_1\alpha_4-\alpha_2\alpha_3-1)f_{\gamma_1}}{(\alpha_1^2\alpha_4^2-\alpha_2^2\alpha_3^2+2\alpha_2\alpha_3-1)}\right.\\
&+ \left.\frac{2\alpha_1\alpha_2 f_{\gamma_2}}{(\alpha_1^2\alpha_4^2-\alpha_2^2\alpha_3^2+2\alpha_2\alpha_3-1)}\right],\\
\dot\mu_1 &= \left[\frac{\alpha_1 f_{\mu_2}}{(\alpha_1\alpha_4-\alpha_2\alpha_3+1)} + \frac{f_{\gamma_1}}{(\alpha_1\alpha_4-\alpha_2\alpha_3+1)} - \frac{\alpha_2 f_{\gamma_2}}{(\alpha_1\alpha_4-\alpha_2\alpha_3+1)}\right],\\
\dot\mu_2 &= \left[\frac{\alpha_1 f_{\mu_1}}{(\alpha_1\alpha_4+\alpha_2\alpha_3-1)} + \frac{\alpha_3 f_{\gamma_1}}{(\alpha_1\alpha_4+\alpha_2\alpha_3-1)} - \frac{f_{\gamma_2}}{(\alpha_1\alpha_4+\alpha_2\alpha_3-1)}\right].
\end{aligned}$$

12.6. Hartree Equation in $\mathbb{R}^3$ - Spectral Theory

We review here results for equations of the form

$$-\Delta\phi_\lambda + \lambda\phi_\lambda - (|\phi_\lambda|^2 * |x|^{-1})\phi_\lambda = 0,$$

near a nonlinear bound state in the Hartree equation, otherwise known as Schrödinger-Poisson. We note there is an implicit scaling

$$\phi_\lambda(x) = \lambda\phi(\lambda^{\frac{1}{2}}x),$$

where $\phi = \phi_1$. Also Linearizing about such a state in the nonlinear Hartree (Schrödinger Poisson) equation, we have

eqn:Hartree

$$\left\{\begin{array}{c} iu_t + \Delta u - V(|x|)u + (|x|^{-1} * |u|^2)u = 0,\ u:\mathbb{R}\times\mathbb{R}^d \rightarrow \mathbb{C}\\ u(0,x) = u_0(x),\ u_0:\mathbb{R}^d \rightarrow \mathbb{C}.\end{array}\right. \tag{12.10}$$

gives a similar non-self-adjoint system to Schrödinger where now looking for a solution $\psi = e^{i\lambda^2 t}(\phi_\lambda + R(x,t))$. gives

$$\begin{aligned} iR_t + \Delta R - \left(V(|x|) + \lambda^2\right) R &= -(|x|^{-1} * (\phi+R)(\phi+\bar{R})(\phi+R) + (|x|^{-1} * \phi^2)\phi + \mathcal{O}(R^2) \\ &= -(|x|^{-1} * (\phi^2 + 2\mathfrak{Re}(R)\phi))(\phi+R) + \mathcal{O}(R^2) \\ &= -(|x|^{-1} * \phi^2)R - 2(|x|^{-1} * (\phi\mathfrak{Re}(R)))\phi + \mathcal{O}(R^2) \\ &= -(|x|^{-1} * \phi^2)\mathfrak{Re}(R) - 2(|x|^{-1} * (\phi\mathfrak{Re}(R)))\phi - (|x|^{-1} * \phi^2)\mathfrak{Im}(R) + \mathcal{O}(R^2), \end{aligned}$$

by splitting R up into its real and imaginary parts, then doing a Taylor Expansion. Hence, if $R = u + iv$, we get

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \mathscr{H} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{12.11}$$

where

$$\mathscr{H} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}, \tag{12.12}$$

where

$$L_- v = -\Delta v + V(|x|)v + \lambda^2 v - (|x|^{-1} * \phi_\lambda)v$$

and

$$L_+ u = -\Delta u + V(|x|)u + \lambda^2 u - (|x|^{-1} * \phi_\lambda^2)u - 2(|x|^{-1} * (\phi u))\phi.$$

Our goal for this section is to understand the spectrum of $\mathscr{H}$. It is known, see Lenzmann (2009) that

$$\mathrm{Ker}(L_-) = \{\phi\}$$

and

$$\mathrm{Ker}(L_+) = \{\partial_1\phi, \partial_2\phi, \partial_3\phi\}.$$

Also, orthogonality to adjoint kernel appears to give a coercive estimate of the form

$$\langle L_- u, u\rangle + \langle L_- v, v\rangle \geq c\|(u,v)\|_{H^1}.$$

Such coercivity is used to apply the orbital stability machinery of Weinstein to allow long time dynamics to be explored in Datchev-Ventura (2010), [**?**].

12.7. Example: Double Well Potentials

The cubic nonlinear Schrödinger/Gross-Pitaevskii (NLS/GP) equation

eqn:nlsdwp

$$i\partial_t u = (-\Delta + V(x))u - |u|^2 u \tag{12.13}$$

plays a central role in the mathematical description of nonlinear optical and quantum many-body phenomena. In the context of nonlinear optics, u denotes the slowly varying envelope of a nearly monochromatic electromagnetic field propagating in a waveguide, t the distance along the wave-guide and $x \in \mathbb{R}^2$ the dimensions transverse to the wave guide [**?**, **?**]. At low intensity, light is guided to a higher refractive index region, corresponding to a potential well $V(x)$. The *Kerr* nonlinear effect gives rise to an increase in the refractive index in regions of higher intensity, $|u|^2$, and therefore a "deeper" effective potential well $V(x) - |u|^2$. In the context of quantum many-body physics, NLS/GP emerges in the mean-field limit of many weakly interacting identical quantum particles

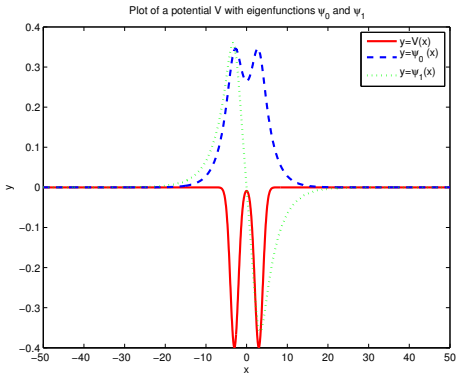

FIGURE 1. Double-well potential, $V(x;3,1)$. Superposed are the two eigenstates, even ground (ψ_0) and odd excited (ψ_1) states, of the Schrödinger operator, $-\partial_x^2 + V(x;3,1)$

fig:pot

obeying Bose statistics, as the number of particles tends to infinity; see [**?**, **?**, **?**] and other recent works. The potential $V(x)$ governs the confining trap for the quantum particles.[1]

In this paper we focus on a class of symmetric *double-well* potentials. A model to keep in mind is the two parameter family of symmetric double well potentials:

dw-gauss

$$V(x) = V(x;\sigma,L) = -\left[\frac{1}{\sqrt{4\pi\sigma^2}}e^{-\frac{(x-L)^2}{4\sigma^2}} + \frac{1}{\sqrt{4\pi\sigma^2}}e^{-\frac{(x+L)^2}{4\sigma^2}}\right], \tag{12.15}$$

which converges as $\sigma \to 0^+$ to a double-delta well at $\pm L$. In figure 1 the case $L = 3$, $\sigma = 1$ is shown.

Double-well potentials are of particular interest in optics as models of coupled parallel wave guides channeling light beams, which interact through evanescent tails. In the quantum context, these are natural and simple systems in which to study quantum tunneling. As discussed below, the combined effects of a confining double-well potential with focusing cubic nonlinearity lead to the phenomenon of spontaneous *symmetry breaking of the ground state* at sufficiently high optical power or particle number. See, for example, [**?**, **?**, **?**] for experimental studies of symmetry breaking. *Our goal in this paper is to investigate the phase space dynamics of NLS/GP near the symmetry breaking point.*

The equation NLS/GP, (12.161), is a Hamiltonian system and expressible in the form:

Ham-u

$$i\partial_t u = \frac{\delta \mathbb{H}}{\delta u^*}\,. \tag{12.16}$$

$\mathbb{H}$ denotes the conserved Hamiltonian energy functional:

$$\mathbb{H}[u] = \int \left(|\nabla u|^2 + V|u|^2 - \frac{1}{4}|u|^4\right) dx.$$

[1] NLS / GP falls into a larger class of models:

eqn:nlsdwp-gK

$$i\partial_t u = (-\Delta + V(x))u + gK\left[|u|^2\right]u, \tag{12.14}$$

allowing for more general nonlinear terms, *e.g.* more general functions of $|u|^2$ or K nonlocal; see, for example, [**?**]. For $g < 0$ (focusing case) our analysis yields very similar results to those obtained in this paper. For $g > 0$ (defocusing) our methods can be adapted to prove a shadowing result for trajectories of the finite dimensional reduction which arises in this case as well.

The conserved squared L^2 norm (particle number / optical power) is denoted:

NNdef (12.17) $$\mathcal{N}[u] \;=\; \int |u|^2 dx.$$

We are interested in the dynamics near special classes of *nonlinear bound states* of NLS/GP. Nonlinear bound states are solutions of the form

$$u(x,t) \;=\; e^{-i\Omega t}\,\Psi_\Omega(x),$$

where Ψ_Ω is spatially localized:

nl-elliptic (12.18) $$(-\Delta + V(x))\Psi_\Omega \; - |\Psi_\Omega|^2\,\Psi_\Omega \;=\; \Omega\,\Psi_\Omega, \quad \Psi_\Omega \in H^1(\mathbb{R}).$$

Consider first the case of the linear eigenvalue problem:

evp (12.19) $$(-\Delta + V(x))\Psi_\Omega \;=\; \Omega\,\Psi_\Omega, \quad \Psi_\Omega \in H^1(\mathbb{R}).$$

In this case, there is a least energy *ground state*, ψ_0 with corresponding simple eigenvalue Ω_0 [**RS78b**]. If the separation between wells is sufficiently large, then the ground state eigenfunction is a *bimodal* positive symmetric state, which reflects the discrete symmetry of the potential [**?**, **?**, **JW04**]; see figure 1. In addition, there is an anti-symmetric (odd) state, ψ_1 with energy Ω_1, such that $\Omega_0 < \Omega_1 < 0$.

In the attractive / focusing *nonlinear* case, (12.13), the character of solutions, and the solution set varies with the solution norm. Indeed, if we consider the set of solutions of (12.18) on the level set

constraintN (12.20) $$\int |\Psi_\Omega|^2 = \mathcal{N},$$

we find that for large enough well-separation, there is a *symmetry breaking threshold* $\mathcal{N}_{cr}$ [**?**]; see figure 2:

(1) If $\mathcal{N} < \mathcal{N}_{cr}$ there is a unique positive, symmetric and bimodal state.
(2) For $\mathcal{N} > \mathcal{N}_{cr}$, (modulo phase) there are three positive localized states: a symmetric state (which exists for all $\mathcal{N} > 0$) and two *asymmetric states*, biased respectively to the right and left wells.
(3) As $\mathcal{N}$ increases beyond $\mathcal{N}_{cr}$ this symmetry broken state becomes increasingly concentrated in one of the wells [**?**, **?**].
(4) The symmetric (bimodal) state is dynamically stable for $\mathcal{N} < \mathcal{N}_{cr}$ and unstable for $\mathcal{N} > \mathcal{N}_{cr}$. For $\mathcal{N} > \mathcal{N}_{cr}$ the asymmetric states are stable.

That symmetry breaking occurs at *sufficiently large* values of $\mathcal{N}$ was studied variationally in [**?**] for the nonlinear Hartree equation. Their method can be adapted to a large class of equations, for which a ground state can be realized as a minimizer of a Hamiltonian, $\mathcal{H}$, subject to fixed $\mathcal{N}$. The above detailed portrait of the symmetry breaking transition and the exchange of stability among branches was studied in detail in [**?**]. In particular, for large well-separation, L,

Ncr-est (12.21) $$\mathcal{N}_{cr}(L) = \mathcal{O}\left(\Omega_1(L) - \Omega_0(L)\right) = \mathcal{O}(e^{-\kappa L}), \;\; \kappa > 0.$$

Figure 3 displays a numerical computation of the symmetry-broken state, occurring for $\mathcal{N} > \mathcal{N}_{cr}$, where $V(x)$ is of the type displayed (12.15).

Our goal is to explore the *detailed general dynamics* near the symmetry breaking transition. In particular, we wish to show that on large but finite time scales, the dynamics is controlled by a finite dimensional dynamical system. Note that this is not a study of soliton stability for the nonlinear system, though the questions are certainly related as shall be observed in the sequel. Toward a formulation of precise results, we first introduce a class of double-well potentials in one dimension. Following [**?**], start with

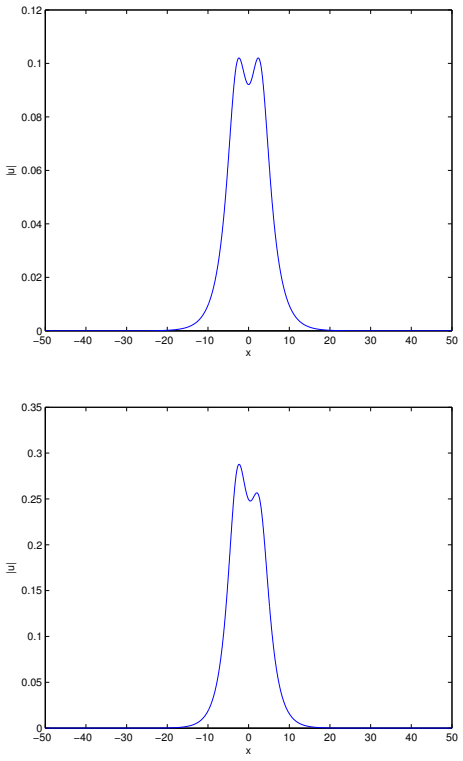

FIGURE 2. A symmetric (bimodal) soliton and an asymmetric soliton, which exists only for $\mathcal{N} > \mathcal{N}_{cr}$.

fig:sols

a single rapidly decaying potential well centered at 0, $V_0(x)$, for which the Schrödiner operator $H_0 = -\partial_x^2 + V_0$ has exactly one (simple) eigenvalue ω. Then, construct a double well potential

$$V_L(x) = V_0(x-L) + V_0(x+L), \quad L > 0 \tag{12.22}$$

and define the Schrödinger operator

HLdef

$$H_L = -\partial_x^2 + V_L. \tag{12.23}$$

There exists $L_0 > 0$ such that for $L > L_0$, H_L has a pair of simple eigenvalues, $\Omega_0 = \Omega_0(L)$ and $\Omega_1 = \Omega_1(L)$ and corresponding eigenfunctions ψ_0 (even) and ψ_1 (odd):

$$\begin{aligned} &H\psi_j = \Omega_j \psi_j, \ j = 0,1; \quad \psi_j \in L^2 \\ &\Omega_0 < \Omega_1 < 0. \end{aligned}$$

As noted above, the symmetry breaking threshold, $\mathcal{N}_{cr}(L)$ (equation (12.21)) is exponentially small for large well-separation. Therefore, to study the dynamics in a neighborhood of the symmetry breaking point, it is natural to use coordinates associated with the *linear* operator $H_0 = -\partial_x^2 + V$. Throughout this result, we will assume that V is such that ψ_0 and ψ_1 are the only discrete eigenfunctions of H_0 and in addition that V is sufficiently smooth and decaying as defined in [**Wed99**] and discussed in Appendix 12.13

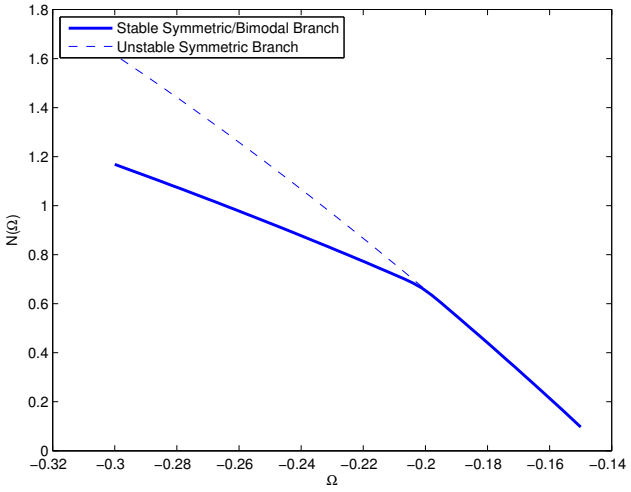

FIGURE 3. A numerical plot of the symmetry breaking in the soliton curve. Specifically, we plot here N versus Ω for a double- (Gaussian) well potential with separation parameter $L = 3$ and $\sigma = 1$. Note the stable curve is invariant under reflection about the y axis, hence the slightly thicker curve represents the resulting degeneracy for the asymmetric states.

fig8

as to guarantee dispersive estimates to do perturbation theory. We note that the essential estimates are also satisfied for double delta-function wells, the we explore in our numerical computations. These ideas will be explored further in a forthcoming note [**?**].

REMARK 12.7.1. *Though we choose L such that the spectral assumptions are satisfied, we note that throughout this paper we work with a fixed L for a particular $V(x)$. Thus our estimates depend on this fixed L. In our small data analysis, various parameters are chosen, independent of L.*

We expand the solution as follows:

decomp

$$u(x,t) = c_0(t)\psi_0(x) + c_1(t)\psi_1(x) + \mathscr{R}(t,x), \tag{12.24}$$
$$\langle \psi_j, \mathscr{R}(\cdot,t) \rangle = 0, \quad j = 0,1.$$

By orthogonality, we have

totalenergy

$$\mathscr{N}[u(\cdot,t)] = |c_0(t)|^2 + |c_1(t)|^2 + \int |\mathscr{R}(x,t)|^2 \, dx = \mathscr{N}[u(\cdot,0)]. \tag{12.25}$$

Referring to this decomposition, we now give an overview of the paper:

(1) In section 12.8 we express the NLS / GP as an equivalent dynamical system governing $c_0(t)$, $c_1(t)$ and $\mathscr{R}(x,t)$. This Hamiltonian system, equivalent to NLS / GP, has the form of two equations governing discrete nonlinear "oscillators", coupled to an equation for a wave field, $\mathscr{R}(x,t)$.

(2) In section 12.9 we study the finite dimensional reduction governing c_0 and c_1, obtained by dropping all terms which couple the oscillator and field variables. This reduction is a finite dimensional Hamiltonian system with conserved Hamiltonian:

$$H(c_0, \bar{c_0}, c_1, \bar{c_1}),$$

given in (12.50), and l^2 invariant

$$N[c_0, c_1] = |c_0|^2 + |c_1|^2.$$

This finite dimensional Hamiltonian undergoes symmetry breaking for a critical value of N, denoted N_{cr}^{FD}.

REMARK 12.7.2. *The reduction described is also obtainable via substitution of the Ansatz* (12.24) *into the Lagrangian of NLS/GP, that is restrict $\mathscr{L}_{NLS-GP}$ to the bound state manifold, and obtain the equations of motion; see, for example, [**HZ07**] and references cited therein.*

Use of these two invariants facilitates an analysis of the finite dimensional phase space; trajectories with prescribed values of H and N lie on a two-dimensional surface. We then discuss some of the rich dynamics of this finite dimensional reduction and our goal is to prove their persistence for non-trivial time scales for the full PDE, NLS/GP.

For the dynamics on level set $N \sim N_{cr}^{FD} \approx \mathscr{N}_{cr}$, we establish the following behavior:

(a) $N < N_{cr}^{FD}$: There is an elliptic fixed point, corresponding to the stable symmetric state of NLS/GP for $\mathscr{N} < \mathscr{N}_{cr}$. A neighorhood of this fixed point is foliated by stable time-periodic solutions.
(b) $N > N_{cr}^{FD}$: The fixed point for $N < N_{cr}^{FD}$ persists, but transitions from being a stable elliptic point to an unstable saddle. This corresponds to the unstable symmetric-bimodal state for $\mathscr{N} > \mathscr{N}_{cr}$. At $N = N_{cr}^{FD}$, two new elliptic equilibria bifurcate and, a neighborhood of each is foliated by stable time-periodic solutions. These stable time-periodic oscillations correspond to stable oscillations around the stable asymmetric standing waves for $\mathscr{N} > \mathscr{N}_{cr}$.
(c) $N > N_{cr}^{FD}$: There are periodic solutions, outside a separatrix, which encircle both new equilibria. In the physical configuration space, these correspond to soliton transport from one well to the other and back, continuing periodically. Furthermore, one can quantify the "energy barrier" that must be exceeded to dislodge a "soliton" from localization about one of the wells. Energy thresholds, such as that described above play an important role in transport of energy in inhomogeneous and discrete systems. Natural directions to pursue beyond this work are transport in systems with many wells and the *Peireles-Nabbaro barrier* for motion of localized coherent structures in discrete lattice systems.

(3) In section 12.11 we prove that *certain* phase space structures for the finite dimensional dynamical systems, persist in character for the full NLS/GP system, *on long, but finite time scales.* Stated nontechnically,

Theorem: For any sufficiently small amplitude periodic solution about an equilibrium state of the finite dimensional reduction (N above or below the bifurcation threshold), there is a solution of the PDE (NLS/GP), whose projection into the finite dimensional phase space, shadows this finite dimension orbit on very long time scales.

A precise statement is given in Theorem 3; see also Figure 4. The time scales on which these results hold enable us to see nearly periodic oscillations for the PDE on long time scales, i.e. through many, many oscillations, but not on an infinite time scale. Indeed, we do not expect the persistence of such oscillations on infinite time scales due to nonlinear coupling of bound to radiation modes for the full system, for example, [**?**, **?**]. We conjecture that in the infinite time limit, the soliton executes a (radiation) damped oscillation to some stable nonlinear bound state, corresponding to the damped oscillatory decay to a stable equilibrium of the finite dimensional reduction. Evidence is presented in section **??**, where numerical simulations are discussed.

We remark that our current theorem does not apply to perturbations of general "large" periodic orbits of the finite dimensional reduction. More detailed information on the Floquet theory of the linearized equations about general periodic orbits is still needed. This is currently being investigated.

(4) In section **??** we provide a summary and discussion of open problems.

ACKNOWLEDGMENTS. JLM was supported, in part, by a U.S. National Science Foundation Postdoctoral Fellowship and a Hausdorff Center Postdoctoral Fellowship. MIW was supported, in part, by U.S. NSF Grants DMS-04-12305 and DMS-07-07850. MIW wishes to thank Vered Rom-Kedar and Eli Shlizerman for stimulating discussions.

12.7.1. Notation.

(1) The spaces $H^s(\mathbb{R}^n)$, $L^p(\mathbb{R}^n)$, $W^{k,p}$ are the standardly defined Sobolev integration spaces.
(2) We have the L^2 inner product: $\langle f, g\rangle = \int_{\mathbb{R}^n} f\bar{g}$.
(3) Projection onto the bound states of $-\Delta + V$:

$$P_j f \quad = \quad \langle \psi_j, f\rangle \, \psi_j \, = \, (\pi_j f)\, \psi_j, \;\; j = 0, 1.$$

(4) Projection onto the continuous spectral part of $-\Delta + V$:

$$P_c f \, = \, (I - P_0 - P_1) f.$$

sec:oscillator-field

12.8. Formulation of NLS/GP as a coupled finite-infinite dimensional system

In this section we derive an equivalent formulation of NLS/GP, appropriate for studying the exchange of energy between the bound and radiative parts of the solution. We substitute the decomposition (12.24) into NLS / GP and, to the resulting equation, apply the projection operators P_0, P_1 and P_c to obtain the coupled system

eqn:sys-id

$$\begin{cases} i\dot{c}_0 - \Omega_0 c_0 + a_{0000}|c_0|^2 c_0 + a_{0011}(c_1^2\bar{c}_0 + 2|c_1|^2 c_0) = F_0(c_0, c_1, \bar{c}_0, \bar{c}_1; \mathscr{R}, \bar{\mathscr{R}}), \\ i\dot{c}_1 - \Omega_1 c_1 + a_{1111}|c_1|^2 c_1 + a_{0011}(c_0^2\bar{c}_1 + 2|c_0|^2 c_1) = F_1(c_0, c_1, \bar{c}_0, \bar{c}_1; \mathscr{R}, \bar{\mathscr{R}}), \\ i\mathscr{R}_t - H\mathscr{R} = P_c F_b(c_0, c_1, \bar{c}_0, \bar{c}_1) + P_c F_R(c_0, c_1, \bar{c}_0, \bar{c}_1; \mathscr{R}, \bar{\mathscr{R}}). \end{cases} \tag{12.26}$$

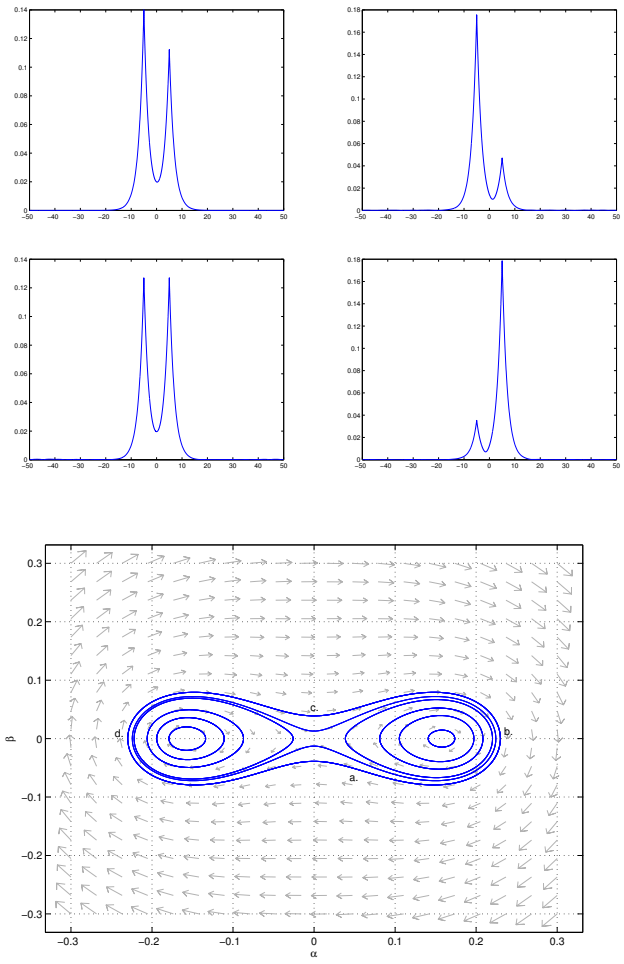

FIGURE 4. A numerical plot of an oscillatory solution to (12.13) at times a, b, c, and d respectively from the corresponding phase plane diagram of the finite dimensional Hamiltonian truncation.

fig:schem

Here,

$$a_{ijkl} = \langle \psi_i \psi_j \psi_k, \psi_l \rangle, \tag{12.27}$$

Fj-def

$$F_j = \pi_j F, \tag{12.28}$$

for $j = 0, 1$ where

eqn:F

$$\begin{aligned} F &= \left[2|c_0|^2\psi_0^2 + 2|c_1|^2\psi_1^2 + 2(c_0\bar{c}_1 + c_1\bar{c}_0)\psi_0\psi_1\right]\mathscr{R} \\ &+ \left[c_0^2\psi_0^2 + c_1^2\psi_1^2 + 2c_0c_1\psi_0\psi_1\right]\bar{\mathscr{R}} \\ &+ \left[\,\bar{c}_1\psi_1 + \bar{c}_0\psi_0\right]\mathscr{R}^2 + \left[2c_0\psi_0 + 2c_1\psi_1\,\right]|\mathscr{R}|^2 + |\mathscr{R}|^2\mathscr{R}, \end{aligned} \tag{12.29}$$

and

Fperp (12.30) $$F_b = |c_0|^2 c_0\psi_0^3 + (c_0^2\bar{c}_1 + 2|c_0|^2 c_1)\psi_0^2\psi_1 + (c_1^2\bar{c}_0 + 2c_0|c_1|^2)\psi_0\psi_1^2 + |c_1|^2 c_1\psi_1^3\,,$$

G-def (12.31)
$$\begin{aligned} F_R = {} & \left[\, 2|c_0|^2\psi_0^2 + 2|c_1|^2\psi_1^2 + 2(c_0\bar{c}_1 + c_1\bar{c}_0)\psi_0\psi_1 \,\right]\mathscr{R} \\ & + \left[\, c_0^2\psi_0^2 + c_1^2\psi_1^2 + 2c_0c_1\psi_0\psi_1 \,\right]\bar{\mathscr{R}} \\ & + \left[\, \bar{c}_1\psi_1 + \bar{c}_0\psi_0 \right]\mathscr{R}^2 + \left[2c_0\psi_0 + 2c_1\psi_1 \,\right]|\mathscr{R}|^2 \\ & + |\mathscr{R}|^2\mathscr{R}\,. \end{aligned}$$

We have that $a_{ijkl} \neq 0 \iff i+j+k+l = 0 \bmod 2$. For simplicity we take

$$a_{ijkl} = 1 \text{ if } i+j+k+l = 0 \bmod 2.$$

REMARK 12.8.1. *In general, it is only possible to scale say a_{0000} and a_{1111} to* 1 *and one should have an arbitrary constant $a_{0011} = a_{0011}(L)$. However, since we fix L as above, the constant a_{0011} will not change scale with our eventual asymptotic parameter. Hence, we will take some liberty and assume in the sequel that $a_{0011} = 1$ though in a true physical model the parameters δ_0, δ_1 and ϵ in Theorem 3 would be modified slightly depending on a_{0011} (and hence L).*

Initial conditions for (12.26) are:

sys-data (12.32) $$c_j(0) = \langle \psi_j, u(\cdot,0)\rangle,\ j = 0,1;\quad \mathscr{R}(\cdot,0) = P_c u(\cdot,0).$$

The system (12.26) may be viewed as an infinite dimensional Hamiltonian system, comprised of two coupled subsystems: one finite dimensional, governing bound state degrees of freedom described by c_0 and c_1, and a second, infinite dimensional, governing a dispersive wave field, $\mathscr{R}(\cdot,t) = P_c u(\cdot,t)$. In the next section, we focus on the finite dimensional truncation of (12.26) obtained by setting $R = 0$. After obtaining a detailed description of the phase space of this truncation in section 12.9, we then turn toward proving the long-time persistence of structures within the finite dimensional system, within the full infinite dimensional problem.

Before embarking on this path, we conclude this section with an alternative coordinate description of (12.26). These coordinates prove very useful in understanding the bifurcations within the finite dynamical subsystem, as $\mathscr{N}$ is varied, and its participation within the infinite dimensional dynamics.

sec:hamco

12.8.1. Alternative coordinates. We require the following decomposition, which is proved in Appendix **??**.

Introduce the following change of coordinates:

$$(\, c_0(t), c_1(t), \bar{c}_0(t), \bar{c}_1(t), \mathscr{R}(\cdot,t)\,) \mapsto (\, A(t), \alpha(t), \beta(t), \theta(t), R(\cdot,t)\,)$$

defined by

eqn:rho0 (12.33) $$c_0(t) = A(t)e^{i\theta(t)},$$

eqn:rho1 (12.34) $$c_1(t) = (\alpha(t) + i\beta(t))e^{i\theta(t)},$$

u-new (12.35) $$u(x,t) = e^{i\theta(t)}(A(t)\psi_0 + (\alpha(t) + i\beta(t))\psi_1 + R),\ \ \mathscr{R} = e^{i\theta(t)}R.$$

Such coordinates have been used for finite dimensional systems in for instance the works [**?**], [**?**]. Substitution of this ansatz into NLS/GP, we see

$$\begin{aligned} (i\dot{A} - \dot{\theta}A - \Omega_0 A)\psi_0 + (i\dot{\alpha} - \dot{\beta} - \dot{\theta}(\alpha + i\beta) - (\alpha + i\beta)\Omega_1)\psi_1 \\ + iR_t - HR - \dot{\theta}R = F_b(A,\alpha,\beta) + F_R(A,\alpha,\beta;R,\bar{R}), \end{aligned}$$

where F_b, F_R is determined similarly to (12.30),(12.31). See Appendix **??** for further details. Hence,

$$\begin{aligned}
\dot{A} &= \mathfrak{Im}(F_0),\\
\dot{\alpha} &= \dot{\theta}\beta + \Omega_1\beta + \mathfrak{Im}(F_1),\\
\dot{\beta} &= -\dot{\theta}\alpha - \Omega_1\alpha - \mathfrak{Re}(F_1),\\
\dot{\theta} &= -\Omega_0 - A^{-1}\mathfrak{Re}(F_0),\\
iR_t &= HR + \dot{\theta}R + P_c(F_b + F_R)
\end{aligned}$$

or

$$\begin{aligned}
\dot{A} &= \mathfrak{Im}(F_0),\\
\dot{\alpha} &= (-\Omega_0 - A^{-1}\mathfrak{Re}(F_0))\beta + \Omega_1\beta + \mathfrak{Im}(F_1),\\
\dot{\beta} &= -(-\Omega_0 - A^{-1}\mathfrak{Re}(F_0))\alpha - \Omega_1\alpha - \mathfrak{Re}(F_1),\\
\dot{\theta} &= -\Omega_0 - A^{-1}\mathfrak{Re}(F_0),\\
iR_t &= (H-\Omega_0)R - A^{-1}\mathfrak{Re}(F_0)R + P_c(F_b + F_R).
\end{aligned}$$

Note, there will be linear terms in R and $\bar{R}$ contained in F, which we must handle very carefully in our eventual iteration argument.

This leads to the system of the form

$$\begin{aligned}
\dot{A} &= -2\alpha\beta A + \text{Error}(R,\bar{R};A,\alpha,\beta),\\
\dot{\alpha} &= [\Omega_1 - (\alpha^2+\beta^2) - A^2 + \dot{\theta}]\beta + \text{Error}(R,\bar{R};A,\alpha,\beta),\\
\dot{\beta} &= -[\Omega_1 - (\alpha^2+\beta^2+A^2) - 2A^2 + \dot{\theta}]\alpha + \text{Error}(R,\bar{R};A,\alpha,\beta),\\
\dot{\theta} &= -\Omega_0 + A^2 + (3\alpha^2+\beta^2) + A^{-1}\text{Error}(R,\bar{R};A,\alpha,\beta),\\
iR_t &= (H-\Omega_0)R - A^{-1}\mathfrak{Re}(\pi_0(F))R + P_cF_b(A,\alpha,\beta) + P_cF_R(A,\alpha,\beta;R,\bar{R}).
\end{aligned}$$

Then, substituting $\dot{\theta}$, the system becomes

eqn:coupledfd1 (12.36) $$\dot{A} = -2\alpha\beta A + \text{Error}_A(R,\bar{R};A,\alpha,\beta),$$

eqn:coupledfd2 (12.37) $$\dot{\alpha} = [\Omega_1 - \Omega_0 + 2\alpha^2]\beta + \text{Error}_\alpha(R,\bar{R};A,\alpha,\beta),$$

eqn:coupledfd3 (12.38) $$\dot{\beta} = -[\Omega_1 - \Omega_0 - 2A^2 + 2\alpha^2]\alpha + \text{Error}_\beta(R,\bar{R};A,\alpha,\beta),$$

eqn:coupledfd4 (12.39) $$\dot{\theta} = -\Omega_0 + A^2 + 3\alpha^2 + \beta^2 + \text{Error}_\theta(R,\bar{R};A,\alpha,\beta),$$

eqn:coupledid (12.40) $$\begin{aligned} iR_t &= (H-\Omega_0)R + (A^2+3\alpha^2+\beta^2)R\\ &\quad + P_cF_b(A,\alpha,\beta) + P_cF_R(A,\alpha,\beta;R,\bar{R}), \end{aligned}$$

with $P_cR = R$ and

eqn:est-ERROR-fperp (12.41) $$\begin{aligned} P_cF_b &= P_c\left[A^3\psi_0^3 + (\alpha^2+\beta^2)(\alpha+i\beta)\psi_1^3 + A(\alpha+i\beta)^2\psi_1^2\psi_0\right.\\ &+ \left. 2A(\alpha^2+\beta^2)\psi_1^2\psi_0 + A^2(\alpha-i\beta)\psi_0^2\psi_1 + 2A^2(\alpha+i\beta)\psi_0^2\psi_1\right], \end{aligned}$$

eqn:est-ERROR-gperp (12.42) $$\begin{aligned} P_cF_R &= P_c\left(\left[2A^2\psi_0^2 + 4A(\alpha)\psi_0\psi_1 + 2(\alpha^2+\beta^2)\psi_1^2\right]R\right.\\ &+ \left[A^2\psi_0^2 + (\alpha+i\beta)^2\psi_1^2 + 2A(\alpha+i\beta)\psi_0\psi_1\right]\bar{R}\\ &+ \left.\left[A\psi_0 + (\alpha-i\beta)\psi_1\right]R^2 + \left[2A\psi_0 + 2(\alpha+i\beta)\psi_1\right]|R|^2 + |R|^2R\right). \end{aligned}$$

For simplicity, set $\vec{\sigma} = (\alpha,\beta,A)$.

prop:app1

PROPOSITION 12.8.2. *Assume $A,\dots,R$ defined as in* (12.36)-(12.40). *Then, we have*

eqn:est-ERROR-A
$$|Error_A| \lesssim |\vec{\sigma}|^2\|R\|_{L^\infty} + |\vec{\sigma}|\|R\|_{L^\infty}^2 + \|R\|_{L^\infty}^3, \tag{12.43}$$

eqn:est-ERROR-alpha
$$|Error_\alpha| \lesssim |\vec{\sigma}|^2\|R\|_{L^\infty} + |\vec{\sigma}|\|R\|_{L^\infty}^2 + \|R\|_{L^\infty}^3, \tag{12.44}$$

eqn:est-ERROR-beta
$$|Error_\beta| \lesssim |\vec{\sigma}|^2\|R\|_{L^\infty} + |\vec{\sigma}|\|R\|_{L^\infty}^2 + \|R\|_{L^\infty}^3, \tag{12.45}$$

eqn:est-ERROR-theta
$$|Error_\theta| \lesssim A^{-1}\left(|\vec{\sigma}|^2\|R\|_{L^\infty} + |\vec{\sigma}|\|R\|_{L^\infty}^2 + \|R\|_{L^\infty}^3\right). \tag{12.46}$$

The error terms in α and β gained from $\dot\theta$ are of the form $A^{-1}\alpha$, $A^{-1}\beta \ll 1$.

we seek to prove the error bounds from Proposition 12.8.2. We assume we are near the symmetry breaking equilibrium point, meaning we may take $\alpha(t),\beta(t) \ll A(t)$ and $A(t) > c > 0$.

Then, we have

$$\begin{aligned}
(i\dot A &- \dot\theta A - \Omega_0 A)e^{i\theta}\psi_0 + (i\dot\alpha - \dot\beta - \dot\theta(\alpha+i\beta) - (\alpha+i\beta)\Omega_1)e^{i\theta}\psi_1 \\
&+ i(\partial_t - H - \dot\theta)R = \\
&- \left[A^3\psi_0^3 + (\alpha^2+\beta^2)(\alpha+i\beta)\psi_1^3 + A(\alpha+i\beta)^2\psi_1^2\psi_0 + 2A(\alpha^2+\beta^2)\psi_1^2\psi_0\right. \\
&\left. + A^2(\alpha-i\beta)\psi_0^2\psi^1 + 2A^2(\alpha+i\beta)\psi_0^2\psi^1\right] \\
&- \left[2A^2\psi_0^2 + 4A(\alpha)\psi_0\psi_1 + 2(\alpha^2+\beta^2)\psi_1^2\right]R \\
&+ \left[A^2\psi_0^2 + (\alpha+i\beta)^2\psi_1^2 + 2A(\alpha+i\beta)\psi_0\psi_1\right]\bar R \\
&- \left[A\psi_0 + (\alpha-i\beta)\psi_1\right]R^2 - \left[2A\psi_0 + 2(\alpha+i\beta)\psi_1\right]|R|^2 - |R|^2R.
\end{aligned}$$

Hence,

$$\begin{aligned}
(i\dot A &- \dot\theta A - \Omega_0 A + A^3 + A(\alpha+i\beta)^2 + 2A(\alpha^2+\beta^2)) = \\
&- \left[A\langle\psi_0^2, R^2\rangle + (\alpha-i\beta)\langle\psi_0\psi_1, R^2\rangle\right] - \left[2A\langle\psi_0^2, |R|^2\rangle + 2(\alpha+i\beta)\langle\psi_0\psi_1, |R|^2\rangle\right] \\
&- \left[\langle\psi_0, |R|^2R\rangle\right].
\end{aligned}$$

As a result, we have

$$\begin{aligned}
(i\dot\alpha - \dot\beta - \dot\theta(\alpha+i\beta) &= \Omega_1(\alpha+i\beta) - (\alpha^2+\beta^2)(\alpha+i\beta) \\
&\quad -2A^2(\alpha+i\beta) - A^2(\alpha-i\beta)) \\
&\quad - \left[A\langle\psi_0\psi_1, R^2\rangle + (\alpha-i\beta)\langle\psi_1^2, R^2\rangle\right]e^{i2\theta} \\
&\quad - \left[2A\langle\psi_0\psi_1, |R|^2\rangle + 2(\alpha+i\beta)\langle\psi_1^2, |R|^2\rangle\right] \\
&\quad - \left[\langle\psi_0, |R|^2R\rangle\right]
\end{aligned}$$

and

$$\begin{aligned}
iR_t - HR - \dot\theta R &= -P_c\left[A^3\psi_0^3 + (\alpha^2+\beta^2)(\alpha+i\beta)\psi_1^3 + A(\alpha+i\beta)^2\psi_1^2\psi_0\right. \\
&\quad \left. +2A(\alpha^2+\beta^2)\psi_1^2\psi_0 + A^2(\alpha-i\beta)\psi_0^2\psi^1 + 2A^2(\alpha+i\beta)\psi_0^2\psi^1\right] \\
&\quad -P_c\left[2A^2\psi_0^2 + 4A(\alpha)\psi_0\psi_1 + 2(\alpha^2+\beta^2)\psi_1^2\right]R \\
&\quad -P_c\left[A^2\psi_0^2 + (\alpha+i\beta)^2\psi_1^2 + 2A(\alpha+i\beta)\psi_0\psi_1\right]\bar R \\
&\quad -P_c\left[A\psi_0 + (\alpha-i\beta)\psi_1\right]R^2 - P_c\left[2A\psi_0 + 2(\alpha+i\beta)\psi_1\right]|R|^2 \\
&\quad -P_c|R|^2R
\end{aligned}$$

or

$$iR_t - HR - \dot\theta R = \left[F_b(A,\alpha,\beta,\theta) + F_R(A,\alpha,\beta,\theta;R,\bar R)\right],$$

where we have assumed

$$P_c F_b = F_b,\ P_c F_R = F_R.$$

Let us take $A > 0$. Then, we see

$$\begin{aligned}
\dot{A} &= -2\alpha\beta A + \mathrm{Error}'_A, \\
\dot{\alpha} &= [\Omega_1 - (\alpha^2+\beta^2) - A^2 + \dot{\theta}]\beta + \mathrm{Error}'_\alpha, \\
\dot{\beta} &= -[\Omega_1 - (\alpha^2+\beta^2+A^2) - 2A^2 + \dot{\theta}]\alpha + \mathrm{Error}'_\beta, \\
A\dot{\theta} &= -\Omega_0 A + A^3 + A(3\alpha^2+\beta^2) + \mathrm{Error}'_\theta \\
iR_t - HR - \dot{\theta}R &= F_b(A,\alpha,\beta) + F_R(A,\alpha,\beta;R,\bar{R}).
\end{aligned}$$

Specifically, we have (12.36)-(12.40) with

$$\begin{aligned}
\mathrm{Error}_A(R,\bar{R},\vec{\alpha}) &= \mathfrak{Im}\big(-\langle\big[2A^2\psi_0^3 + 4A(\alpha)\psi_0^2\psi_1 + 2(\alpha^2+\beta^2)\psi_1^2\psi_0\big], R\rangle \\
&- \langle\big[A^2\psi_0^3 + (\alpha+i\beta)^2\psi_1^2\psi_0 + 2A(\alpha+i\beta)\psi_0^2\psi_1\big], \bar{R}\rangle \\
&- \big[A\langle\psi_0^2, R^2\rangle + (\alpha - i\beta)\langle\psi_0\psi_1, R^2\rangle\big] \\
&- \big[2A\langle\psi_0^2, |R|^2\rangle + 2(\alpha+i\beta)\langle\psi_0\psi_1, |R|^2\rangle\big] + \langle\psi_0, |R|^2 R\rangle\big),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Error}_\alpha(R,\bar{R},\vec{\alpha}) &= \mathfrak{Im}\big(-\langle\big[2A^2\psi_0^2\psi_1 + 4A(\alpha)\psi_0\psi_1^2 + 2(\alpha^2+\beta^2)\psi_1^3\big], R\rangle \\
&- \langle\big[A^2\psi_0^2\psi_1 + (\alpha+i\beta)^2\psi_1^3 + 2A(\alpha+i\beta)\psi_0\psi_1^2\big], \bar{R}\rangle \\
&- \big[A\langle\psi_0\psi_1, R^2\rangle + (\alpha - i\beta)\langle\psi_1^2, R^2\rangle\big] \\
&- \big[2A\langle\psi_0\psi_1, |R|^2\rangle + 2(\alpha+i\beta)\langle\psi_1^2, |R|^2\rangle\big] + \langle\psi_1, |R|^2 R\rangle\big) \\
&- \beta A^{-1}\mathfrak{Re}\big(-\langle\big[2A^2\psi_0^3 + 4A(\alpha)\psi_0^2\psi_1 + 2(\alpha^2+\beta^2)\psi_1^2\psi_0\big], R\rangle \\
&- \langle\big[A^2\psi_0^3 + (\alpha+i\beta)^2\psi_1^2\psi_0 + 2A(\alpha+i\beta)\psi_0^2\psi_1\big], \bar{R}\rangle \\
&- \big[A\langle\psi_0^2, R^2\rangle + (\alpha - i\beta)\langle\psi_0\psi_1, R^2\rangle\big] \\
&- \big[2A\langle\psi_0^2, |R|^2\rangle + 2(\alpha+i\beta)\langle\psi_0\psi_1, |R|^2\rangle\big] + \langle\psi_0, |R|^2 R\rangle\big),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Error}_\beta(R,\bar{R},\vec{\alpha}) &= -\mathfrak{Re}\big(-\langle\big[2A^2\psi_0^2\psi_1 + 4A(\alpha)\psi_0\psi_1^2 + 2(\alpha^2+\beta^2)\psi_1^3\big], R\rangle \\
&- \langle\big[A^2\psi_0^2\psi_1 + (\alpha+i\beta)^2\psi_1^3 + 2A(\alpha+i\beta)\psi_0\psi_1^2\big], \bar{R}\rangle \\
&- \big[A\langle\psi_0\psi_1, R^2\rangle + (\alpha - i\beta)\langle\psi_1^2, R^2\rangle\big] \\
&- \big[2A\langle\psi_0\psi_1, |R|^2\rangle + 2(\alpha+i\beta)\langle\psi_1^2, |R|^2\rangle\big] + \langle\psi_1, |R|^2 R\rangle\big) \\
&- \alpha A^{-1}\mathfrak{Re}\big(-\langle\big[2A^2\psi_0^3 + 4A(\alpha)\psi_0^2\psi_1 + 2(\alpha^2+\beta^2)\psi_1^2\psi_0\big], R\rangle \\
&- \langle\big[A^2\psi_0^3 + (\alpha+i\beta)^2\psi_1^2\psi_0 + 2A(\alpha+i\beta)\psi_0^2\psi_1\big], \bar{R}\rangle \\
&- \big[A\langle\psi_0^2, R^2\rangle + (\alpha - i\beta)\langle\psi_0\psi_1, R^2\rangle\big] \\
&- \big[2A\langle\psi_0^2, |R|^2\rangle + 2(\alpha+i\beta)\langle\psi_0\psi_1, |R|^2\rangle\big] + \langle\psi_0, |R|^2 R\rangle\big),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Error}_\theta(R,\bar{R},\vec{\alpha}) &= -A^{-1}\mathfrak{Re}\big(-\langle\big[2A^2\psi_0^3 + 4A(\alpha)\psi_0^2\psi_1 + 2(\alpha^2+\beta^2)\psi_1^2\psi_0\big], R\rangle \\
&- \langle\big[A^2\psi_0^3 + (\alpha+i\beta)^2\psi_1^2\psi_0 + 2A(\alpha+i\beta)\psi_0^2\psi_1\big], \bar{R}\rangle \\
&- \big[A\langle\psi_0^2, R^2\rangle + (\alpha - i\beta)\langle\psi_0\psi_1, R^2\rangle\big] \\
&- \big[2A\langle\psi_0^2, |R|^2\rangle + 2(\alpha+i\beta)\langle\psi_0\psi_1, |R|^2\rangle\big] + \langle\psi_0, |R|^2 R\rangle\big).
\end{aligned}$$

For convenience, we rewrite the system (12.36)-(12.40) in compact form as:

eqn:simpcoupledsystem

$$\left\{\begin{array}{c} \dot{\vec{\sigma}} = \vec{F}_{FD}(\vec{\sigma}) + \vec{G}_{FD}(\vec{\sigma}; R, \bar{R}), \\ \dot{\theta} = -\Omega_0 + A^2 + 3\alpha^2 + \beta^2 + G_\theta(R, \bar{R}; A, \alpha, \beta), \\ (i\partial_t - (H - \Omega_0) - (A^2 + 3\alpha^2 + \beta^2))R = P_c F_b(\vec{\sigma}) + P_c F_R(\vec{\sigma}; R, \bar{R}). \end{array}\right. \tag{12.47}$$

For simplicity, we take $F_b = P_c F_b$ and $F_R = P_c F_R$ in the sequel.

sec:finite-dim

12.9. Phase space of the finite dimensional Hamiltonian truncation

In this section we study the finite-dimensional system obtained by setting the dispersive part of the solution, $R(x,t)$, equal to zero in (12.26). We denote the solution of the resulting system by $(\rho_0(t), \rho_1(t)) \in \mathbb{C}^2$:

eqn:sys-fd

$$\left\{\begin{array}{c} i\dot{\rho_0} = \Omega_0\rho_0 - (\rho_0^2\bar{\rho_0} + 2\rho_1\bar{\rho_1}\rho_0 + \rho_1^2\bar{\rho_0}), \\ i\dot{\bar{\rho_0}} = -\Omega_0\bar{\rho_0} + (\bar{\rho_0}^2\rho_0 + 2\rho_1\bar{\rho_1}\bar{\rho_0} + \bar{\rho_1}^2\rho_0), \\ i\dot{\rho_1} = \Omega_1\rho_1 - (\rho_1^2\bar{\rho_1} + 2\rho_0\bar{\rho_0}\rho_1 + \rho_0^2\bar{\rho_1}), \\ i\dot{\bar{\rho_1}} = -\Omega_1\bar{\rho_1} + (\bar{\rho_1}^2\rho_1 + 2|\bar{\rho_0}|^2\bar{\rho_1} + \bar{\rho_0}^2\rho_1). \end{array}\right. \tag{12.48}$$

Symmetry breaking for a system of this type arising from a general class of *defocusing* nonlinearities was considered recently in [**?**].

This is a two degree of freedom Hamiltonian system, with time-translation and phase invariances inherited from NLS/GP. The associated time-conserved Hamiltonian and L^2 (optical power or particle number) functionals are:

Ndef

$$H = \Omega_0|\rho_0|^2 + \Omega_1|\rho_1|^2 - \frac{1}{2}|\rho_0|^4 - \frac{1}{2}|\rho_1|^4 - 2|\rho_1|^2|\rho_0|^2 - \frac{1}{2}(\rho_1^2\bar{\rho_0}^2 + \bar{\rho_1}^2\rho_0^2), \tag{12.49}$$

Hdef

$$N = |\rho_0|^2 + |\rho_1|^2. \tag{12.50}$$

In terms of H, system (12.48) can be expressed in Hamiltonian form

Hdef

$$i\partial_t\vec{\rho} = J\nabla_{\vec{\rho}}H, \tag{12.51}$$

where

$$\vec{\rho} = \begin{bmatrix} \rho_0 \\ \bar{\rho_0} \\ \rho_1 \\ \bar{\rho_1} \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{12.52}$$

In terms of the alternative coordinates of section (12.8.1),

$$\rho_0 = Ae^{i\theta},\ \rho_1 = (\alpha + i\beta)e^{i\theta},$$

hence the system (12.48) takes the form:

eqn:ode-sys-alpha

$$\left\{\begin{array}{c} \dot{\alpha} = \left[\ \Omega_1 + \dot{\theta} - (\alpha^2 + \beta^2) - A^2\ \right]\beta, \\ \dot{\beta} = -\left[\ \Omega_1 + \dot{\theta} - (\alpha^2 + \beta^2) - 3A^2\ \right]\alpha, \\ \dot{A} = -2\alpha\beta A, \\ \dot{\theta} = -\Omega_0 + A^2 + 3\alpha^2 + \beta^2. \end{array}\right. \tag{12.53}$$

Recall that, for simplicity, we have set $a_{ijkl} = 1$ for all $i,\ j,\ k,\ l = 0,\ 1$. Note that θ completely decouples from the A, α, β equations, meaning we have

eqn:alpha-reduced1

$$\left\{\begin{array}{c} \dot{\alpha} = \left[\ \Omega_1 - \Omega_0 + 2\alpha^2\ \right]\beta, \\ \dot{\beta} = -\left[\ \Omega_1 - \Omega_0 - 2A^2 + 2\alpha^2\ \right]\alpha, \\ \dot{A} = -2\alpha\beta A, \end{array}\right. \tag{12.54}$$

where now the θ equation is decoupled to give

eqn:theta-decoupled

$$\dot{\theta} = -\Omega_0 + A^2 + 3\alpha^2 + \beta^2. \tag{12.55}$$

One can verify, by changing coordinates in (12.49-12.50) or directly from (12.53), the time-invariant quantities:

$$\begin{aligned} N &= A^2+\alpha^2+\beta^2, \\ H &= \Omega_0 A^2+\Omega_1(\alpha^2+\beta^2)-\frac{1}{2}A^4-\frac{1}{2}(\alpha^2+\beta^2)^2 \\ &\quad -2A^2(\alpha^2+\beta^2)-A^2(\alpha^2-\beta^2). \end{aligned}$$

We obtain a closed system for (α,β) using that $N=A^2+\alpha^2+\beta^2$ is conserved. Then, we may reduce the system to

qn:ode-sys-alpha-reduced

$$\left\{ \begin{array}{l} \dot\alpha = \left[\ \Omega_{10}+2\alpha^2\right]\ \beta, \\ \dot\beta = -\left[\ \Omega_{10}+2(2\alpha^2+\beta^2-N)\ \right]\alpha \end{array} \right. , \quad \Omega_{10}=\Omega_1-\Omega_0 \tag{12.56}$$

In addition, the system has the conserved quantity

eqn:reduced-ham

$$H=\frac{1}{2}\left[\frac{1}{2}\left(N-\frac{\Omega_{10}}{2}\right)^2+\Omega_{10}(\alpha^2+\beta^2)-2N\alpha^2+(\alpha^2+\beta^2)^2+\alpha^4-\beta^4\right]. \tag{12.57}$$

In this case, we have

eqn:little-ham

$$\begin{pmatrix} \dot\alpha \\ \dot\beta \end{pmatrix} = J\nabla H, \tag{12.58}$$

where

$$J=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Now, let us define the matrices B_{eq} and $\tilde B_{eq}$ to be those related to linearization about the equilibrium solution $(A_{eq},\alpha_{eq},\beta_{eq},\theta_{eq})$ and the decoupled reduced system $(A_{eq},\alpha_{eq},\beta_{eq})$ respectively. Similarly, let

eqn:monodromy

$$M(t),\ \tilde M(t) \tag{12.59}$$

be the resulting monodromy matrices for the linearization about nearby time dependent periodic orbits $(A(t),\alpha(t),\beta(t),\theta(t))$ and the decoupled reduced system $(A(t),\alpha(t),\beta(t))$ respectively.

In the following section, we actually discuss the relevant bounds on the operators $e^{B_{eq}t}$ and $e^{\tilde B t}$.

sec:fd-phasespace

12.9.1. Bifurcation of equilibria and Symmetry Breaking. Recall that $N=N[A,\alpha,\beta]$ is a constant of the motion for the system (12.53), corresponding to the physical quantities optical power or particle number. Thus it is natural to explore the nature of the phase space restricted to the level sets of N. We are interested in time-periodic states of frequency Ω, corresponding in the physical space to solutions of NLS/GP of the form $u(x,t)=e^{-i\Omega t}U$. Thus, we transform the system to a rotating frame by setting

Theta-def

$$\theta(t)\ =\ \Theta(t)-\Omega t \tag{12.60}$$

and obtain

eqn:alpha-rotate

$$\begin{aligned} \dot\alpha &= \left[\ \Omega_1-\Omega+\dot\Theta(t)-(\alpha^2+\beta^2)-A^2\ \right]\beta, \\ \dot\beta &= -\left[\ \Omega_1-\Omega+\dot\Theta(t)-(\alpha^2+\beta^2)-3A^2\ \right]\alpha, \\ \dot A &= -2\alpha\beta A, \\ \dot\Theta &= \Omega-\Omega_0+A^2+3\alpha^2+\beta^2. \end{aligned} \tag{12.61}$$

The states we seek are equilibria in this rotating frame. Thus, we require

e-rotate.a (12.62) $$\left[\ \Omega_1 - \Omega - (\alpha^2+\beta^2) - A^2\ \right]\ \beta = 0,$$

e-rotate.b (12.63) $$\left[\ \Omega_1 - \Omega - (\alpha^2+\beta^2) - 3A^2\ \right]\alpha = 0,$$

e-rotate.c (12.64) $$\alpha\beta A = 0,$$

e-rotate.d (12.65) $$\Omega - \Omega_0 + A^2 + 3\alpha^2 + \beta^2 = 0,$$

whose solutions we consider on the level set

e-rotate.e (12.66) $$A^2 + \alpha^2 + \beta^2 = N.$$

It is easy to observe an equilibrium corresponding to the
Symmetric states:

sym-eq (12.67) $$\text{For } N \geq 0: \quad A_* = N^{\frac{1}{2}},\ \alpha_* = \beta_* = 0,\ \Omega_* = \Omega_0 - N.$$

Via (12.35) we identify this with the symmetric ground state of NLS/GP;

$$u(x,t)\ \sim\ N^{\frac{1}{2}}\ e^{i(-\Omega_0+N)t}\ \psi_0(x).$$

Due to its correspondence with the symmetric state, we refer to this equilibrium of the finite dimensional reduction as the symmetric equilibrium.

A second, bifurcating family can be found explicitly as follows. Define

Ncr-def (12.68) $$N_{cr}^{FD}\ =\ \frac{\Omega_{10}}{2}\ =\ \frac{\Omega_1 - \Omega_0}{2}.$$

REMARK 12.9.1. *Had we set the nonlinearity coefficient $g = -1$ and the "interaction weights", a_{ijkl} equal to one, we would have*

NcrFD (12.69) $$N_{cr}^{FD} = \frac{\Omega_{10}}{g(\int \psi_0^4 dx - 3\int \psi_0^2\psi_1^2 dx)} > 0.$$

Positivity in (12.69) *is observed by using the asymptotic relation of ψ_0 and ψ_1 to the eigenstate of the single / isolated potential well; see [**?**].*[2]

Using (12.65) we first eliminate Ω from (12.62) and obtain $\left[\ \Omega_1 - \Omega_0 + 2\alpha^2\ \right]\beta = 0$. Since $\Omega_1 - \Omega_0 > 0$ we conclude $\beta = 0$. Thus, (12.64) is satisfied. We now use (12.65) again to eliminate Ω from (12.63). Thus we have, since $\beta = 0$

$$\begin{aligned} A^2\ -\ \alpha^2\ &=\ N_{cr}^{FD}, \\ A^2\ +\ \alpha^2\ &=\ N. \end{aligned}$$

Solving for A and α we have the following equilibria, corresponding to symmetry broken states, which bifurcate for at $N = N_{cr}^{FD}$:
Symmetry broken states:

$$\text{For } N \geq N_{cr}^{FD} \equiv \frac{\Omega_{10}}{2}:$$

sym-brk (12.70) $$A_* = \left(\frac{N + N_{cr}^{FD}}{2}\right)^{\frac{1}{2}},\ \alpha_* = \left(\frac{N - N_{cr}^{FD}}{2}\right)^{\frac{1}{2}},\ \beta_* = 0,\ \Omega_* = -2\Omega_0 + N_{cr}^{FD} - N.$$

[2] Though N_{cr}^{FD} is a good first order approximation to $\mathcal{N}_{cr}$, the true symmetry breaking point in the nonlinear problem (12.13). As in this note we will be studying existence of solutions close to those described by the finite dimensional dynamics of (12.48), we will work from here on using $\mathcal{N}_{cr} = N_{cr}^{FD}$.

Via (12.35) we identify this with the asymmetric ground state of NLS/GP;

$$u(x,t) \sim e^{i(-2\Omega_0+N_{cr}^{FD}-N)t}\left(\left(\frac{N+N_{cr}^{FD}}{2}\right)^{\frac{1}{2}}\psi_0(x)+\left(\frac{N-N_{cr}^{FD}}{2}\right)^{\frac{1}{2}}\psi_1(x)\right).$$

Due to its correspondence with the asymmetric (or symmetry broken) states of NLS/GP, we refer to these equilibria of the finite dimensional reduction as the asymmetric equilibria.

12.9.2. Near-equilibrium periodic solutions of the finite dimensional system.

prop:perorb

PROPOSITION 12.9.2. *For $N < N_{cr}^{FD}$, a neighborhood of the elliptic fixed point* $(0,0)$ *and, for $N > N_{cr}^{FD}$, neighborhoods of the elliptic fixed points* $(\pm\sqrt{(N-N_{cr}^{FD})/2},0)$*, is foliated by periodic orbits of* (12.56) .

PROOF. The strategy of proof is to exploit the convexity of H near an elliptic fixed point. Note that, with the definition $N_{cr}^{FD} = \Omega_{10}/2 = (\Omega_1-\Omega_0)/2$, we have

$$\begin{aligned} H_\alpha &= 2\alpha[2\alpha^2+\beta^2-(N-N_{cr}^{FD})],\ H_\beta = \beta[N_{cr}^{FD}+2\alpha^2], \\ H_{\alpha,\alpha} &= 2[(6\alpha^2+\beta^2-(N-N_{cr}^{FD})],\ H_{\beta,\beta} = [\Omega_{10}+2\alpha^2],\ H_{\alpha,\beta} = 4\alpha\beta. \end{aligned}$$

We find that the linearly stable elliptic equilibria are $(\alpha,\beta) = (0,0)$ for $N < N_{cr}^{FD}$ and $(\pm\sqrt{\frac{N-N_{cr}^{FD}}{2}},0)$ for $N > N_{cr}^{FD}$. At these equilibria, D^2H is strictly convex with minimum value equal to zero. Hence, for a fixed $\delta > 0$, the level set $H(\alpha,\beta)$ is a bounded, closed curve in the (α,β) plane, encircling the equilibrium, a periodic orbit for (12.56) . □

REMARK 12.9.3. *We note that the dichotomy in the types of equilibria found for $N > N_{cr}^{FD}$ and $N < N_{cr}^{FD}$ exactly describe the phenomenon of symmetry breaking for the profile of the stable bound state for* (12.13) *at small amplitudes as seen in [**?**].*

We can estimate the period of small amplitude periodic solutions about equilibria by linearization:

$$\begin{aligned} \alpha(t) &= \sqrt{\frac{|N-N_{cr}^{FD}|}{2}}+\varepsilon h_1(t), \\ \beta(t) &= \varepsilon h_2(t). \end{aligned}$$

Substitution into (12.53) gives

$$\ddot{h}_1 = -4(N-N_{cr}^{FD})\left(N_{cr}^{FD}+\frac{(N-N_{cr}^{FD})}{2}\right)h_1+\varepsilon\mathcal{O}(|\vec{h}|^2). \tag{12.71}$$

and therefore a period of oscillation

eqn:Teps

$$T = \frac{\pi}{|N^2-N_{cr}^2|^{\frac{1}{2}}}+\mathcal{O}(\varepsilon) \equiv T_+(N-N_{cr}^{FD})+\mathcal{O}(\varepsilon). \tag{12.72}$$

Similarly, for $N-N_{cr}^{FD} < 0$ we have

$$\alpha = \varepsilon h_1, \tag{12.73}$$

$$\beta = \varepsilon h_2, \tag{12.74}$$

which when plugged into (18.33) gives

$$\ddot{h}_1 = -4|N-N_{cr}^{FD}|N_{cr}^{FD}h_1+\varepsilon\mathcal{O}(|\vec{h}|^2). \tag{12.75}$$

and the period of oscillation near the equilibrium point is expect to be:

$$T = \frac{\pi}{(|N - N_{cr}^{FD}| N_{cr}^{FD})^{\frac{1}{2}}} + \mathcal{O}(|N - N_{cr}^{FD}|) = T_{-}(N - N_{cr}^{FD}) + \mathcal{O}(\varepsilon). \tag{12.76}$$

REMARK 12.9.4. *It is on a time scale of a large number of periods, $T_{\pm}$, that we hope to control the difference between the observed finite dimensional periodic solutions and the full solution to the PDE.*

In Appendix 12.12 we apply the implicit function theorem (Poincaré continuation approach) to (12.56) to construct approximations to these small amplitude periodic solutions and their periods. In terms of the system (12.48) for $\rho_0(t), \rho_1(t)$, we have:

prop:perorb-period

PROPOSITION 12.9.5. *Fix $(N - N_{cr}^{FD}) > 0$ or $(N - N_{cr}^{FD}) < 0$ such that $|N - N_{cr}^{FD}| \ll 1$. Let us assume $N_{cr}^{FD} \gg |N - N_{cr}^{FD}|$. Take $(\rho_0^{eq}, \rho_1^{eq})$ be the corresponding equilibrium solution for* (12.48)*. For any periodic solution $\rho_0(t), \rho_1(t)$ of* (12.48) *such that*

eqn:perorbmag

$$|(\rho_0(t), \rho_1(t)) - (\rho_0^{eq}, \rho_1^{eq})| \ll |N - N_{cr}^{FD}|, \tag{12.77}$$

we have

$$(\rho_0(t+T), \rho_1(t+T)) = (\rho_0(t), \rho_1(t)), \tag{12.78}$$

where

$$T \leq 2T_{\pm}. \tag{12.79}$$

REMARK 12.9.6. *For small perturbations of the equilibrium point (for either $(N - N_{cr}^{FD}) < 0$ or $(N - N_{cr}^{FD}) > 0$, it is for precisely the period $T_{\pm}(N - N_{cr}^{FD})$ on which we must control the coupling to the continuous spectrum for the full solution to* (12.13) *in order to prove these finite dimensional structures are observable over many oscillations, see Section 12.11. In order to generalize our result to any periodic solution predicted by the finite dimensional dynamics, we must understand fully the period of each full oscillation. This will be discussed further in Section* **??**.

sec:stability-finite-dim

12.9.3. Stability of equilibria; finite dimensional analysis. In this section, we consider the stability of the solution branches obtained in the previous section. We rewrite the system (12.61), using the last equation to eliminate $-\Omega + \dot{\Theta}$ from the equations for α and β. Thus we have

syst-rot

$$\begin{cases} \dot{\alpha} = \left[\Omega_{10} + 2\alpha^2 \right] \beta, \\ \dot{\beta} = -\left[\Omega_{10} + 2\alpha^2 - 2A^2 \right] \alpha, \\ \dot{A} = -2\alpha\beta A, \\ \dot{\Theta} = \Omega - \Omega_0 + A^2 + 3\alpha^2 + \beta^2. \end{cases} \tag{12.80}$$

Note that in these coordinates the equations for α, β and A decouple from the equation for Θ.

The finite dimensional system has a phase portrait, equivalent to (12.48); see Figure 5. In particular, we observe elliptic and hyperbolic equilibria, and periodic orbits. We now embark on detailed linear stability analysis of these states.

Linearization about an arbitrary solution

$$(\alpha_*(t), \beta_*(t), A_*(t), \theta_*(t)) \tag{12.81}$$

gives the linearized perturbation equation

$$\partial_t \begin{bmatrix} \delta\alpha \\ \delta\beta \\ \delta A \\ \delta\theta \end{bmatrix} = \begin{bmatrix} 4\alpha_*\beta_* & (\Omega_{10}+2\alpha_*^2) & 0 & 0 \\ -(\Omega_{10}+6\alpha_*^2-2A_*^2) & 0 & -4\alpha_* A_* & 0 \\ -2A_*\beta_* & -2\alpha_* A_* & 2\alpha_*\beta_* & 0 \\ 6\alpha_* & 2\beta_* & 2A_* & 0 \end{bmatrix} \begin{bmatrix} \delta\alpha \\ \delta\beta \\ \delta A \\ \delta\theta \end{bmatrix}$$

eqn:sys-lin-alpha

$$= B(t) \begin{bmatrix} \delta\alpha \\ \delta\beta \\ \delta A \\ \delta\theta \end{bmatrix}. \tag{12.82}$$

Since the evolution of α, β and A, decouple from that for Θ, we consider the behavior of the reduced system

$$\partial_t \begin{bmatrix} \delta\alpha \\ \delta\beta \\ \delta A \end{bmatrix} = \begin{bmatrix} 4\alpha_*\beta_* & 2(N_{cr}^{FD}+\alpha_*^2) & 0 \\ -2(N_{cr}^{FD}+3\alpha_*^2-A_*^2) & 0 & -4\alpha_* A_* \\ -2A_*\beta_* & -2\alpha_* A_* & -2\alpha_*\beta_* \end{bmatrix} \begin{bmatrix} \delta\alpha \\ \delta\beta \\ \delta A \end{bmatrix}$$

red-pert-eqn

$$= \tilde{B}(t) \begin{bmatrix} \delta\alpha \\ \delta\beta \\ \delta A \end{bmatrix}. \tag{12.83}$$

Then, $\tilde{B}$ is the 3 by 3 block in matrix of equation (12.82). In obtaining (12.83), we used that $N_{cr}^{FD} = \Omega_{10}/2$.

Linearized dynamics about the symmetric equilibrium state:
For the symmetric equilibrium, as displayed in (12.67) we have

eqn:eqsym

$$(\alpha_-^{eq}, \beta_-^{eq}, A_-^{eq}, \theta_-^{eq}(t)) = (0, 0, N^{\frac{1}{2}}, (-\Omega_0 + N)t). \tag{12.84}$$

Hence,

$$B = B_- = 2 \begin{bmatrix} 0 & N_{cr}^{FD} & 0 & 0 \\ N - N_{cr}^{FD} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & N^{\frac{1}{2}} & 0 \end{bmatrix}.$$

For the reduced system, we have

tB-symm

$$\tilde{B}_- = 2 \begin{bmatrix} 0 & N_{cr}^{FD} & 0 \\ N - N_{cr}^{FD} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{12.85}$$

whose eigenvalues and implied linear stability character is as follows:

tB-symm-eig

$$N < N_{cr}^{FD}: \ \lambda_0 = 0, \ \lambda_\pm(N) = \pm 2i \left[\ (N_{cr}^{FD} - N) \ N_{cr}^{FD} \ \right]^{\frac{1}{2}}, \text{ (stable elliptic point)}, \tag{12.86}$$

tB-symm-eig-unstable

$$N > N_{cr}^{FD}: \ \lambda_0 = 0, \ \lambda_\pm(N) = \pm 2 \left[\ (N - N_{cr}^{FD}) \ N_{cr}^{FD} \ \right]^{\frac{1}{2}}, \text{ (unstable saddle)}. \tag{12.87}$$

see Fig. 5. *Thus, the symmetric state transitions from stable to unstable as N increases beyond N_{cr}^{FD}.*

Furthermore, $\tilde{B}_-$ can be diagonalized

$$\tilde{B}_- = \begin{bmatrix} 2N_{cr}^{FD} & 2N_{cr}^{FD} & 0 \\ \lambda_+ & \lambda_- & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_+ & 0 & 0 \\ 0 & \lambda_- & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2N_{cr}^{FD} & 2N_{cr}^{FD} & 0 \\ \lambda_+ & \lambda_- & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

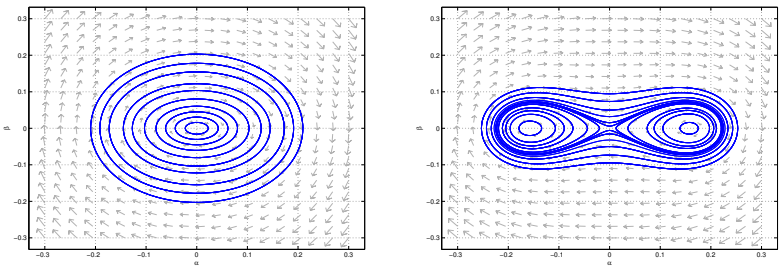

FIGURE 5. Symmetric state for $N-N_{cr}^{FD}<0$ corresponds to an equilibrium elliptic point $(\alpha,\beta)=(0,0)$. Plotted phase portrait corresponds to parameter values $N_{cr}=.1$ and $N-N_{cr}^{FD}=-.05$ (left) and $N_{cr}=.1$ and $N-N_{cr}^{FD}=.05$ (right). It is clear for $N-N_{cr}^{FD}<0$ the equilibrium sotion is stable and for $N-N_{cr}^{FD}>0$ the equilibrium sotion is unstable. fig7

and the linear evolution is given by

(12.88)
$$e^{\tilde{B}_- t} = \begin{bmatrix} 2N_{cr}^{FD} & 2N_{cr}^{FD} & 0 \\ \lambda_+ & \lambda_- & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{\lambda_+ t} & 0 & 0 \\ 0 & e^{\lambda_- t} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2N_{cr}^{FD} & 2N_{cr}^{FD} & 0 \\ \lambda_+ & \lambda_- & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}.$$

In other words, for $N<N_{cr}^{FD}$ we have the bound

$$e^{\tilde{B}_- t}\vec{\delta\alpha} = \begin{bmatrix} \cos(|\lambda_+|t)\delta\alpha + \left(\frac{N_{cr}^{FD}}{|N-N_{cr}^{FD}|}\right)^{\frac{1}{2}} \sin(|\lambda_+|t)\delta\beta \\ \cos(|\lambda_+|t)\delta\beta + \left(\frac{|N-N_{cr}^{FD}|}{N_{cr}^{FD}}\right)^{\frac{1}{2}} \sin(|\lambda_+|t)\delta\alpha \\ \delta A \end{bmatrix}.$$

The full linearized dynamics are governed by the matrix

(12.89)
$$B_- = \begin{bmatrix} * & * & * & 0 \\ * & \tilde{B}_{3\times 3} & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 2N^{\frac{1}{2}} & 0 \end{bmatrix},$$

where B has the same eigenvalues as $\tilde{B}$, with $\lambda=0$ now a generalized eigenvalue of multiplicity two, implying linear growth of $\delta\theta(t)$. Explicitly, we have

$$B_-^2 = 4\begin{bmatrix} N_{cr}^{FD}(N-N_{cr}^{FD}) & 0 & 0 & 0 \\ 0 & N_{cr}^{FD}(N-N_{cr}^{FD}) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, it is clear

$$\begin{aligned} e^{B_- t} &= M_0 + 2N^{\frac{1}{2}} t M_1 + \cos(2\sqrt{N_{cr}^{FD}|N-N_{cr}^{FD}|}\,t) M_2 \\ &\quad + \sin(2\sqrt{N_{cr}^{FD}|N-N_{cr}^{FD}|}\,t) M_3, \end{aligned}$$

where

$$
\begin{aligned}
M_0 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
M_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\
M_2 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
M_3 &= \begin{bmatrix} 0 & \frac{\sqrt{N_{cr}^{FD}}}{\sqrt{|N-N_{cr}^{FD}|}} & 0 & 0 \\ \frac{\sqrt{|N-N_{cr}^{FD}|}}{\sqrt{N_{cr}^{FD}}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

Note here the leading order behavior is then a system which oscillates at the correct period and grows linearly only in the phase term. On a component basis, we may say

$$
e^{B_- t}\vec{\eta} \approx \vec{\eta} + \sin(2\sqrt{N_{cr}^{FD}|N-N_{cr}^{FD}|}\,t)\frac{\sqrt{N_{cr}^{FD}}}{\sqrt{N-N_{cr}^{FD}}}\eta_2\vec{e}_4 + 2N^{\frac{1}{2}}t\eta_3\vec{e}_4.
$$

Alternatively, we can use the spectrum of B_- to define the matrix

$$
P_- = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -i\left(\frac{N_{cr}^{FD}-N}{N_{cr}^{FD}}\right)^{\frac{1}{2}} & i\left(\frac{N_{cr}^{FD}-N}{N_{cr}^{FD}}\right)^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

Then, we have

$$
P_-^{-1}B_-P_- = \begin{bmatrix} -2i\left((N_{cr}^{FD}-N)N_{cr}^{FD}\right)^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 2i\left((N_{cr}^{FD}-N)N_{cr}^{FD}\right)^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2N^{\frac{1}{2}} & 0 \end{bmatrix}.
$$

Hence, we see

$$e^{B_- t} = P_- \begin{bmatrix} e^{-2i\left((N_{cr}^{FD}-N)N_{cr}^{FD}\right)^{\frac{1}{2}}t} & 0 & 0 & 0 \\ 0 & e^{2i\left((N_{cr}^{FD}-N)N_{cr}^{FD}\right)^{\frac{1}{2}}t} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} P_-^{-1} + 2N^{\frac{1}{2}} t \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Linearized dynamics about asymmetric equilibrium states, $N > N_{cr}^{FD}$:
For either asymmetric equilibrium, displayed in (12.67) we have

$$(\alpha_+^{eq}, \beta_+^{eq}, A_+^{eq}, \theta_+^{eq}(t)) = \left(\left(\frac{N - N_{cr}^{FD}}{2} \right)^{\frac{1}{2}}, 0, \left(\frac{N + N_{cr}^{FD}}{2} \right)^{\frac{1}{2}}, (-2\Omega_0 + N_{cr}^{FD} - N)t \right). \tag{12.90}$$

eqn:eqanti

Hence,

$$B_+ = \begin{bmatrix} 0 & (N + N_{cr}^{FD}) & 0 & 0 \\ -2(N - N_{cr}^{FD}) & 0 & 2(N^2 - (N_{cr}^{FD})^2)^{\frac{1}{2}} & 0 \\ 0 & -(N^2 - (N_{cr}^{FD})^2)^{\frac{1}{2}} & 0 & 0 \\ \frac{6}{\sqrt{2}}(N - N_{cr}^{FD})^{\frac{1}{2}} & 0 & \sqrt{2}(N + N_{cr}^{FD})^{\frac{1}{2}} & 0 \end{bmatrix}. \tag{12.91}$$

In this case, we substitute (12.70) into the expression for $\tilde{B}$ in (12.83) and obtain:

$$\tilde{B} = \begin{bmatrix} 0 & N + N_{cr}^{FD} & 0 \\ -2(N - N_{cr}^{FD}) & 0 & 2\left(N^2 - (N_{cr}^{FD})^2\right)^{\frac{1}{2}} \\ 0 & -\left(N^2 - (N_{cr}^{FD})^2\right)^{\frac{1}{2}} & 0 \end{bmatrix}, \tag{12.92}$$

tB-asymm

where $N > N_{cr}^{FD} = \frac{\Omega_{10}}{2}$. The eigenvalues of $\tilde{B}$ are:

$$\lambda_0 = 0, \ \lambda_\pm(N) = \pm 2i \left(N^2 - (N_{cr}^{FD})^2\right)^{\frac{1}{2}}, \quad N > N_{cr}^{FD}. \tag{12.93}$$

tB-asymm-eig

Therefore, the bifurcating asymmetric states are stable elliptic points. Also, $\tilde{B}_+$ is diagonalizable, resulting in

$$e^{\tilde{B}_+ t}\vec{\delta\alpha} = \begin{bmatrix} \frac{1}{2}(1 + \cos(|\lambda_+|t))\delta\alpha + \tilde{\nu}\sin(|\lambda_+|t)\delta\beta + \frac{1}{2}\tilde{\nu}(1 - \cos(|\lambda_+|t))\delta A \\ \cos(|\lambda_+|t)\delta\beta + \tilde{\nu}^{-1}\sin(|\lambda_+|t)\delta\alpha + \sin(|\lambda_+|t)\delta A \\ \frac{1}{2}\tilde{\nu}^{-1}(1 - \cos(|\lambda_+|t))\delta\alpha + \frac{1}{2}\sin(|\lambda_+|t)\delta\beta + \frac{(}{1} + \cos(|\lambda_+|t))\delta A \end{bmatrix},$$

where

$$\tilde{\nu} = \left(\frac{N + N_{cr}^{FD}}{|N - N_{cr}^{FD}|} \right)^{\frac{1}{2}}.$$

In a manner analogous to the case of symmetric bound states, we have

$$B_+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0, \ B_+ \begin{bmatrix} \left(\frac{N+N_{cr}^{FD}}{N-N_{cr}^{FD}}\right)^{\frac{1}{2}} \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, it follows

$$\begin{aligned} e^{B_+ t}\vec{\eta} &\approx \vec{\eta} + \sin(2\left(N^2 - N_{cr}^2\right)^{\frac{1}{2}} t)\left(N^2 - N_{cr}^2\right)^{\frac{1}{2}} \eta_2 \vec{e}_1 \\ &\quad + (1 - \cos(2\left(N^2 - N_{cr}^2\right)^{\frac{1}{2}} t))\left(\frac{N + N_{cr}^{FD}}{N - N_{cr}^{FD}}\right)^{\frac{1}{2}} \eta_3 \vec{e}_1 \\ &\quad + (N + N_{cr}^{FD})^{\frac{1}{2}}(1 + t)\eta_3 \vec{e}_4. \end{aligned}$$

Once again, we see the dominant behavior be oscillations in α, β, A with a linear growth accompanied by a factor of order $N^{\frac{1}{2}}$ in θ.

As before, we can use the spectrum of B_+ to define the matrix

$$P_+ = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -2i\left(\frac{N - N_{cr}^{FD}}{N + N_{cr}^{FD}}\right)^{\frac{1}{2}} & 2i\left(\frac{N - N_{cr}^{FD}}{N + N_{cr}^{FD}}\right)^{\frac{1}{2}} & 0 & 0 \\ -\left(\frac{N - N_{cr}^{FD}}{N + N_{cr}^{FD}}\right)^{\frac{1}{2}} & -\left(\frac{N - N_{cr}^{FD}}{N + N_{cr}^{FD}}\right)^{\frac{1}{2}} & \left(\frac{N - N_{cr}^{FD}}{N + N_{cr}^{FD}}\right)^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$P_+^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4}\left(\frac{N + N_{cr}^{FD}}{N - N_{cr}^{FD}}\right)^{\frac{1}{2}} & -\frac{1}{4}\left(\frac{N + N_{cr}^{FD}}{N - N_{cr}^{FD}}\right)^{\frac{1}{2}} & 0 \\ \frac{1}{4} & \frac{1}{4i}\left(\frac{N + N_{cr}^{FD}}{N - N_{cr}^{FD}}\right)^{\frac{1}{2}} & -\frac{1}{4}\left(\frac{N + N_{cr}^{FD}}{N - N_{cr}^{FD}}\right)^{\frac{1}{2}} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2}\left(\frac{N + N_{cr}^{FD}}{N - N_{cr}^{FD}}\right)^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, we have

$$P_+^{-1} B_+ P_+ = \begin{bmatrix} -2i\left((N^2 - (N_{cr}^{FD})^2)\right)^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 2i\left((N^2 - (N_{cr}^{FD})^2)\right)^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{4}{\sqrt{2}}(N - N_{cr}^{FD})^{\frac{1}{2}} & \frac{4}{\sqrt{2}}(N - N_{cr}^{FD})^{\frac{1}{2}} & \frac{8}{\sqrt{2}}(N - N_{cr}^{FD})^{\frac{1}{2}} & 0 \end{bmatrix}.$$

Hence, we see

$$
\begin{aligned}
e^{B_+ t} \;=\; P_+ &\begin{bmatrix} e^{2i\left((N_{cr}^{FD}+N)(N-N_{cr}^{FD})\right)^{\frac{1}{2}}t} & 0 & 0 & 0 \\ 0 & e^{-2i\left((N_{cr}^{FD}-N)(N-N_{cr}^{FD})\right)^{\frac{1}{2}}t} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} P_+^{-1} \\
&+4\sqrt{2}(N-N_{cr}^{FD})^{\frac{1}{2}}(e^{2i(N^2-N_{cr}^2)^{\frac{1}{2}}t}-1)P_+ \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0 \end{bmatrix} P_+^{-1} \\
&+4\sqrt{2}(N-N_{cr}^{FD})^{\frac{1}{2}}(e^{-2i(N^2-N_{cr}^2)^{\frac{1}{2}}t}-1)P_+ \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix} P_+^{-1} \\
&+8\sqrt{2}(N-N_{cr}^{FD})^{\frac{1}{2}}t \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\ \frac{1}{2}&0&\frac{1}{2}\frac{\sqrt{N_{cr}^{FD}+N}}{\sqrt{N-N_{cr}^{FD}}}&0 \end{bmatrix}.
\end{aligned}
$$

REMARK 12.9.7. *The analysis of $e^{B_\pm t}$ provides partial understanding of the inherent instability in the phase of the finite dimensional periodic orbits. Specifically, an orbit (A_1,α_1,β_1) will oscillate with period T_1 while a different orbit (A_2,α_2,β_2) will oscillate with period T_2. As a result, if two orbits begin quite close in phase, they will naturally oscillate out of phase with one another, which from the analysis above gives the linear shift in the $\delta\theta$ component. However, the oscillations will be purely described by the (A,α,β) system, meaning we must decouple the phase from the equation to get stability.*

sec:per

12.9.4. Stability of Periodic Orbits Near Equilibria. In this subsection we discuss the linearize evolution about near-equilibrium periodic orbits.

prop:monodromy-bounds

PROPOSITION 12.9.8. *Fix $\epsilon>0$. There exists a $\delta>0$ such that if a given periodic orbit solution of* (12.80)*, $(A(t),\alpha(t),\beta(t),\theta(t))$, with period T with*

$$|(A(0),\alpha(0),\beta(0),0)-(A_{eq},\alpha_{eq},\beta_{eq},\theta_{eq})|<\delta, \tag{12.94}$$

then given M and $\tilde{M}$ the monodromy matrices as in (12.59) *we have*

$$\|M(t)\|_{L^\infty\to L^\infty}<(1+\epsilon)\|e^{B_{eq}t}\|_{L^\infty\to L^\infty}, \tag{12.95}$$

$$\|\tilde{M}(t)\|_{L^\infty\to L^\infty}<(1+\epsilon)\|e^{\tilde{B}_{eq}t}\|_{L^\infty\to L^\infty}. \tag{12.96}$$

PROOF. The proof follows from continuity of the Floquet multipliers, which we set here to be $\lambda_1,\lambda_2,\lambda_3,\lambda_4$. We discuss the case for B_{eq} and $M(t)$ here. The analysis for the case for $\tilde{B}_{eq}$ and $\tilde{M}$ will follow similarly.

It is clear that $(\dot{A}(t),\dot{\alpha}(t),\dot{\beta}(t),\dot{\theta}(t))$ is a solution to (12.82), giving at least one Floquet multiplier $\lambda_1=1$. Similarly, diffentiation with respect to the period gives a similar result, meaning we in fact have $\lambda_1=\lambda_2=1$.

From the analysis of the phase diagram for periodic orbits in α,β, we have that $\alpha(t)=\alpha(-t)$ and $\beta(t)=-\beta(-t)$. Hence

$$\int_0^T \alpha(s)\beta(s)ds=0. \tag{12.97}$$

We know from standard Floquet theory that given a monodromy matrix M_A for the system

$$\dot{X} = A(t)X$$

for $A(t+T) = A(t)$, we have

$$\det M_A(t) = e^{\int_0^T \mathrm{Tr}(A)(s)ds}.$$

Hence, given $\mathrm{tr}(B(t)) = \alpha\beta$, we have

$$\prod_{j=1}^{4} \lambda_j = 1.$$

As a result either $\lambda_3 = \bar{\lambda}_4$ with $|\lambda_3| = |\lambda_4| = 1$ or $\lambda_3, \lambda_4 \in \mathbb{R}$ with $\lambda_3 = \lambda_4^{-1}$. By continuity of the Floquet multipliers, near $\vec{\alpha}_{eq}$, we have the only the degeneracy at $\lambda = 1$ resulting in a similar growth behavior to that of $e^{B_{eq}t}$ as seen in Section 12.9.3. □

12.10. An Ansatz for the Coupled System

sec:ansatz

Starting with the finite dimensional system given in (12.56), we consider a periodic orbit, $(\tilde{\alpha}(t), \tilde{\beta}(t))$, near equilibrium point and construct a periodic orbit of the extended system (12.54):

$$\sigma_*(t) = \left(\tilde{\alpha}(t), \tilde{\beta}(t), \tilde{A}(t)\right).$$

Below we shall specify how near the equilibrium σ_* need be.

We now write the system (12.47) by centering around the orbit σ_* of the finite dimensional truncation:

$$\begin{aligned} \sigma(t) &= \sigma_*(t) + \eta, \\ &\equiv \left(\tilde{A}(t) + \eta_A(t), \tilde{\alpha}(t) + \eta_\alpha(t), \tilde{\beta}(t) + \eta_\beta(t)\right). \end{aligned} \tag{12.98}$$

eta-def

This corresponds to a solution of the form:

$$\begin{aligned} u(x,t) &= e^{i\theta(t)}\big((\tilde{A}(t) + \eta_A(t))\psi_0 + [(\tilde{\alpha}(t) + \eta_\alpha(t)) + i(\tilde{\beta}(t) + \eta_\beta(t))]\psi_1 \\ &\quad + R(x,t)\big) \end{aligned}$$

with initial conditions

$$u_0(x) = e^{i\theta(0)}\left(\ \tilde{A}(0)\ \psi_0(x) + [\tilde{\alpha}(0) + i\tilde{\beta}(0)]\ \psi_1(x)\ \right).$$

Centered about σ_*, the system (12.47) becomes:

$$\begin{aligned} \dot{\vec{\eta}} &= D_\sigma F_{FD}(\sigma_*(t))\,\vec{\eta} + [\vec{F}_{FD}(\vec{\sigma}_* + \vec{\eta}) - \vec{F}_{FD}(\vec{\sigma}_*) - D_{\vec{\sigma}}\vec{F}_{FD}(\vec{\sigma}_*)\vec{\eta}] \\ &\quad + \vec{G}_{FD}(\vec{\sigma}_*, \vec{\eta}; R, \bar{R}), \\ \dot{\theta} &= -\Omega_0 + A^2 + 3\alpha^2 + \beta^2 + G_\theta(R, \bar{R}; A, \alpha, \beta), \\ iR_t &= (H - \Omega_0)R + (\tilde{A}^2 + 3\tilde{\alpha}^2 + \tilde{\beta}^2)R + F_b\left(\sigma_* + \eta\right) + F_R\left(\sigma_* + \eta; R, \bar{R}\right) \\ &\quad + \left(\ A^2 - \tilde{A}^2\ +\ 3(\alpha^2 - \tilde{\alpha}^2)\ +\ \beta^2 - \tilde{\beta}^2\ \right)R\,. \end{aligned}$$

We have the estimates:

$$\left|\ \vec{F}_{FD}(\vec{\sigma}_* + \vec{\eta}) - \vec{F}_{FD}(\vec{\sigma}_*) - D_{\vec{\sigma}}F_{FD}(\vec{\sigma}_*)\vec{\eta}\ \right| = \mathcal{O}\left(\ \tilde{A}|\vec{\eta}|^2 + |\vec{\eta}|^3\ \right).$$

See Appendix **??** for the explicit expressions in the system.

We denote by $\tilde{R}$ the leading order part of R, driven by the periodic solution $\sigma_*(t)$:

$$i\tilde{R}_t = (H - \Omega_0)\tilde{R} + (\tilde{A}^2 + 3\tilde{\alpha}^2 + \tilde{\beta}^2)\tilde{R} + P_c F_b(\vec{\sigma}_*). \tag{12.99}$$

eqn:tildeR

The correction to $\tilde{R}$ is given by w, which satisfies:

tR-def

$$R = \tilde{R} + w. \tag{12.100}$$

Introduce, $\tilde{M}(t)$, a fundamental solution matrix for the system of ODEs with time-periodic coefficients:

$$\partial_t \eta = D_\sigma F(\sigma_*(t))\,\eta.$$

.

We shall study the following system of integral equations for $\eta(t)$, $\theta(t)$, $w(x,t)$:

eqn:intsys

$$\begin{aligned}
\vec{\eta}(t) &= \int_0^t \tilde{M}(t)\tilde{M}^{-1}(s)\left[(\vec{F}_{FD}(\vec{\sigma}_* + \eta) - \vec{F}_{FD}(\vec{\sigma}_*) - D_\sigma \vec{F}_{FD}(\vec{\sigma}_*)\vec{\eta}\right) \\
&\quad + \vec{G}_{FD}(\vec{\sigma}_*, \vec{\eta}; R, \bar{R})\Big]\, ds, \\
\theta &= \theta_0 + \int_0^t \left[-\Omega_0 + A^2 + 3\alpha^2 + \beta^2 + G_\theta(R, \bar{R}; \vec{\sigma}_*, \vec{\eta})\right] ds, \\
w(t) &= \int_0^t e^{iH(t-s) - i\Omega_0(t-s) + i\int_s^t (\tilde{A}^2 + 3\tilde{\alpha}^2 + \tilde{\beta}^2)(s')ds'} \\
&\quad \times P_c\left[(F_b(\vec{\sigma}_* + \vec{\eta}) - F_b(\vec{\sigma}_*)) + F_R(\vec{\sigma}_*, \vec{\eta}; R, \bar{R})\right].
\end{aligned} \tag{12.101}$$

We view a solution, $(\vec{\eta}, R)$, of this system of integral equations as fixed point of a mapping, $\mathscr{M}$:

fixedpoint

$$(\vec{\eta}, w) = \mathscr{M}(\vec{\eta}, w). \tag{12.102}$$

In the next section we formulate and solve this fixed point problem in a function space, which yields the existence of solutions to NLS/GP which "shadow" the periodic orbit, $\sigma_*(t)$, on the time scale of many periods.

sec:persist

12.11. Main Results

Recall that the period of the orbits described in Proposition 12.9.2 satisfies

$$T_{period} \sim (|N_{cr}^{FD} - N|N_{cr}^{FD})^{-\frac{1}{2}}.$$

In this section we shall construct solutions to the full PDEs, which "shadow" the finite dimensional orbits for many periods.

Let $\tau > 0$ denote a number to be chosen sufficiently small. And define the region of parameter space, about the symmetry breaking point, in which we conduct our study by

periodN

$$\begin{aligned}
\left|N_{cr}^{FD} - N\right| &= \tau, \\
N_{cr}^{FD} &= \tau^\gamma.
\end{aligned} \tag{12.103}$$

For now $0 < \gamma < 1$, but we shall place further constraints will be placed on γ.

remark:dd-small

REMARK 12.11.1. *Recall that $\tau^\gamma = N_{cr}^{FD} \sim \mathscr{N}_{cr}(L)$, where L is the well-spacing parameter for the double-well. Since, for L sufficiently large, $\mathscr{N}_{cr}(L) \sim \Omega_1(L) - \Omega_0(L) \lesssim e^{-\kappa L}$, the eigenvalues splitting , and τ can be made small by choosing L sufficiently large.*

In terms of τ, we now describe the periodic solutions discussed in Propositions 12.9.2 and 12.9.5:

prop:delta-perorb

PROPOSITION 12.11.2. *There exists $\tau_0 > 0$, such that for $0 < \tau < \tau_0$, the system* (12.54) *has periodic solutions, which are small perturbations of the equilibria:*

$$|\tilde{A}(t)|^2 \sim \tau^\gamma, \ |\tilde{\alpha}(t)|^2 + |\tilde{\beta}(t)|^2 \ \sim \tau, \quad N > N_{cr}^{FD}, \tag{12.104}$$

orbit-size-tau

$$|\tilde{A}(t)|^2 \sim \tau^\gamma, \ |\tilde{\alpha}(t)|^2 + |\tilde{\beta}(t)|^2 \ \sim \tau^{1+\delta}, \quad N < N_{cr}^{FD}, \tag{12.105}$$

where $\delta > 0$ is to be chosen below.

The period of oscillations of the periodic solutions of Proposition 12.9.2 is

Tdelta

$$T_{period}(\tau) \sim \tau^{-\frac{1+\gamma}{2}}. \tag{12.106}$$

We shall seek to construct solutions on a time interval of the form

Texist

$$0 \le t \le T_*(\tau), \quad T_*(\tau) \sim \tau^{-\epsilon} \times T_{period}(\tau) \ = \ \tau^{-\frac{1+\gamma}{2}-\epsilon}, \tag{12.107}$$

where $\epsilon > 0$ is to be chosen below.

12.11.1. Notation. It should be noted, notationally when we refer to

Texist

$$A \lesssim B, \tag{12.108}$$

we mean

$$A \le CB \tag{12.109}$$

for some C of order 1. Also, for

$$A \ll B, \tag{12.110}$$

we mean

$$A \le cB \tag{12.111}$$

for some $c > 0$ much less than 1.

sec:thm

12.11.2. Statement of Theorem.

thm:main-eq

THEOREM 3. *Assume $0 < \tau < \tau_0$ and $\frac{7}{9} < \gamma < 1$.*
Denote by

$$\sigma_*(t) \ = \ \left(\tilde{A}(t), \tilde{\alpha}(t), \tilde{\beta}(t) \right),$$

a periodic solution of (12.53)*, for which:*

(1) *whose period $T_{period}(\tau)$ satisfies*

$$T_{period}(\tau) \lesssim (|N_{cr}^{FD} - N| N_{cr}^{FD})^{-\frac{1}{2}} = \tau^{-\frac{1+\gamma}{2}}$$

(2) *whose fundamental matrix, $\tilde{M}(t)$ of the linearized dynamics about $\sigma_*(t)$ satisfies the norm bound:*

$$0 < s, t < T_{period}(\tau) \implies \|\tilde{M}(t)\tilde{M}^{-1}(s)\| \le C \left(\frac{N + N_{cr}^{FD}}{|N - N_{cr}^{FD}|} \right)^{\frac{1}{2}} \ = \ C \, \tau^{\frac{\gamma-1}{2}},$$

(3) *and which is a small orbit about an equilibrium $\left(A^{eq}, \alpha^{eq}, \beta^{eq} \right)$ as in Proposition 12.11.2.*

Take initial data for NLS/GP of the form

$$u_0(x) = e^{i\theta(0)}\left(\tilde{A}(0)\psi_0(x) + [\tilde{\alpha}(0) + i\tilde{\beta}(0)]\psi_1(x) \right), \tag{12.112}$$

where $\theta(0) \in \mathbb{R}$ *is chosen arbitrarily.*

Then there exists a solution $u(x,t)$ *of* (12.13) *of the form*

$$\begin{aligned} u(x,t) = e^{i\theta(t)}\big((\tilde{A}(t) + \eta_A(t))\psi_0(x) + [(\tilde{\alpha}(t) + \eta_\alpha(t)) + i(\tilde{\beta}(t) + \eta_\beta)]\psi_1(x) \\ + \tilde{R}(x,t) + w(x,t)\big), \end{aligned}$$

where $\tilde{R}$ *as in* (12.99).
Furthermore, $\eta(t) \equiv \left(\eta_A(t), \eta_\alpha(t), \eta_\beta(t) \right)$, $\theta(t) \in C^1([0, T_*(\tau)])$, $w \in L_t^\infty H_x^1 \cap L_t^4 L_x^\infty$ *and* w *satisfies the bounds*

$$\|\eta\|_{L_t^\infty[0,T_*(\tau)]} + \|w\|_{L_t^\infty([0,T_*(\tau)];H_x^1)} + \|w\|_{L_t^4([0,T_*(\tau)];L_x^\infty)} \lesssim \tau^{\frac{1}{2}+\delta_1}, \tag{12.113}$$

for all $t \in I = [0, T_*(\tau)] = [0, T_{period}(\tau)\ \tau^{-\epsilon}]$ *where* $\epsilon > 0$, $\delta_1 > \delta$ *are specified in the proof.*

sec:proof

12.11.3. Proof of theorem. In the sequel, we will often use the contracted notation

$$L_t^p W_x^{k,q} = L^p([0,T^*]; W^{k.q}(\mathbb{R})),$$

where T^* is given by (12.107), (12.103).

We now show that the map

$$(\vec{\eta}, w) = \mathscr{M}(\vec{\eta}, w)$$

defined in section 12.10 is a contraction.

Define the space

$$\begin{aligned} X(I) \ &= \ X([0,T_*(\tau)]) = \\ &\left\{ (\vec{\eta}, w) : \eta \in L_t^\infty([0,T_*(\tau)]),\ \ w \in L_t^\infty([0,T_*(\tau)]; H_x^1) \cap L_t^4([0,T_*(\tau)]; L_x^\infty) \right\} \end{aligned}$$

equipped with the natural norm

$$\|(\vec{\eta}, w)\|_{X(I)} = \|\vec{\eta}\|_{L_t^\infty(I)} + \|w\|_{L_t^4(I;L_x^\infty)} + \|w\|_{L_t^\infty(I;H_x^1)},$$

$$\text{where} \quad I = [0, T_*(\tau)].$$

We define $B_\tau(I) \subset X(I)$ such that $(\vec{\eta}, R) \in B_\tau(I)$ if and only if

eqn:contmapball

$$\|(\vec{\eta}, w)\|_{X(I)} \leq \tau^{\frac{1}{2}+\delta_1}, \tag{12.114}$$

where $\delta_1 > 0$ will be chosen later.

We must prove the following:

PROPOSITION 12.11.3. *The mapping* $\mathscr{M} : X(I) \to X(I)$, *defined in* (12.102), *has the properties*

(1) $\mathscr{M} : B_\tau(I) \to B_\tau(I)$.
(2) *There exists* $\kappa < 1$ *such that given* $(\vec{\eta}_j, w_j) \in B_\tau(I)$ *for* $j = 1, 2$, *we have*

$$d(\mathscr{M}(\vec{\eta}_1, w_1), \mathscr{M}(\vec{\eta}_2, w_2)) \leq \kappa\ d((\vec{\eta}_1, w_1), (\vec{\eta}_2, w_2)).$$

Thus, there exists a unique solution $(\vec{\eta}, w)$ *in* $B_\tau(I)$.

PROOF. We begin by proving necessary bounds on $\tilde{R}$.

lem:tildeR

PROPOSITION 12.11.4. *Given a solution $\vec{\sigma}_* = (\tilde{A}(t), \tilde{\alpha}(t), \tilde{\beta}(t))$ to* (12.54)*, as in Proposition 12.11.2, there exists $\tau_0 > 0, \delta_0 > \delta$, such that for all $\tau < \tau_0$ we have the following: for all $t \in I$, and defining*

$$\tilde{R} = \int_0^t e^{-iH(t-s) - i\Omega_0(t-s) + i\int_s^t (\tilde{A}^2 + 3\tilde{\alpha}^2 + \tilde{\beta}^2)(s')ds'} P_b F(\vec{\sigma}_*) ds \tag{12.115}$$

for F_b defined in (12.26)*, we have*

$$\|\tilde{R}\|_{L_t^\infty(I;L_x^\infty)} \lesssim \tau^{1+\delta_0}.$$

PROOF. Note that it is only the magnitude of the components of σ_*, via $F(\sigma_*(t))$, which factor in to the bounds of $\tilde{R}$.

From Proposition 12.11.2 and the finite dimensional conservation laws

$$\tilde{A}(t) = \tau^{\frac{\gamma}{2}} + \epsilon_0(t), \tag{12.116}$$

where $|\epsilon_0| \sim \tau^{\frac{1+\delta}{2}}$. Based on the expansion in the ansatz, we have

$$\begin{aligned} F_b(\sigma_*) \quad = \quad & P_c[\tilde{A}^3\psi_0^3 + (\tilde{A}^2(\tilde{\alpha} - i\tilde{\beta}) + 2\tilde{A}^2(\tilde{\alpha} + i\tilde{\beta}))\psi_0^2\psi_1 \\ & + ((\tilde{\alpha} + i\tilde{\beta})^2\tilde{A} + 2\tilde{A}(\tilde{\alpha}^2 + \tilde{\beta}^2))\psi_0\psi_1^2 \\ & + (\alpha^2 + \beta^2)(\alpha + i\beta)\psi_1^3]. \end{aligned}$$

The term of largest order in this expansion is

$$\tilde{A}^3\psi_0^3. \tag{12.117}$$

The bounds on the remaining terms will follow similarly, so we look at

$$\begin{aligned} & \int_0^t e^{-i(H-\Omega_0)(t-s) - i\int_s^t (\tilde{A}^2 + 3\tilde{\alpha}^2 + \tilde{\beta}^2)(s')ds'} P_c(\tau^{\frac{\gamma}{2}} + \epsilon_0)^3\psi_0^3 ds \\ & \quad \sim \tau^{\frac{3\gamma}{2}} \int_0^t e^{-i(H-\Omega_0)(t-s) - i\int_s^t (\tilde{A}^2 + 3\tilde{\alpha}^2 + \tilde{\beta}^2)(s')ds'} P_c\psi_0^3 ds. \end{aligned}$$

In particular, we will show there exists $\delta_0 > \delta$, such that

eqn:n

$$\|\tau^{\frac{3\gamma}{2}} \int_0^t e^{-i(H-\Omega_0)(t-s) - i\int_s^t (\tilde{A}^2 + 3\tilde{\alpha}^2 + \tilde{\beta}^2)(s')ds'} P_c\psi_0^3 ds\|_{L_{t,x}^\infty} \lesssim \tau^{1+\delta_0}. \tag{12.118}$$

From (12.53), we know

$$\dot{\tilde{\theta}} = -\Omega_0 + \tilde{A}^2 + 3\tilde{\alpha}^2(t) + \tilde{\beta}^2. \tag{12.119}$$

Hence, the leading order constant terms from the derivative are $N_{cr}^{FD} - \Omega_0$. We write

$$\begin{aligned} & \tau^{\frac{3\gamma}{2}} \quad \int_0^t e^{-i(H-\Omega_0)(t-s) - i\int_s^t (\tilde{A}^2 + 3\tilde{\alpha}^2 + \tilde{\beta}^2)(s')ds'} P_c\psi_0^3 ds = \\ & \tau^{\frac{3\gamma}{2}} \quad e^{-iHt - i\tilde{\theta}(t)} \int_0^t e^{i\tilde{\theta}(s) + i\Omega_0 s - iN_{cr}^{FD}s} e^{iHs - i\Omega_0 s + iN_{cr}^{FD}s} P_c\psi_0^3 ds = \\ & \tau^{\frac{3\gamma}{2}} \quad e^{-iHt - i\tilde{\theta}(t)} \int_0^t e^{i\tilde{\theta}(s) + i\Omega_0 s - iN_{cr}^{FD}s} \\ & \qquad \times \frac{d}{ds}\frac{1}{i}(H - \Omega_0 + N_{cr}^{FD})^{-1} e^{iHs - i\Omega_0 s + iN_{cr}^{FD}s} P_c\psi_0^3 ds, \end{aligned}$$

where the resolvent $(H - \Omega_0 + N_{cr}^{FD})^{-1}$ is a well-defined operator in H^1 on $P_c\psi_0^3$, see Appendix 12.13. Using the estimates from Appendix 12.13 and integration by parts, we

have

$$\begin{aligned}
\tau^{\frac{3\gamma}{2}} \quad & \left| e^{-iHt-i\theta(t)} \int_0^t e^{i\tilde{\theta}(s)+i\Omega_0 s-iN_{cr}^{FD}s} e^{iHs-i\Omega_0 s+iN_{cr}^{FD}s} P_c \psi_0^3 ds \right| \\
\lesssim \quad & \tau^{\frac{3\gamma}{2}} \left| (H-\Omega_0+N_{cr}^{FD})^{-1} P_c \psi_0^3 \right| \\
& + \tau^{\frac{3\gamma}{2}} \left| e^{-iHt} e^{-i\tilde{\theta}(t)} (H-\Omega_0+N_{cr}^{FD})^{-1} P_c \psi_0^3 \right| \\
& + \tau^{\frac{3\gamma}{2}} \mathcal{O}(\tau) \int_0^t \| e^{iHs} (H-\Omega_0+N_{cr}^{FD})^{-1} P_c \psi_0^3 \|_{L^\infty} ds \\
\lesssim \quad & \tau^{\frac{3\gamma}{2}} \| (H-\Omega_0+N_{cr}^{FD})^{-1} P_c \psi_0^3 \|_{H^1} \\
& + \tau^{\frac{3\gamma}{2}} \| e^{-iHt} e^{-i\tilde{\theta}(t)} (H-\Omega_0+N_{cr}^{FD})^{-1} P_c \psi_0^3 \|_{H^1} \\
& + \tau^{\frac{3\gamma}{2}} \mathcal{O}(\tau) \int_0^t \| e^{iHs} (H-\Omega_0+N_{cr}^{FD})^{-1} P_c \psi_0^3 \|_{H^1} ds \\
\lesssim \quad & \tau^{\frac{3\gamma}{2}} + \tau^{\frac{3\gamma}{2}} + \tau^{\frac{3\gamma}{2}} \mathcal{O}(\tau)\ t.
\end{aligned}$$

By selecting $\gamma > \frac{2}{3}$, we ensure that all terms resulting from integration by parts are bounded by $\tau^{1+\delta_0}$ for all $t \in I = [0, \tau^{-\frac{1+\gamma}{2}-\epsilon}]$. □

Using dispersive estimates, we have additional bounds for $\tilde{R}$ given by the following

lem:tildeRstr

LEMMA 12.11.5. *Given $\tilde{R}$ defined as in* (12.99)*, we have*

$$\|\tilde{R}\|_{H_x^1} \lesssim |\tilde{A}|^3 |I|^{\frac{3}{4}}$$

and

$$\|\tilde{R}\|_{L_t^p(I;L_x^q)} \lesssim |\tilde{A}|^3 |I|^{\frac{3}{4}}$$

for any Strichartz pair (p,q).

PROOF. To begin, we first note that by the L_x^2 boundedness of wave operators discussed in Appendix 12.13, we need only consider

$$\|\langle H \rangle \tilde{R}\|_{L_x^2},$$

where $\langle x \rangle = (1+(x)^2)^{1/2}$.

As in Lemma 12.11.4, we look at the the worst term, namely

$$\begin{aligned}
\|\langle H^s \rangle \tilde{R}\|_{L_t^\infty L_x^2} \quad & \leq \quad \| \int_0^t e^{iH(t-s)} \tilde{A}^3(s) \langle H^s \rangle P_c \psi_0^3 ds \|_{L^\infty L^2} \\
& \leq \quad \tau^{\frac{3\gamma}{2}} \|\psi_0^3\|_{L^1} |I|^{\frac{3}{4}} \\
& \lesssim \quad \tau^{\frac{3\gamma}{2}} |I|^{\frac{3}{4}},
\end{aligned}$$

where we have used the Strichartz estimate (13.43) with dual Strichartz norm $L^{\frac{4}{3}}(I;L^1)$.

The Strichartz estimate follows similarly. □

REMARK 12.11.6. *Though such estimates do not arise in this work, let us also point out the following simple estimate*

$$|\langle \tilde{R}, \chi_1 \rangle| \quad \lesssim \quad \tau^{\frac{3\gamma}{2}} \int_0^t \langle \chi_1, e^{-iH(t-s)} \chi_2 \rangle ds \tag{12.120}$$

$$\lesssim \quad \tau^{\frac{3\gamma}{2}} \int_0^t \langle t-s \rangle^{-\frac{3}{2}} ds \lesssim \tau^{\frac{3\gamma}{2}} t^{-\frac{1}{2}}, \tag{12.121}$$

which is proved in [?] and may be applicable when trying to prove long or infinite time results on similar problems to the one studied here.

Note, by standard Sobolev embeddings and the contraction assumption, we have

$$\|R\|_{L^\infty_{t,x}} \lesssim \tau^{\frac{1}{2}+\delta_1}. \tag{12.122}$$

Since we have proper bounds on $\tilde{R}$, we must now bound

$$\begin{aligned}\|\vec{\eta}\|_{L^\infty_t} &= \|\int_0^t M(t)M^{-1}(s)\big[\vec{F}_{FD}(\vec{\sigma}_*+\vec{\eta})-\vec{F}_{FD}(\vec{\sigma}_*)-D_{\sigma^*}\vec{F}_{FD}(\vec{\sigma}_*)\vec{\eta}\\ &\quad +\vec{G}_{FD}(\vec{\sigma}_*+\vec{\eta};R,\bar{R})\big]ds\|_{L^\infty}\end{aligned}$$

and

$$\begin{aligned}\|w\|_{L^\infty_t H^1_x\cap L^4_t L^\infty_x} &= \|\int_0^t e^{iH(t-s)}e^{i\Omega_0(t-s)}e^{i\int_s^t(\tilde{A}^2+3\tilde{\alpha}^2+\tilde{\beta}^2)(s')ds'}\\ &\times\ P_c\left[(F_b(\vec{\sigma}_*+\vec{\eta})-F_b(\vec{\sigma}_*))+F_R(\vec{\sigma}_*,\vec{\eta};R,\bar{R})\right]ds\|_{L^\infty H^1\cap L^4L^\infty}.\end{aligned}$$

From Appendix 12.13, we have for any Strichartz pair (p,q) that

$$\|\int_0^t e^{iH(t-s)}P_c f\|_{L^p_t W^{1,q}_x} \lesssim \|f(x,t)\|_{L^{\tilde{p}}_t W^{1,\tilde{q}}_x}, \tag{12.123}$$

where $(\tilde{p},\tilde{q})$ is a dual Strichartz pair. In one dimension, it is useful to note we may take $\tilde{p}=\frac{4}{3}+\mu_1$ and $\tilde{q}=1+\mu_2$ for $\mu_j>0$ small, $j=1,2$. In other words, we can be as close to the endpoint estimate of $\tilde{p}=\frac{4}{3}$ and $\tilde{q}=1$ as we need to be.

By reducing the system and including the lowest order terms from the expansions above, we can reduce to controlling a model problem of the form

$$\begin{aligned}\eta &= \int_0^t\left(\frac{N+N^{FD}_{cr}}{|N-N^{FD}_{cr}|}\right)^{\frac{1}{2}}[\tilde{A}\eta^2+\eta^3\\ &+ \tilde{A}^2\langle\tilde{R},\chi\rangle+\tilde{A}^2\langle w,\chi\rangle+2\tilde{A}\eta\langle\chi,\tilde{R}\rangle+\eta^2\langle\tilde{R},\chi\rangle\\ &+ \tilde{A}\eta\langle w,\chi\rangle+\eta^2\langle w,\chi\rangle]ds\\ &= Term^\eta_1+Term^\eta_2+\cdots+Term^\eta_8],\end{aligned}$$

where $\chi\in\mathscr{S}$, and

$$\begin{aligned}w &= \int_0^t e^{iH(t-s)}e^{i\Omega_0(t-s)}e^{i\int_s^t(\tilde{A}^2+3\tilde{\alpha}^2+\tilde{\beta}^2)(s')ds'}\\ &\times\ P_c[(3\tilde{A}^2\eta\psi_0^3+2\tilde{A}\tilde{R}\eta+\tilde{A}\psi_0^2\tilde{R}\eta)\\ &+ (\tilde{A}^2\psi_0^2w+w^3+2\tilde{A}\eta w+\tilde{A}\langle\chi,w\rangle\tilde{R}+\tilde{A}\langle\chi,\tilde{R}\rangle w)\\ &+ (\tilde{A}^2\psi_0^2\tilde{R}+\tilde{A}\langle\chi,\tilde{R}\rangle\tilde{R})]ds\\ &= Term^w_1+Term^w_2+\cdots+Term^w_{10}.\end{aligned}$$

For our model problem, we now take $(\eta,w)\in X$, $\|(\eta,w)\|_X\le\tau^{\frac{1}{2}+\delta_1}$. In addition, take $(\eta_j,w_j)\in X$, $\|(\eta_j,w_j)\|_X\le\tau^{\frac{1}{2}+\delta_1}$ for $j=1,2$.

It follows

$$\begin{aligned}\|Term^\eta_1\|_{L^\infty_t} &= \left(\frac{N+N^{FD}_{cr}}{|N-N^{FD}_{cr}|}\right)^{\frac{1}{2}}\|\int_0^t\tilde{A}(s)\eta^2(s)ds\|_{L^\infty_t}\\ &\le \tau^{\frac{\gamma}{2}-\frac{1}{2}+\frac{\gamma}{2}-\frac{1+\gamma}{2}-\epsilon}\|\eta\|^2_{L^\infty_t}\\ &\le \tau^{\frac{\gamma}{2}+2\delta_1-\epsilon}\ \le\ \tau^{\frac{1}{2}+\delta_1}\end{aligned}$$

provided we choose

constraint-1 (12.124) $$\delta_1 > \frac{1-\gamma}{2} + \epsilon.$$

In addition, it is clear from the analysis there that

$$\|Term_1^{\eta_1} - Term_1^{\eta_2}\|_{L_t^\infty} \leq \tau^{\frac{\gamma-1}{2}+\delta_1-\epsilon}\|\eta_1-\eta_2\|_{L_t^\infty}.$$

Next, for $Term_2^\eta$ we get

$$\begin{aligned}\|Term_2^\eta\|_{L_t^\infty} &= \left(\frac{N+N_{cr}^{FD}}{|N-N_{cr}^{FD}|}\right)^{\frac{1}{2}}\|\int_0^t \eta^3(s)ds\|_{L_t^\infty} \\ &\leq \tau^{\frac{\gamma}{2}-\frac{1}{2}-\frac{1+\gamma}{2}-\epsilon}\|\eta\|_{L^\infty}^3 \\ &\leq \tau^{\frac{1}{2}+3\delta_1-\epsilon}\end{aligned}$$

provided

constraint-2 (12.125) $$2\delta_1 > \epsilon.$$

Similarly, we have

$$\|Term_2^{\eta_1} - Term_2^{\eta_2}\|_{L_t^\infty} \leq \tau^{2\delta_1-\epsilon}\|\eta_1-\eta_2\|_{L_t^\infty}.$$

For $Term_3^\eta$, we have

$$\begin{aligned}\|Term_3^\eta\|_{L_t^\infty} &= \left(\frac{N+N_{cr}^{FD}}{|N-N_{cr}^{FD}|}\right)^{\frac{1}{2}}\|\int_0^t \tilde{A}^2\langle\tilde{R},\chi\rangle ds\|_{L_t^\infty} \\ &\leq \tau^{\frac{\gamma}{2}-\frac{1}{2}+\gamma+1-\frac{1}{2}-\frac{\gamma}{2}-\epsilon} \sim \tau^{\gamma-\epsilon} \leq \tau^{\frac{1}{2}+\delta_1}\end{aligned}$$

provided

constraint-3 (12.126) $$\gamma - \epsilon > \frac{1}{2} + \delta_1.$$

Since $Term_3^\eta$ is independent of (η, w), it follows easily that

$$\|Term_3^{\eta_1} - Term_3^{\eta_2}\|_{L_t^\infty} = 0,$$

meaning this term does not factor into the contraction.

The bound on $Term_4^\eta$ is of the form

$$\begin{aligned}\|Term_4^\eta\|_{L_t^\infty} &= \left(\frac{N+N_{cr}^{FD}}{|N-N_{cr}^{FD}|}\right)^{\frac{1}{2}}\|\int_0^t \tilde{A}^2\langle w,\chi\rangle ds\|_{L_t^\infty} \\ &\leq \tau^{\frac{\gamma}{2}-\frac{1}{2}+\gamma}\|\chi\|_{L_t^{\frac{4}{3}}L_x^1}\|w\|_{L_t^4L_x^\infty} \\ &\leq \tau^{\frac{\gamma}{2}-\frac{1}{2}+\gamma-\frac{3}{8}-\frac{3\gamma}{8}-\frac{3\epsilon}{4}}\|w\|_{L_t^4L_x^\infty} \\ &\leq \tau^{\frac{9\gamma}{8}-\frac{7}{8}-\frac{3\epsilon}{4}}\|w\|_{L_t^4L_x^\infty} \leq \tau^{\frac{1}{2}+\delta_1},\end{aligned}$$

provided

constraint-4 (12.127) $$\gamma > \frac{7}{9} + \frac{2}{3}\epsilon.$$

It follows immediately that

$$\begin{aligned}\|Term_4^{\eta_1} - Term_4^{\eta_2}\|_{L_t^\infty} &\leq \tau^{\frac{\gamma}{2}-\frac{1}{2}+\gamma-\frac{3}{8}-\frac{3\gamma}{8}-\frac{3\epsilon}{4}}\|w\|_{L_t^4L_x^\infty}\\ &\leq \tau^{\frac{9\gamma}{8}-\frac{7}{8}-\frac{3\epsilon}{4}}\|w_1-w_2\|_{L_t^4L_x^\infty}.\end{aligned}$$

For $Term_5^\eta$, we have

$$\begin{aligned}\|Term_5^\eta\|_{L_t^\infty} &= \left(\frac{N+N_{cr}^{FD}}{|N-N_{cr}^{FD}|}\right)^{\frac{1}{2}}\|\int_0^t 2\tilde{A}\eta\langle\chi,\tilde{R}\rangle ds\|_{L_t^\infty}\\ &\leq \tau^{\frac{\gamma}{2}-\frac{1}{2}+\frac{\gamma}{2}-\frac{1}{2}-\frac{\gamma}{2}-\epsilon}\|\eta\|_{L_t^\infty}\|\tilde{R}\|_{L_{x,t}^\infty}\\ &\leq \tau^{\frac{\gamma}{2}-\epsilon}\|\eta\|_{L^\infty}\leq\tau^{\frac{1}{2}+\delta_1},\ (\gamma>2\epsilon),\end{aligned}$$

which follows from previous constraints on γ. Once again, we have as well

$$\|Term_5^{\eta_1} - Term_5^{\eta_2}\|_{L_t^\infty} \leq \tau^{\frac{\gamma}{2}-\epsilon}\|\eta_1-\eta_2\|_{L_t^\infty}.$$

For $Term_6^\eta$, we have using Proposition 12.11.4

$$\begin{aligned}\|Term_6^\eta\|_{L_t^\infty} &= \left(\frac{N+N_{cr}^{FD}}{|N-N_{cr}^{FD}|}\right)^{\frac{1}{2}}\|\int_0^t \eta^2\langle\tilde{R},\chi\rangle ds\|_{L_t^\infty}\\ &\leq \tau^{\frac{\gamma}{2}-\frac{1}{2}+1-\frac{1}{2}-\frac{\gamma}{2}-\epsilon}\|\eta\|_{L_t^\infty}^2\\ &\leq \tau^{\frac{1}{2}+\delta_1-\epsilon}\|\eta\|_{L_t^\infty}\leq\tau^{\frac{1}{2}+\delta_1},\end{aligned}$$

which follows if $\delta_1>\epsilon$. Furthermore,

$$\|Term_6^{\eta_1} - Term_6^{\eta_2}\|_{L_t^\infty} \leq \tau^{\frac{1}{2}+\delta_1-\epsilon}\|\eta_1-\eta_2\|_{L^\infty}.$$

For $Term_7^\eta$, we have

$$\begin{aligned}\|Term_7^\eta\|_{L_t^\infty} &= \left(\frac{N+N_{cr}^{FD}}{|N-N_{cr}^{FD}|}\right)^{\frac{1}{2}}\|\int_0^t \tilde{A}\eta\langle w,\chi\rangle ds\|_{L_t^\infty}\\ &\leq \tau^{\frac{\gamma}{2}-\frac{1}{2}+\frac{\gamma}{2}-\frac{3}{8}-\frac{3\gamma}{8}-\frac{3\epsilon}{4}}\|\eta\|_{L_t^\infty}\|w\|_{L_t^4L_x^\infty}\\ &\leq \tau^{\frac{5\gamma}{8}-\frac{3}{8}-\frac{3\epsilon}{4}+\delta_1}\|\eta\|_{L_t^\infty}\leq\tau^{\frac{1}{2}+\delta_1}\end{aligned}$$

provided

constraint-5 (12.128) $$\gamma>\frac{3}{5}+\frac{6}{5}\epsilon-\frac{8}{5}\delta_1.$$

Furthermore,

$$\|Term_7^{\eta_1} - Term_7^{\eta_2}\|_{L_t^\infty} \leq \tau^{\frac{5\gamma}{8}-\frac{3}{8}-\frac{3\epsilon}{4}+\delta_1}\|(\eta_1-\eta_2,w_1-w_2)\|_X.$$

For $Term_8^\eta$, we have

$$\begin{aligned}\|Term_8^\eta\|_{L_t^\infty} &= \left(\frac{N+N_{cr}^{FD}}{|N-N_{cr}^{FD}|}\right)^{\frac{1}{2}}\|\int_0^t \eta^2\langle w,\chi\rangle ds\|_{L_t^\infty}\\ &\leq \tau^{\frac{\gamma}{2}-\frac{1}{2}-\frac{3}{8}-\frac{3\gamma}{8}-\frac{3\epsilon}{4}}\|\eta\|_{L_t^\infty}^2\|w\|_{L_t^4L_x^\infty}\\ &\leq \tau^{\frac{\gamma}{8}+\frac{1}{8}-\frac{3\epsilon}{4}+2\delta_1}\|w\|_{L_t^4L_x^\infty}\leq\tau^{\frac{1}{2}+\delta_1}\end{aligned}$$

by previous constraints on γ. Again, it follows that

$$\|Term_8^{\eta_1} - Term_8^{\eta_2}\|_{L_t^\infty} \quad \leq \quad \tau^{\frac{\gamma}{8}+\frac{1}{8}-\frac{3\epsilon}{4}+2\delta_1}\|w_1 - w_2\|_{L_t^4 L_x^\infty}.$$

We now study the map on the dispersive part, w. For $Term_1^w$,using the Strichartz estimates (13.43), we have

$$\begin{aligned}
\| \quad Term_1^w \quad \|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} &= \\
&\|\int_0^t e^{iH(t-s)} e^{i\Omega_0(t-s)} e^{i\int_s^t (\tilde{A}^2+3\tilde{\alpha}^2+\tilde{\beta}^2)(s')ds'} P_c 3\tilde{A}^2\eta\psi_0^3 ds\|_{L^\infty H^1 \cap L^4 L^\infty} \\
&\leq \tau^\gamma \|\eta\|_{L^\infty}\|\psi_0^3\|_{L^{\frac{4}{3}}W^{1,1}} \\
&\leq \tau^{\gamma-\frac{3}{8}-\frac{3\gamma}{8}-\frac{3\epsilon}{4}}\|\eta\|_{L^\infty} \leq \tau^{\frac{1}{2}+\delta_1}, \ (\gamma > \frac{3}{5}),
\end{aligned}$$

by previous constraints on γ. Hence,

$$\|Term_1^{w_1} - Term_1^{w_2}\|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \quad \leq \quad \tau^{\gamma-\frac{3}{8}-\frac{3\gamma}{8}-\frac{3\epsilon}{4}}\|\eta_1 - \eta_2\|_{L_t^\infty}.$$

For $Term_2^w$, we have using bounds on $\tilde{R}$ in H^1 (Lemma 12.11.5)

$$\begin{aligned}
\| \quad Term_2^w \quad \|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} &= \\
&\|\int_0^t e^{iH(t-s)} e^{i\Omega_0(t-s)} e^{i\int_s^t (\tilde{A}^2+3\tilde{\alpha}^2+\tilde{\beta}^2)(s')ds'} P_c 2\tilde{A}\tilde{R}\eta ds\|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \\
&\leq \tau^{\frac{\gamma}{2}}\|\tilde{R}\|_{L_t^1 H_x^1}\|\eta\|_{L_t^\infty} \\
&\leq \tau^{2\gamma-\frac{7}{8}-\frac{7\gamma}{8}-\frac{7\epsilon}{4}}\|\eta\|_{L_t^\infty} \leq \tau^{\frac{9\gamma}{8}-\frac{7}{8}-\frac{7\epsilon}{4}}\|\eta\|_{L_t^\infty} \leq \tau^{\frac{1}{2}+\delta_1},
\end{aligned}$$

which follows using $\gamma > \frac{7}{9} + \frac{14}{9}\epsilon$. Hence, it follows that

$$\|Term_2^{w_1} - Term_2^{w_2}\|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \quad \leq \quad \tau^{\frac{9\gamma}{8}-\frac{7}{8}-\frac{7\epsilon}{8}}\|\eta_1 - \eta_2\|_{L_t^\infty}.$$

For $Term_3^w$, using Lemma 12.11.5 once again we have

$$\begin{aligned}
\| \quad Term_3^w \quad \|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} &= \\
&\|\int_0^t e^{iH(t-s)} e^{i\Omega_0(t-s)} e^{i\int_s^t (\tilde{A}^2+3\tilde{\alpha}^2+\tilde{\beta}^2)(s')ds'} P_c \tilde{A}\psi_0^2\tilde{R}\eta ds\|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \\
&\leq \tau^{\frac{\gamma}{2}}\|\tilde{R}\psi_0^2\|_{L_t^1 H_x^1}\|\eta\|_{L_t^\infty} \\
&\leq \tau^{2\gamma-\frac{7}{8}-\frac{7\gamma}{8}-\frac{7\epsilon}{4}}\|\eta\|_{L_t^\infty} \leq \tau^{\frac{9\gamma}{8}-\frac{7}{8}-\frac{7\epsilon}{4}}\|\eta\|_{L_t^\infty} \leq \tau^{\frac{1}{2}+\delta_1},
\end{aligned}$$

which follows easily from previous constraints on γ. Hence, it follows directly that

$$\|Term_3^{w_1} - Term_3^{w_2}\|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \quad \leq \quad \tau^{\frac{9\gamma}{8}-\frac{7}{8}-\frac{7\epsilon}{4}}\|\eta_1 - \eta_2\|_{L_t^\infty}.$$

For $Term_4^w$, we have

$$\begin{aligned}
\| \quad Term_4^w \quad \|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} &= \\
&\|\int_0^t e^{iH(t-s)} e^{i\Omega_0(t-s)} e^{i\int_s^t (\tilde{A}^2+3\tilde{\alpha}^2+\tilde{\beta}^2)(s')ds'} P_c \tilde{A}^2\psi_0^2 w ds\|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \\
&\leq \tau^\gamma \ \|\psi_0^2 w\|_{L_t^{\frac{4}{3}}W_x^{1,1}} \leq \tau^{-\frac{3}{8}+\frac{5\gamma}{8}-\frac{3\epsilon}{4}}\|w\|_{L_t^\infty H_x^1} \leq \tau^{\frac{1}{2}+\delta_1}, \quad (\gamma > \frac{3}{5}),
\end{aligned}$$

which follows from previous constraints on γ. Furthermore,

$$\|Term_4^{w_1} - Term_4^{w_2}\|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \quad \leq \quad \tau^{\frac{5}{8}+\frac{\gamma}{8}-\frac{3\epsilon}{4}} \|w_1 - w_2\|_{L_t^\infty H_x^1}.$$

For $Term_5^w$, we have

$$\begin{aligned} \| \quad Term_5^w \quad \|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} = \\ \| \int_0^t e^{iH(t-s)} e^{i\Omega_0(t-s)} e^{i\int_s^t (\tilde{A}^2+3\tilde{\alpha}^2+\tilde{\beta}^2)(s')ds'} P_c w^3 ds \|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \\ \leq \tau^{-\frac{1}{2}-\frac{\gamma}{2}-\epsilon} \|w\|^3_{L_t^\infty H_x^1} \\ \leq \tau^{\frac{1-\gamma}{2}-\epsilon+2\delta_1} \|w\|_{L_t^\infty H_x^1} \leq \tau^{\frac{1}{2}+\delta_1}, \end{aligned}$$

which follows easily by previous constraints on γ. Moreover,

$$\|Term_5^{w_1} - Term_5^{w_2}\|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \quad \leq \quad \tau^{\frac{1-\gamma}{2}-\epsilon+2\delta_1} \|w_1 - w_2\|_{L_t^\infty H_x^1}.$$

For $Term_6^w$, we have

$$\begin{aligned} \| \quad Term_6^w \quad \|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} = \\ \| \int_0^t e^{iH(t-s)} e^{i\Omega_0(t-s)} e^{i\int_s^t (\tilde{A}^2+3\tilde{\alpha}^2+\tilde{\beta}^2)(s')ds'} 2P_c \tilde{A}\eta w ds \|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \\ \leq \tau^{\frac{\gamma}{2}-\frac{1}{2}-\frac{\gamma}{2}-\epsilon} \|\eta\|_{L_t^\infty} \|w\|_{L_t^\infty H_x^1} \leq \tau^{\delta_1-\epsilon} \|w\|_{L_t^\infty H_x^1}, \end{aligned}$$

which follows easily from previous constraints on δ_1. Hence, it follows directly that

$$\|Term_6^{w_1} - Term_6^{w_2}\|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \quad \leq \quad \tau^{\delta_1-\epsilon} \|w_1 - w_2\|_{L_t^\infty H_x^1}.$$

For $Term_7^w$, we have from Lemma 12.11.5

$$\begin{aligned} \| \quad Term_7^w \quad \|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} = \\ \| \int_0^t e^{iH(t-s)} e^{i\Omega_0(t-s)} e^{i\int_s^t (\tilde{A}^2+3\tilde{\alpha}^2+\tilde{\beta}^2)(s')ds'} P_c \tilde{A} \langle \chi, w \rangle \tilde{R} ds \|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \\ \leq \tau^{\frac{\gamma}{2}-\frac{1}{2}-\frac{\gamma}{2}-\epsilon} \|\tilde{R}\|_{L_t^\infty H_x^1} \|w\|_{L_t^\infty H_x^1} \\ \leq \tau^{\frac{9\gamma}{8}-\frac{7}{8}-\frac{7}{4}\epsilon} \|w\|_{L_t^\infty H_x^1} \leq \tau^{\frac{1}{2}+\delta_1}, \quad (\gamma > \frac{7}{9}), \end{aligned}$$

which follows from previous constraints on γ. It follows directly that

$$\|Term_7^{w_1} - Term_7^{w_2}\|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \quad \leq \quad \tau^{\frac{9\gamma}{8}-\frac{7}{8}-\frac{7}{4}\epsilon} \|w_1 - w_2\|_{L_t^\infty H_x^1}.$$

The bounds for $Term_8^w$ follows in a similar manner.

For $Term_9^w$ and $Term_{10}^w$, we have

$$\begin{aligned} \| \quad Term_9^w \quad \|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} = \\ \| \int_0^t e^{iH(t-s)} e^{i\Omega_0(t-s)} e^{i\int_s^t (\tilde{A}^2+3\tilde{\alpha}^2+\tilde{\beta}^2)(s')ds'} P_c \tilde{A}^2 \psi_0^2 \tilde{R} ds \|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \\ \leq \tau^\gamma \|\psi_0^2 \tilde{R}\|_{L_t^1 H_x^1} \leq \tau^{\frac{13}{8}\gamma-\frac{7}{8}-\frac{11}{8}\epsilon} \leq \tau^{\frac{1}{2}+\delta_1} \end{aligned}$$

and

$$\begin{aligned}\| \ Term_{10}^{w} \ \|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} = \\ \| \int_0^t e^{iH(t-s)} e^{i\Omega_0(t-s)} e^{i\int_s^t (\tilde{A}^2+3\tilde{\alpha}^2+\tilde{\beta}^2)(s')ds'} P_c \tilde{A}\langle \chi, \tilde{R}\rangle \tilde{R} ds \|_{L_t^\infty H_x^1 \cap L_t^4 L_x^\infty} \\ \leq \tau^\gamma \|\tilde{R}\|_{L_t^1 H_x^1} \|\tilde{R}\|_{L_{x,t}^\infty} \leq \tau^{2\gamma+1-\frac{3}{8}\gamma-\frac{3}{8}-\frac{3}{4}\epsilon} \leq \tau^{\frac{1}{2}+\delta_1}.\end{aligned}$$

Both of these terms are independent of (η, w), meaning the contraction mapping follows easily.

Hence, choosing γ, δ_1, and ϵ such that the constraints (12.124), (12.125), (12.126), (12.127), (12.128) are satisfied, the contraction argument follows and the result holds.

Once we have solved for $(\vec{\sigma}, R)$, it is clear that the resulting function $\theta \in C(I)$ by construction. □

12.12. Poincaré Continuation and Control of the Period for the Reduced System of ODE's

sec:period

In this Appendix, we construct periodic solutions, as (Poincaré) continuations of the elliptic equilibrium states. The argument (see [**?**], Chapter 14) is based on the implicit function theorem. Consequences of this analysis are leading order approximations of periodic solutions and their associated periods as stated in Proposition 12.9.5.

Recall the reduced system of ODE's for α, β in the alternative coordinates:

$$\partial_t \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} (\Omega_{10}+2\alpha^2)\beta \\ -\left[\Omega_{10}+2(2\alpha^2+\beta^2-N)\right]\alpha \end{pmatrix} = f(\alpha,\beta).$$

We will work in the case $N > N_{cr}^{FD}$, as the result for $N < N_{cr}^{FD}$ follows from similar analysis. Hence, we have an equilibrium solution given by

$$\alpha_* = \pm\sqrt{\frac{N-N_{cr}^{FD}}{2}}, \ \ \beta_* = 0.$$

We first center our calculation about the equilibrium state. Let ε be a parameter, to be chosen small, which sets the scale of the neighborhood about the equilibrium and define:

$$\alpha = \alpha_* + \varepsilon h_1, \beta = \varepsilon h_2$$

(In this section, ε is completely unrelated to the small ε parameter in Theorem 3.) Then, we have

$$\begin{aligned}\partial_t \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \ &= \ Df(\alpha_*,\beta_*)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \varepsilon\sqrt{\frac{N-N_{cr}^{FD}}{2}}\begin{pmatrix} 4h_1h_2 \\ -(6h_1^2-2h_2^2) \end{pmatrix} \\ &\quad + \varepsilon^2 \begin{pmatrix} 2h_1^2h_2 \\ -(h_1^3-h_1h_2^2) \end{pmatrix},\end{aligned}$$

h-eqn

$$\left(\partial_t - Df(\alpha_*,\beta_*)\right)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \varepsilon\sqrt{\frac{N-N_{cr}^{FD}}{2}} N_2(h_1,h_2) + \varepsilon^2 N_3(h_1,h_2), \tag{12.129}$$

where

$$\begin{aligned} Df(\alpha_*,\beta_*) = JD^2H(\alpha_*,\beta_*) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 4(N-N_{cr}^{FD}) & 0 \\ 0 & N+N_{cr}^{FD} \end{pmatrix} \\ &= \begin{pmatrix} 0 & N+N_{cr}^{FD} \\ -4(N-N_{cr}^{FD}) & 0 \end{pmatrix}, \end{aligned}$$

with eigenvalues $\pm 2i(N^2-(N_{cr}^{FD})^2)^{\frac{1}{2}}$.

Therefore, at leading order in ε, solutions have period

T0 (12.130) $$T_0 = \frac{\pi}{|N^2-(N_{cr}^{FD})^2|^{\frac{1}{2}}}.$$

Since (12.129) is nonlinear and autonomous, the period will vary with amplitude. We make the unknown period explicit by introducing a new independent variable via:

s-def (12.131) $$t \;=\; T(\varepsilon)s$$

and seek a solution of (12.129) in the form:

$$\begin{aligned} \vec{H}(s) &= \begin{pmatrix} H_1(s) \\ H_2(s) \end{pmatrix} = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} \\ \vec{H}(s+1) &= \vec{H}(s), \end{aligned}$$

corresponding to a solution which is $T(\varepsilon)-$ periodic in t.

We motivate our rigorous construction of periodic solutions by making a formal expansion of the solution and period in powers of the small parameter, ε:

Hexpand (12.132) $$\begin{aligned} H_1 &= H_1^{(0)} + \varepsilon H_1^{(1)} + \varepsilon^2 H_1^{(2)} + \dots \\ H_2 &= H_2^{(0)} + \varepsilon H_2^{(1)} + \varepsilon^2 H_2^{(2)} + \dots . \end{aligned}$$

Texpand (12.133) $$T(\varepsilon) = T_0(1+\varepsilon\mu_1+\varepsilon^2\mu_2+\dots)$$

Here, T_0 is the leading order approximation to the period, given in (12.130), and $H_j^{(k)}, \mu_j$, $j \geq 1, k \geq 0$ are to be determined.

The system for $\vec{H}(s) = \vec{H}(s;\varepsilon)$ is then

Hs-eqn (12.134) $$\partial_s \vec{H} = T(\varepsilon)Df(\alpha_*,\beta_*)\vec{H} + T(\varepsilon)\varepsilon\sqrt{\frac{N-N_{cr}^{FD}}{2}}N_2(\vec{H}) + T(\varepsilon)\varepsilon^2 N_3(\vec{H}).$$

Substitution of the expansions (12.132) and (12.133) into (12.134) yields the hierarchy of equations:

$$\begin{aligned} \varepsilon^0 &: \left(\partial_s - T_0Df(\alpha_*,\beta_*)\right)\vec{H}^{(0)} &&= 0, \\ \varepsilon^1 &: \left(\partial_s - T_0Df(\alpha_*,\beta_*)\right)\vec{H}^{(1)} &&= T_0\,\mu_1 Df(\alpha_*,\beta_*)\,\vec{H}^{(0)} \;+\; N^{(1)} \\ \varepsilon^2 &: \left(\partial_s - T_0Df(\alpha_*,\beta_*)\right)\vec{H}^{(2)} &&= T_0\,\mu_2 Df(\alpha_*,\beta_*)\vec{H}^{(0)} \;+\; N^{(2)} \\ &\qquad\vdots \\ \varepsilon^m &: \left(\partial_s - T_0Df(\alpha_*,\beta_*)\right)\vec{H}^{(m)} &&= T_0\,\mu_m Df(\alpha_*,\beta_*)\vec{H}^{(0)} \;+\; N^{(m)} \quad m \geq 3, \end{aligned}$$

where $N^{(m)} = N^{(m)}(\vec{H}^{(j)}, \mu_j;\, 0 \leq j \leq m-1)$.

We now discuss the solution of the above hierarchy. Beginning with the equation for $\vec{H}^{(0)}$, we determine the kernel of $\partial_s - T_0 Df(\alpha_*,\beta_*)$. In particular,

$$\begin{aligned} e^{Bs} &= \begin{bmatrix} 1 & 1 \\ 2i\sqrt{\frac{N-N_{cr}^{FD}}{N+N_{cr}^{FD}}} & -2i\sqrt{\frac{N-N_{cr}^{FD}}{N+N_{cr}^{FD}}} \end{bmatrix} \begin{bmatrix} e^{is} & 0 \\ 0 & e^{-is} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{i}{4}\sqrt{\frac{N+N_{cr}^{FD}}{N-N_{cr}^{FD}}} \\ \frac{1}{2} & \frac{i}{4}\sqrt{\frac{N+N_{cr}^{FD}}{N-N_{cr}^{FD}}} \end{bmatrix} \\ &= \begin{bmatrix} \cos(t) & \frac{1}{2}\sqrt{\frac{N+N_{cr}^{FD}}{N-N_{cr}^{FD}}}\sin(t) \\ -2\sqrt{\frac{N-N_{cr}^{FD}}{N+N_{cr}^{FD}}}\sin(t) & \cos(t) \end{bmatrix} \end{aligned}$$

and therefore

$$\mathrm{Ker}((\partial_s - T_0 Df(\alpha_*,\beta_*))) = \mathrm{span}\left\{\vec{\xi}(s), \partial_s\vec{\xi}(s)\right\},$$

where

xi-def (12.135) $$\vec{\xi}(s) = \begin{pmatrix} \cos(s) \\ -2\sqrt{\frac{N-N_{cr}^{FD}}{N+N_{cr}^{FD}}}\sin(s) \end{pmatrix}.$$

We take $\vec{H}^{(0)}(s) = \vec{\xi}(s)$.

We now turn to the equation for $\vec{H}^{(1)}$. In general, the equation

$$(\partial_s - T_0 Df(\alpha_*,\beta_*))\vec{H}^{(j)} = \vec{f},\ \ \vec{f}(s+1) = \vec{f}(s)$$

is solvable if and only if

solvability (12.136) $$\int_0^1 \vec{y}^T(s)\ \vec{f}(s)\ ds = 0 \text{ for all } y \in \mathrm{Ker}\left(\partial_t + T_0 Df(\alpha_*,\beta_*)^T\right).$$

Due to the Hamiltonian structure, $Df(\alpha_*,\beta_*) = JD^2H(\alpha_*,\beta_*)$, the adjoint kernel is given by

$$\mathrm{Ker}((\partial_s - T_0 Df(\alpha_*,\beta_*))^*) = \mathrm{span}\left\{J\vec{\xi}(t), J\partial_t\vec{\xi}(t)\right\},$$

It is elementary to check however that if $H^{(0)}(s)$ is chosen as above, and therefore

$$\begin{pmatrix} H_1^{(0)}(s) \\ H_2^{(0)}(s) \end{pmatrix} = \begin{pmatrix} \text{even}(s) \\ \text{odd}(s) \end{pmatrix},$$

then the right hand sides of each equation in the hierarchy for $H^{(m)}(s)$ satisfies

$$\begin{pmatrix} N_1^{(m)}(s) \\ N_2^{(m)}(s) \end{pmatrix} = \begin{pmatrix} \text{odd}(s) \\ \text{even}(s) \end{pmatrix},$$

It follows, on symmetry grounds, that the full right hand side of each equation is orthogonal to the second adjoint vector, $J\partial_s\xi(s)$. Orthogonality to $J\xi(s)$ in implies

$$T_0\ \langle J\vec{\xi}\ JD^2H(\alpha_*,\beta_*)\vec{\xi}\rangle_{L^2([0,1])}\ \mu_m\ +\ \langle J\vec{\xi}, N^{(m)}\rangle_{L^2([0,1])} = 0.$$

This determines μ_m uniquely, provided the following non-degeneracy condition holds:

ortho (12.137) $$\langle J\vec{\xi}\ ,\ JD^2H(\alpha_*,\beta_*)\vec{\xi}\rangle_{L^2([0,1])} \neq 0$$

To verify this, we use $J^TJ = -I$ to simplify (12.137), and we obtain for the case under consideration, $N > N_{cr}^{FD}$, that

nondegen (12.138) $$\langle J\vec{\xi}\ ,\ JD^2H(\alpha_*,\beta_*)\vec{\xi}\rangle_{L^2([0,1])} = 4(N - N_{cr}^{FD}) \neq 0.$$

The calculation implies

$$\mu_1 \sim \tau^{-\frac{1}{2}} \text{ and therefore } \varepsilon\mu_1 \sim \tau^{\frac{1}{2}} \ll 1$$

The above formal expansion is made rigorous via the implicit function theorem. Define

$$\begin{aligned}
&\vec{F}(\vec{H}, \delta T; \varepsilon) = (F_1, F_2)(H_1, H_2, \delta T; \varepsilon) = 0, \\
&\delta T = T(\varepsilon) - T_0.
\end{aligned}$$

where

$$\begin{aligned}
F_1(\vec{H}, \delta T; \varepsilon) &= \dot{\vec{H}} - T_0 Df(\alpha_*, \beta_*)\vec{H} - \delta T Df(\alpha_*, \beta_*)\vec{H} \\
&\quad + (T_0 + \delta T)\varepsilon\sqrt{\frac{N - N_{cr}^{FD}}{2}} N_2(\vec{H}) + (T_0 + \delta T)\varepsilon^2 N_3(\vec{H}) \\
F_2(\vec{H}, \delta T; \varepsilon) &= \delta T \langle J\vec{\xi}\ ,\ JD^2 H(\alpha_*, \beta_*)\vec{\xi}\rangle_{L^2([0,1])} \\
&\quad - \varepsilon(T_0 + \delta T)\int_0^1 \left[\sqrt{\frac{N - N_{cr}^{FD}}{2}} N_2(\vec{H}) + \varepsilon N_3(\vec{H})\right] \cdot J\vec{\xi}(s)ds
\end{aligned}$$

Hence, $\vec{F} : C^1_{even}(S^1) \times C^1_{odd}(S^1) \times \mathbb{R} \to C^0_{odd}(S^1) \times C^1_{even}(S^1) \times \mathbb{R}$, where $C^k(S^1)$ is the space of k times differentiable functions of period 1. Note that $(H, \delta T) = (\vec{\xi}, 0)$ is a solution for $\varepsilon = 0$. To apply the Implicit Function Theorem to conclude the existence of a solution $\varepsilon \mapsto (\vec{H}(\varepsilon), \delta T(\varepsilon))$ for ε sufficiently small, we need to check that the Jacobian, $d_{\vec{H}, \delta T} F(\vec{H}_0, 0; 0)$ is an isomorphism between $C^1_{even}(S^1) \times C^1_{odd}(S^1) \times \mathbb{R}$ and $C^0_{odd}(S^1) \times C^1_{even}(S^1) \times \mathbb{R}$.

Hence, we derive

$$\begin{aligned}
(dF)_0 &\equiv d_{\vec{H}, \delta T} F(\vec{H}_0, 0; 0) \\
&= \begin{bmatrix} \partial_t - T_0 Df(\alpha_*, \beta_*) & -Df(\alpha_*, \beta_*)\vec{H}_0 \\ 0 & \langle J\vec{\xi}\ ,\ JD^2 H(\alpha_*, \beta_*)\vec{\xi}\rangle_{L^2([0,1])} \end{bmatrix}.
\end{aligned}$$

We claim that since, by (12.138), $\langle J\vec{\xi}\ ,\ JD^2 H(\alpha_*, \beta_*)\vec{\xi}\rangle_{L^2([0,1])} \neq 0$, $(dF)_0$ is invertible. To see this, consider the equation $(dF)_0(Y, \rho)^T = (Z, \tau)^T \in C^0(S^1) \times \mathbb{R}$. Solvability of the second equation requires $\langle J\vec{\xi}\ ,\ JD^2 H(\alpha_*, \beta_*)\vec{\xi}\rangle_{L^2([0,1])} \neq 0$. Furthermore, this non-degeneracy condition ensures that δT can be chosen so that the first equation is solvable for $Y \in C^1_{even}(S^1) \times C^1_{odd}(S^1)$. This completes the proof.

12.13. Dispersive Estimates

sec:est

In this Appendix, we follow closely the work [**Wed99**] on wave operators for Schrödigner operators defined on $\mathbb{R}$. For proofs and further exposition see [**Wed99**] and the references contained within. First of all, let us define $H_0 = -\Delta$ and $H = -\Delta + V$ with the constraints on V to be discussed in the sequel.

The wave operators, $W_\pm$ are defined by

$$W_\pm = \lim_{t\to\infty} e^{itH} e^{-itH_0}. \tag{12.139}$$

Similarly, their adjoints are defined by

$$W_\pm^* = \lim_{t\to\infty} e^{itH_0} e^{-itH} P_c, \tag{12.140}$$

where P_c is the projection onto the continuous spectrum of H. The notion of wave operators is intimately related to the idea of distorted Fourier bases, which are discussed in detail in [**Agm75a**], [**Hör05**], [**RS78b**]. In one dimension, this is directly related to the Jost solutions. These objects are studied in general in [**RS78b**] and generalized to even a certain class of non-self-adjoint operators in [**KS06a**].

We define a space L^1_γ the space of all complex-valued measurable functions ϕ defined on $\mathbb{R}$ such that

$$\|\phi\|_{L^1_\gamma} = \int |\phi(x)|(1+|x|)^\gamma dx < \infty. \tag{12.141}$$

Also, take the space $W^{k,p}$ to be the standard Sobolev space defined by having k derivatives bounded in the L^p norm. Then, we have the following

thm:w

THEOREM 4 (Weder). *Suppose that $V \in L^1_\gamma$ for $\gamma > \frac{5}{2}$ and that for some $k = 1,2,\ldots$, $V^{(l)} \in L^1$ for $l = 0,1,2,\ldots,k-1$. Then $W_\pm$ and $W^*_\pm$ originally defined on $W_{k,p} \cap L^2$, $1 \le p \le \infty$, have extensions to bounded operators on $W_{k,p}$, $1 < p < \infty$. Moreover, there are constants C_p such that:*

$$\|W_\pm f\|_{W^{k,p}} \le C_p \|f\|_{W^{k,p}},\ \|W^*_\pm f\|_{W^{k,p}} \le C_p \|f\|_{W^{k,p}}, \tag{12.142}$$

where $f \in W^{k,p} \cap L^2$, $1 < p < \infty$.

Note, there are specific requirements on the potential V which allow Theorem 4 to be extended to the cases $p = 1$ and $p = \infty$, however we will not discuss them here. Also, the theorem holds for $V \in L^1_\gamma$ with $\gamma > \frac{3}{2}$ provided H has no resonance at the endpoint of the continuous spectrum.

An important property of wave operators is that for any Borel function f, we have

$$f(H)P_c = W_\pm f(H_0)W^*_\pm,\ f(H_0) = W^*_\pm f(H)P_c W_\pm. \tag{12.143}$$

Hence, we have

$$\|e^{iHt}P_c f\|_{L^p} = \|W_\pm e^{itH_0} W^*_\pm f\|_{L^p} \tag{12.144}$$

and using standard dispersive estimates for the linear Schrödinger operator (see for instance [**SS99**] for a concise overview) arrive at

$$\|e^{iHt}P_c f\|_{L^p} \le C_p t^{-(\frac{1}{2}-\frac{1}{p})} \|f\|_{W^{k,p}}. \tag{12.145}$$

Define a Strichartz pair (q,r) to be admissible if

strnum

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{r} \tag{12.146}$$

with $2 \le r < \infty$. Then, we arrive at the celebrated Strichartz estimates

eqn:strich1

$$\|e^{iHt}P_c u_0\|_{L^q W^{k,r}} \lesssim \|u_0\|_{W^{k,2}} \tag{12.147}$$

and, using duality techniques and once again the boundedness of the wave operators, we have

eqn:strich2

$$\|\int_0^t e^{iH(t-s)}P_c f\,ds\|_{L^q W^{k,r}} \lesssim \|f(x,t)\|_{L^{\tilde{q}'}_t W^{k,\tilde{r}'}_x}, \tag{12.148}$$

where (q,r) and $(\tilde{q},\tilde{r})$ satisfy (12.146).

As a side note, using positive commutators and well crafted local smoothing spaces, from [**?**], we have the full Strichartz estimate

eqn:str1

$$\|\int_0^t e^{iH(t-s)}P_c f\,ds\|_{L^\infty L^2} \lesssim \|f(x,t)\|_{L^{\tilde{p}'}_t L^{\tilde{q}'}_x}, \tag{12.149}$$

where $(\tilde{p},\tilde{q})$ is any allowable pair as in (12.146). Now, implementing the boundedness of wave operators on $W^{k,p}$ spaces from [**Wed99**], we have the following useful relation

$$\|\int_0^t e^{iH(t-s)}P_c f\,ds\|_{L^\infty H^1} \lesssim \|f(x,t)\|_{L_t^{\tilde{p}'}W_x^{1,\tilde{q}'}}, \tag{12.150}$$

where again $(\tilde{p},\tilde{q})$ is a Strichartz pair as in (12.146) without first going through the dispersive estimates.

Note, as mentioned in the introduction the discussion above may be extended to the case of V having delta function type singularities using formalism discussed in [**?**].

12.14. Exercises

sec:res

12.14.1. Commutator / Resolvent type bounds. In [**DW11**], where homogenization of high contrast oscillatory structures with defects is studied, bounds on $(H_0+1)^{-1}(H_{\tilde{q},\vec{y}}+1)$ are required to estimate a Lipmann Schwinger equation. We have, by our main theorem that

$$(H_0+1)^{-1}(H_{\tilde{q},\vec{y}}+1)P_c = (H_0+1)^{-1}W_+(H_0+1)W_+^*: L^2\to L^2.$$

strich

12.14.2. Dispersive and Strichartz estimates in H^1 for δ-Schrödinger. We may represent

$$e^{itH}P_c f = \frac{1}{2\pi}\int\int_0^\infty e^{-\frac{itk^2}{2}}\left(e_+(x,k)\,\overline{e_+(x,k)}+e_-(x,k)\,\overline{e_-(x,k)}\right)f(y)\,dk\,dy. \tag{12.151}$$

From here, we may use direct computations to arrive at Strichartz estimates and apply Weder's results on wave operators since the potentials are all in L^1 with compact support.

Using the properties of wave operators, we have

$$\|e^{iHt}P_c f\|_{L^p} = \|W_\pm e^{itH_0}W_\pm^* f\|_{L^p} \tag{12.152}$$

and using standard dispersive estimates for the linear Schrödinger operator (see for instance [**SS99**] for a concise overview) arrive at

$$\|e^{iHt}P_c f\|_{L^p} \le C_p t^{-(\frac{1}{2}-\frac{1}{p})}\|f\|_{W^{1,p}}. \tag{12.153}$$

Define a Strichartz pair (q,r) to be admissible if

$$\frac{2}{q} = \frac{1}{2}-\frac{1}{r} \tag{12.154}$$

with $2\le r<\infty$. Then, we arrive at the celebrated Strichartz estimates

eqn:strichdelta1

$$\|e^{iHt}P_c u_0\|_{L^qW^{1,r}} \lesssim \|u_0\|_{W^{1,2}} \tag{12.155}$$

and

eqn:strichdelta2

$$\left\|\int_0^t e^{iH(t-s)}P_c f\right\|_{L^qW^{1,r}} \lesssim \|f(x,t)\|_{L_t^{\tilde{q}}W_x^{1,\tilde{r}}} \tag{12.156}$$

using duality techniques and once again the boundedness of the wave operators.

As a side note, using positive commutators and well crafted local smoothing spaces, from [**MMT08**] we have the Strichartz estimate

eqn:str1

$$\left\|\int_0^t e^{iH(t-s)}P_c f\right\|_{L^\infty L^2} \lesssim \|f(x,t)\|_{L_t^{\tilde{p}}L_x^{\tilde{q}}}. \tag{12.157}$$

Now, by boundedness of wave operators on $W^{1,p}$ spaces for singular potentials as proved in Theorem 11.3, we have the following useful relation

$$\left\| \int_0^t e^{iH(t-s)} P_c f \right\|_{L^\infty H^1} \lesssim \| f(x,t) \|_{L_t^{\tilde{p}} W_x^{1,\tilde{q}}}, \tag{12.158}$$

where $(\tilde{p}, \tilde{q})$ is a dual Strichartz pair without first going through the dispersive estimates.

sec:lwp

12.14.3. Local Well-Posedness in H^1 for δ-NLS. Consider the nonlinear Schrödinger / Gross-Pitaevskii, with a potential consisting of a finite set of Dirac delta functions:

$$\begin{cases} i\partial_t u + H_{\vec{q},\vec{y}} u - |u|^{2\sigma} u = 0, \\ u(x,0) = u_0(x) \in H^1, \end{cases}$$

for $0 < \sigma < \infty$. We seek a solution in the following sense:

$$u = \Lambda[u],$$

where

nls-int

$$\Lambda[u](t) = e^{-iH_{\vec{q},\vec{y}} t} u_0 - i \int_0^t e^{-iH_{\vec{q},\vec{y}}(t-s)} |u|^{2\sigma} u(s) ds. \tag{12.159}$$

We claim that local well-posedness can be established via the contraction mapping principle in the space $C^0([0,T); H^1(\mathbb{R}))$ for T sufficiently small. To prove the necessary boundedness and contraction estimates, it is natural to apply the operator $(I + H_{\vec{q},\vec{y}})^{\frac{1}{2}} P_c$, which commutes with the group $e^{-iH_{\vec{q},\vec{y}} t}$ to (12.159). Then, estimates follow in a straightforward way, using that $H^1(\mathbb{R})$ is an algebra, provided the space

$$\mathscr{H}^1(\mathbb{R}) = \left\{ f : (I + H_{\vec{q},\vec{y}})^{\frac{1}{2}} P_c f \in L^2(\mathbb{R}) \right\} \tag{12.160}$$

is equivalent to the classical Sobolev space H^1. This follows from the relations

$$(I+H)^{\frac{1}{2}} P_c = W (I - \partial_x^2)^{\frac{1}{2}} W^*, \;\; W^* (I+H)^{\frac{1}{2}} W = (I - \partial_x^2)^{\frac{1}{2}}$$

and our results on the boundedness of wave operators associated with $H_{\vec{q},\vec{y}}$ on H^1.

sec:dwp

12.14.4. Long time dynamics for NLS with a double δ well potential. In [**MW10**], the long time dynamics of solutions to the nonlinear Schrödinger / Gross-Pitaevskii equation

eqn:nlsdwp-gK

$$i\partial_t u = (-\Delta + V(x)) u \; + \; gK\left[|u|^2\right] u, \tag{12.161}$$

where V is a symmetric, double well potential, are studied. In particular, under appropriate spectral assumptions on the operator $H = -\partial_x^2 + V(x)$, in a neighborhood of a symmetry breaking bifurcation point, there are different classes of oscillating solutions (12.161) which *shadow* periodic orbits of a finite dimensional reduction on very long, but finite, time scales. These solutions correspond to states with mass concentrations oscillating between the two wells of a symmetric potential well. The proof requires dispersive / Strichartz type estimates. The results of this paper imply that the results of [**MW10**] extend to (12.161) for the case of singular potentials, such as

$$V(x) = -q[\delta(x-L) + \delta(x+L)].$$

CHAPTER 13

Non-self-adjoint Operators

In this chapter, we analytically and numerically describe the spectrum on a linearized Hamiltonian resulting from a saturated NLS equation. It will become clear that while much is still unknown about the spectrum, in general, one can see that they are in fact admissible as defined below.

13.1. Spectral Properties of the Linearized Hamiltonian

spec:lin

To begin, let us write down the form of the linearized equation. First of all, let us assume we are looking for a solution $\psi = e^{i\lambda t}(R_\lambda + \phi(x,t))$. For simplicity, set $R = R_\lambda$. Inserting this into the equation we know that since ϕ is a soliton solution we have

$$i(\phi)_t + \Delta(\phi) = -\beta(R^2)\phi - 2\beta'(R^2)R^2\mathrm{Re}(\phi) + O(\phi^2), \tag{13.1}$$

by splitting ϕ up into its real and imaginary parts, then doing a Taylor Expansion. Hence, if $\phi = u + iv$, we get

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \mathscr{H}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{13.2}$$

where

$$\mathscr{H} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}, \tag{13.3}$$

where

$$L_- = -\Delta + \lambda - \beta(R_\lambda)$$

and

$$L_+ = -\Delta + \lambda - \beta(R_\lambda) - 2\beta'(R_\lambda^2)R_\lambda^2.$$

There are many things we can immediately say about L_-, L_+ and $\mathscr{H}$. For a reference on the spectral theory involved, see Hislop-Sigal [**HS96**] or Reed-Simon [**RS78b**]. First of all, both L_- and L_+ are self-adjoint operators. Also, L_- is a non-negative definite operator and its null space is span$\{R\}$. Note also that the functions $\frac{\partial R}{\partial x_j}$ for $j = 1, 2, ..., n$ are in the null space of L_+. By comparison with the operator $\Delta + \lambda$ and using the fact that R decays exponentially, we see that the essential spectrum of $\mathscr{H}$ is the set $(-\infty, \lambda] \cup [\lambda, \infty)$ from Weyl's Theorem, see for instance [**RS78b**]. Finally, using the fact that L_- is non-negative definite and looking at eigenvalues $\mathscr{H}^2$, we see

$$L_-L_+u = \nu^2 u. \tag{13.4}$$

However, this can be rewritten as

$$Tg = L_-^{\frac{1}{2}}L_+L_-^{\frac{1}{2}}g = \nu^2 g$$

for $g = L_-^{\frac{1}{2}}u$. Since the operator T is self-adjoint, we must have $\nu \in \mathbb{R} \cup i\mathbb{R}$.

We formalize this heuristic discussion with the following theorem from [**ES06**]. Before we begin, note that we also have

$$\mathscr{H} = \mathscr{H}_0 + V = \begin{bmatrix} -\Delta+\lambda^2 & 0 \\ 0 & \Delta+\lambda^2 \end{bmatrix} + \begin{bmatrix} -V_1 & -V_2 \\ V_2 & V_1 \end{bmatrix},$$

when we act on u and $\bar{u}$ instead of u_1 and u_2 where $u = u_1 + iu_2$.

Theorem 13.1 (Erdogan-Schlag)
Assume there are no embedded eigenvalues in the continuous spectrum of spec($\mathscr{H}$). The essential spectrum of $\mathscr{H}$ equals $(-\infty,-\lambda^2]\cup[\lambda^2,\infty)$. Moreover, spec($\mathscr{H}$) $= -$spec($\mathscr{H} = \overline{spec(\mathscr{H})} = spec(\mathscr{H}^*)$ and spec($\mathscr{H}$) $\subset \mathbb{R}\cup i\mathbb{R}$. The discrete spectrum consists of eigenvalues $\{z_j\}_{j=1}^N$, $0 \le N \le \infty$, of finite multiplicity. For each $z_j \neq 0$, the algebraic and geometric multiplicities coincide and Ran($\mathscr{H} - z_j$) is closed. The zero eigenvalue has finite multiplicity.

PROOF. Using standard techniques, $\mathscr{H}$ is closed on $W^{2,2}\times W^{2,2}$. Since $\mathscr{H}_0^* = \mathscr{H}_0$, we have spec$\mathscr{H}_0 \subset \mathbb{R}$. For $\mathfrak{Re}(z) \neq 0$, we have

eqn:es1
$$(\mathscr{H}_0 - z)^{-1} = -(\mathscr{H}_0 + z)\begin{bmatrix} (H_0^2 - z^2)^{-1} & 0 \\ 0 & (H_0^2 - z^2)^{-1} \end{bmatrix} \tag{13.5}$$

$$= -\begin{bmatrix} (H_0^2 - z^2)^{-1} & 0 \\ 0 & (H_0^2 - z^2)^{-1} \end{bmatrix}(\mathscr{H}_0 + z), \tag{13.6}$$

and

eqn:es2
$$(\mathscr{H} - z)^{-1} = (\mathscr{H}_0 - z)^{-1}[I - U_1[I - U_2J(\mathscr{H}_0 - z)^{-1}U_1]^{-1}U_2J(\mathscr{H}_0 - z)^{-1}U_1]^{-1}], \tag{13.7}$$

where

$$J = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},\ U_1 = \begin{bmatrix} |V_1|^{\frac{1}{2}} & 0 \\ 0 & |V_2|^{\frac{1}{2}} \end{bmatrix},\ U_2 = \begin{bmatrix} |V_1|^{\frac{1}{2}} sign(V_1) & 0 \\ 0 & |V_2|^{\frac{1}{2}} sign(V_2) \end{bmatrix}.$$

and $W_1 = V_1 - V_2$ and $W_2 = V_1 + V_2$. From (13.5), we see that $\sigma_{ess}\mathscr{H}_0 = (-\infty,-\lambda^2]\cup[\lambda^2,\infty)$ since the resolvent can be written in terms of $(H_0^2 - z^2)$. Then, using Weyl's Theorem from [**RS78b**], since we have sufficient decay on V_1 and V_2, $\mathscr{H}$ is relatively compact to $\mathscr{H}_0$, so $\sigma_{ess}(\mathscr{H}) = \sigma_{ess}\mathscr{H}_0$. By analytic Fredholm theory, we have that $(\mathscr{H}-z)^{-1}$ is meromorphic on $\mathbb{C}\setminus(-\infty,-\lambda^2]\cup[\lambda^2,\infty)$. Further, the poles of the resolvent are eigenvalues of finite multiplicity and Ran($\mathscr{H} - z_j$) is closed at each pole z_j.

For the matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},\ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

we have $\sigma_j\mathscr{H}\sigma_j = \mathscr{H}$ and $\sigma_j^2 = I$ for $j = 1,2,3$. Hence, the symmetries of the spectrum arise naturally by looking at

$$\sigma_j\mathscr{H}\sigma_j u = u.$$

Let us now look at $\mathscr{H}u = Eu$ for $E \neq 0$. This gives that $u_1,\ u_2 \neq 0$ and $u_1 \perp \mathrm{Ker}(L_-)$. Hence, for $g = L_-^{\frac{1}{2}}u_1$, we get

$$L_-^{\frac{1}{2}}L_+L_-^{\frac{1}{2}}g = Eg.$$

CLAIM 13.1.1. *The operator $L_-^{\frac{1}{2}}L_+L_-^{\frac{1}{2}} : H^4(\mathbb{R}^d) \to H^4(\mathbb{R}^d)$ is self-adjoint.*

PROOF. Suppose that $\langle L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} f, g\rangle = \langle f, h\rangle$ for all $f \in H^4$ and fixed g, $h \in L^2$. Let $f \in \mathrm{Ker}(L_-)$, then we have $h \in \{\mathrm{Ker}(L_-)\}^\perp)$. Applying the Fredholm alternative to $L_-^{\frac{1}{2}}$, we have $h = L_-^{\frac{1}{2}} h_1$ for some $h_1 \in D(L_-^{\frac{1}{2}}) = H^1$. Since h_1 is defined only modulo $\mathrm{Ker}(L_-)$, we have

$$\langle L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} f, g\rangle = \langle f, L_-^{\frac{1}{2}}(h_1 + c\phi)\rangle = \langle L_-^{\frac{1}{2}} f, h_1 + c\phi\rangle$$

for any $c \in \mathbb{C}$ and all $f \in H^4$. Set $f_1 = L_-^{\frac{1}{2}} f$, then

$$\langle L_-^{\frac{1}{2}} L_+ f_1, g\rangle = \langle f_1, h_1 + c\phi\rangle.$$

Note that we have $f_1 \in H^3$ and $f_1 \perp \phi$. In order to remove the latter restriction, take c to be such that

$$\langle L_-^{\frac{1}{2}} L_+ \phi, g\rangle = \langle \phi, h_1 + c\phi\rangle,$$

then

$$\langle L_-^{\frac{1}{2}} L_+ \phi, g\rangle = \langle f_1 + \lambda\phi, h_1 + c\phi\rangle.$$

Taking now $h_2 = h_1 + c\phi$, we have

$$\langle L_-^{\frac{1}{2}} L_+ f_1, g\rangle = \langle f_1, h_2\rangle$$

for all $f_1 \in H^3$. Since this now implies that $h_1 \perp \mathrm{Ker}(L_+)$, we have $h_1 = L_+(h_2 + \sum_{j=1}^d c_j\phi_j)$, $h_2 \in H^3$. Once again, choose the c_j's such that

$$\langle L_-^{\frac{1}{2}}(L_+ f_1 + \sum_{j=1}^d \lambda_j\phi_j), g\rangle = \langle L_+ f_1 + \sum_{j=1}^d \lambda_j\phi_j, h_2 + c\sum_{j=1}^d c_j\phi_j\rangle$$

for all λ_j. Finally, we have

$$\langle L_-^{\frac{1}{2}} f_2, g\rangle = \langle f_2, h_2\rangle$$

for all $f_2 \in H^1$ where $h = L_-^{\frac{1}{2}} L_+ h_2$, $h_2 \in H^3$. Since L_- is self-adjoint, $h_2 = L_-^{\frac{1}{2}} g$. Hence, $g \in H^4$ and $h = L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$. □

Let us discuss the generalized null space of a Hamiltonian resulting from linearizing about a minimal mass soliton. From the discussion of the spectrhm of $\mathscr{H}$, we know that the vectors

$$\begin{bmatrix} 0 \\ R \end{bmatrix}, \begin{bmatrix} R_j \\ 0 \end{bmatrix},$$

for all $j = 1, \ldots, d$ are contained in $Ker(\mathscr{H})$. Now, as $Q(R_\lambda)$ is differentiable with respect to λ, we have by a simple calculation that $L_+\partial_\lambda R = -R$ and $L_-(x\phi) = -2\nabla R$. Hence, the vectors

$$\begin{bmatrix} 0 \\ x_j R \end{bmatrix}, \begin{bmatrix} (\partial_\lambda R)_{\lambda_0} \\ 0 \end{bmatrix}$$

in the generalized null space of order 2. Notice that so far we have constructed at $2d+2$ dimensional null space. Since we know the null spaces of L_- and L_+ exactly, these are unique. For the higher dimensional null spaces, we would have solutions of form

eqn:orth1 (13.8) $$L_- L_+ L_- v = 0,$$

eqn:orth2 (13.9) $$L_+ L_- L_+ u = 0.$$

From Equation (13.9), we would have $L_-L_+u = a\cdot\nabla R$ or $L_+u = Ca\cdot xR$, where $a\in\mathbb{R}^d$, $C\in\mathbb{R}$. Hence, we use an orthogonality argument by taking the inner product of (13.9) with u. Namely, we have

$$\langle u, 0\rangle = 0,$$

but

$$\langle a\cdot\nabla R, Ca\cdot xR\rangle \neq 0.$$

For simplicity, let us say that the minimal mass occurs at 1 and say $R_1 = R$. Since we are at a minimal mass soliton, we have the chance to actually solve Equation (13.8) since

$$(\partial_\lambda Q)(\lambda_0) = 0,$$

or

$$\langle \partial_\lambda R, R\rangle = 0.$$

Hence, if we look at the equation

$$L_-\alpha = (\partial_\lambda\phi),$$

there exists a solution α. Note that we can choose α to be spherically symmetric. To move one step higher, we have the equations

eqn:orth3 (13.10) $$L_+L_-L_+L_-v = 0,$$

eqn:orth4 (13.11) $$L_-L_+L_-L_+u = 0.$$

Now, for Equation (13.10), we would have

$$L_-L_+L_-v = a\cdot\nabla\phi.$$

Then, we would have

$$L_+L_-v = \tilde{a}\cdot xR.$$

Hence, no such solution can exist by orthogonality. For Equation (13.11), we have

$$L_+u = \alpha.$$

Since the kernel of L_+ consists only of non-sperically symmetric functions, again there exists a β such that $L_+\beta = \alpha$. Using a comparable technique, it is easy to see that the generalized null space ends at dimension 4.

As a result, we have the following theorem

Theorem 13.2
There exists a $2d+4$ dimensional null space for $\mathscr{H}$ consisting of the span of the vectors

$$\left\{\begin{bmatrix}0\\R\end{bmatrix},\begin{bmatrix}R_j\\0\end{bmatrix},\begin{bmatrix}0\\x_jR\end{bmatrix},\begin{bmatrix}(\partial_\lambda R)_{\lambda_0}\\0\end{bmatrix},\begin{bmatrix}0\\\alpha\end{bmatrix},\begin{bmatrix}\beta\\0\end{bmatrix}\right\}.$$

The generalized null space of the adjoint can be found by reversing the location of the non-zero elements in the above vectors.

Suppose that there exists a generalized eigenspace for the eigenvalue $E\neq 0$. Then, there exists $\chi\neq 0$ and $\psi\neq 0$ such that $(\mathscr{H}-E)\psi = \chi$ and $(\mathscr{H}-E)\chi = 0$. Then, note that

$$\begin{aligned}(\mathscr{H}^2-E^2)\psi &= (A+E)\chi = 2E\chi,\\ (\mathscr{H}^2-E^2)\chi &= 0.\end{aligned}$$

Hence, $\mathscr{H}^2$ has a generalized eigenspace at E^2. As a result, we see that $T = L_+L_-$ has a generalized eigenspace at E^2. Let $T\chi = E^2\chi$ and $(T-E^2)\psi = c\chi$, for some $c \neq 0$. Hence,

$$\begin{aligned}(L_-^{\frac{1}{2}}L_+L_-^{\frac{1}{2}} - E^2)L_-^{\frac{1}{2}}\psi_1 &= c\chi_1,\\ (L_-^{\frac{1}{2}}L_+L_-^{\frac{1}{2}} - E^2)^2L_-^{\frac{1}{2}}\psi_1 &= cL_-^{\frac{1}{2}}(L_+L_- - E^2)\chi_1 = 0,\end{aligned}$$

where given $P_R^c = I - P_R$, we have $\chi_1 = P_R^c\chi \neq 0$ since $T\chi = E^2\chi$ and $\psi_1 = P_R^c\psi \neq 0$ since $(T-E^2)\psi = c\chi$. However, this means that the self-adjoint operator $L_-^{\frac{1}{2}}L_+L_-^{\frac{1}{2}}$ has a generalized eigenvalue, which is impossible by an orthogonality argument.

Since we have assumed there are no eigenvalues at the endpoints of the continuous spectrum, there can be no accumulation and the number of discrete eigenvalues is finite. □

13.2. Large Eigenvalues

spec:eig

We seek to prove the absence of large L^2 eigenvalues imbedded in the continuous spectrum for Hamiltonian operators that result from linearizing a Nonlinear Schrödinger Equation about a soliton, ϕ. The operator is a non-selfadjoint matrix operator whose entries are second order differential operators. For simplicity, we look at the fourth order equation comes from squaring the operator. Therefore, we look for eigenvalues $u \in L^2$ of the differential operator:

eqn:lplm1

$$L_-L_+u = \mu^4 u, \tag{13.12}$$

where

$$L_- = -\Delta + \lambda - f(x)$$

and

$$L_+ = -\Delta + \lambda - g(x).$$

where μ is in the continuous spectrum for L_-L_+, or $\mu \in (\lambda^2, \infty)$. The dispersive estimates that result from the matrix operator depend heavily on resolvent estimates which heavily rely on spectral properties. Hence, it is necessary to fully understand the spectral properties of such an operator in order to determine linear stability for solutions to the original NLS equation. From the properties of the soliton and the nonlinearity, we have that any solution to Equation 13.12 is locally smooth by using an iteration argument. Asymptotic analysis gives exponential decay as has been discussed above in Chapter (REFERENCE). As a result, we have that an analysis of the possible range of frequencies limits us to smooth solutions that are bounded in frequency by μ or

spec:eqfreq

$$\|\nabla u\|_{L^2} \leq \mu \|u\|_{L^2}. \tag{13.13}$$

See [**Tat00**], [**Ste93a**] for references on microlocal analysis. Hence, we have the following theorem:

Theorem 13.3
There exists a $\mu_0 > \lambda$ such that for all $\mu \geq \mu_0$, the eigenvalue equation (13.12) *has only the trivial zero solution in L^2.*

PROOF. Define the standard Mourre commutator to be:

$$M = x\cdot\nabla + \nabla\cdot x.$$

From the equation itself and the fact that M is skew-adjoint and L_-, L_+ are self-adjoint, we have the following two identities:

$$\begin{aligned}\langle [ML_-L_+ &- L_+L_-M]u,u\rangle = 0,\\ \langle [L_-L_+ &- \mu^4]u,u\rangle = 0,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the standard L^2 inner product.

By combining these two equations and using the frequency bound in (13.13), we see that for μ large enough, L^2 solutions are not possible. We have:

$$\begin{aligned}\langle (\Delta^2 - 2\lambda\Delta &+ \Delta(g(x)) + f(x)\Delta - \lambda^2(f+g) + (\lambda^4-\mu^4) + fg)u,u\rangle = 0, \ (*)\\ ML_-L_+ &- L_+L_-M = [M,L_-]L_+ + L_-[M,L_+] + [L_-,L_+]M,\\ [M,-\Delta] &= 4\Delta,\ [M,f] = 2x\cdot\nabla f.\end{aligned}$$

Hence,

$$\begin{aligned}\langle (-8\Delta^2 &+ 8\lambda^2\Delta - 4(f+g)\Delta + 2(x\cdot\nabla f + x\cdot\nabla g)\Delta\\ &- 2\lambda^2(x\cdot\nabla f + x\cdot\nabla g) + 2(x\cdot\nabla f)g + 2(x\cdot\nabla g)f \ (**)\\ &- [(f-g)\Delta - \Delta(f-g)][d + 2x\cdot\nabla])u,u\rangle = 0.\end{aligned}$$

Note that the last term is a product of skew-adjoint operators and hence, they commute in the current setting. Therefore, by looking at $4(*) + (**)$ and using the fact that $\|\nabla u\|_{L^2} \le \mu^2\|u\|_{L^2}$ from the frequency bound dictated by the operator, we have for $\mu > \mu_0$ a negative definite system and hence no eigenvalues. Note that by this estimate and standard Sobolev embeddings, we have $u \in H^k$ for any k and hence, u is smooth.

The system we deal with is:

$$\begin{aligned}\int [-4(\Delta u)^2 &- 4\lambda^2(f+g)u^2 - 4(\mu^4-\lambda^4)u^2] + [4fg - 2\lambda^2(x\cdot\nabla f + x\cdot\nabla g)\\ &+ 2(x\cdot\nabla f)g + 2(x\cdot\nabla g)f + \Delta(x\cdot\nabla f + x\cdot\nabla g) - x\cdot\nabla(\Delta f - \Delta g)\\ &- d(\Delta f - \Delta g)]u^2 + [4(x\cdot\nabla u)(\nabla(f-g)\cdot\nabla u)\\ &- 2(x\cdot\nabla f + x\cdot\nabla g)\nabla u\cdot\nabla u]\ dx = 0.\end{aligned}$$

Hence,

$$\begin{aligned}\int (-4(\Delta u)^2 &- 4\lambda^2(f+g)u^2)dx - 4(\mu^4-\lambda^4)\|u\|_{L^2}^2 + \|F\|_{L^\infty}\|u\|_{L^2}^2\\ &+ (2\|x\cdot\nabla f + x\cdot\nabla g\|_{L^\infty} + C_d\max_{j,k}\|\partial_j(f-g)x_k\|)\|\nabla u\|_{L^2}^2\\ &\le \int (-4(\Delta u)^2 - 4\lambda^2(f+g)u^2)dx - 4(\mu^4 - \lambda^4 - C_1 - C_2\mu^2)\|u\|_{L^2}^2,\end{aligned}$$

where

$$\begin{aligned}F &= 4fg - 2\lambda^2(x\cdot\nabla f + x\cdot\nabla g) + 2(x\cdot\nabla f)g\\ &+ 2(x\cdot\nabla g)f + \Delta(x\cdot\nabla f + x\cdot\nabla g)\\ &- x\cdot\nabla(\Delta f - \Delta g) - d(\Delta f - \Delta g),\end{aligned}$$

and $C_j = C_j(f,g,\lambda,d)$ for $j = 1,2$. Hence, for μ large, we have that:

$$\int (-4(\Delta u)^2 - 4\lambda^2(f+g)u^2 - C_3(\mu,\lambda,f,g)u^2)\ dx \le 0,$$

for $C_3 > 0$ and u a smooth function, hence $u = 0$.

□

13.3. Spherical Harmonics

spec:sphhar

Since our potential functions are radially symmetric, we expand our solution into a sum of radially symmetric functions multiplied by eigenfunctions on the $(d-1)$-sphere. Note, this method only applies for dimension $d \geq 2$. Using separation of variables, we have that the spherical harmonic eigenfunctions have the form

$$\sum_k u_k(r)\phi_k(\theta),$$

where

$$\Delta_S \phi_k(\theta) = (k^2 + (d-1)d)\phi_k(\theta).$$

See [**Tay97b**] for a description of the eigenspaces of the spherical Laplacian, Δ_S. Then, we have the following ODE eigenvalue problem:

eqn:lplm2

$$\left[\left(-\frac{\partial^2}{\partial r} - \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{\alpha^2}{r^2} + \lambda^2 - f(r)\right)\times\right. \tag{13.14}$$

$$\left.\left(-\frac{\partial^2}{\partial r} - \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{\alpha^2}{r^2} + \lambda^2 - g(r)\right) - \mu^4\right] u(r) = 0, \tag{13.15}$$

where $\alpha^2 = k^2 + (d-1)d$ for $k = 0, 1, 2, \ldots$.

From this framework, we have following theorem:

Theorem 13.4
There exists some $\alpha_0 > 0$ such that for all $\alpha \geq \alpha_0$, the eigenvalue equation (13.14) *has only the trivial zero solution in L^2.*

PROOF. So, for the radial inner product defined by:

$$\langle u, v\rangle_r = \int uvr^{d-1}dr,$$

we define the operators:

$$\begin{aligned}\Delta_r &= r^{1-d}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\right),\\ P_r &= d + 2r\frac{\partial}{\partial r}.\end{aligned}$$

Using the same commutator approach from Section 13.2, we have:

$$\begin{aligned}\langle[4\Delta_r &- \frac{4\alpha^2}{r^2} - 2rf'(r)][-\Delta_r + \frac{\alpha^2}{r^2} + \lambda^2 - g(r)]\\ &+ [-\Delta_r + \frac{\alpha^2}{r^2} + \lambda^2 - f(r)][4\Delta_r - \frac{4\alpha^2}{r^2} - 2rg'(r)]\\ &+ [(f-g)\Delta_r - \Delta_r(f-g)][d + 2r\frac{\partial}{\partial r}]u, u\rangle_r = 0.\end{aligned}$$

Thus, we have

$$\begin{aligned}\langle -8(\Delta_r)^2 u &+ 16\Delta_r(\frac{\alpha^2}{r^2}u) + 8\lambda^2\Delta_r u - 4(f+g)\Delta_r u - \frac{8\alpha^4}{r^4}u - \frac{8\lambda^2\alpha^2}{r^2}u \\ &+ \frac{4\alpha^2}{r^2}(f+g)u + 2(rf'(r)+rg'(r))\Delta_r u - 2\frac{\alpha^2}{r}(f'(r)+g'(r))u \\ &- 2\lambda^2 r(f'(r)+g'(r))u + 2rf'(r)g(r)u + 2rf(r)g'(r)u \\ &+ (d+2r\frac{\partial}{\partial r})((f-g)\Delta_r - \Delta_r(f-g))u, u\rangle_r = 0.\end{aligned}$$

This implies:

$$\begin{aligned}\int [-8(\Delta_r u)^2 &- (16\frac{\alpha^2}{r^2} + 8\lambda^2)(u_r)^2 - (\frac{8\alpha^4}{r^4} + \frac{8\lambda^2\alpha^2}{r^2})]u^2 r^{d-1}dr \\ &+ \int [4(f+g) + 2rf_r - 6rg_r](u_r)^2 r^{d-1}dr \\ &+ \int [\frac{4\alpha^2}{r^2}(f+g) - (d-4)\frac{\alpha^2}{r^4} - \frac{d(d-1)+2\alpha^2}{r}f_r \\ &+ \frac{(d-1)(d-2)-2\alpha^2}{r}(g_r) - (d-2)f_{rr} + 3dg_{rr} + 2rg_{rrr} \\ &+ 2r(g_r f + fg_r) - 2\lambda^2 r(f_r + g_r)]u^2 r^{d-1}dr \leq 0.\end{aligned}$$

Note that we have integrated by parts several times above. To justify this, u must be 0 at $r = 0$ to sufficiently high order to compensate for the singular terms. However, spherical harmonics result from eigenvalues of the spherical Laplacian. These take the values

$$\nu_k = k^2 + (d-2)k,$$

and for each k, the eigenfunctions (and hence the spherical harmonics) are traces of harmonic polynomials of degree k. Hence, in order to give a smooth solution as guaranteed above, u_k must be 0 of order k at the origin. Hence, for $k \geq \max 0, 5-d$, the behavior of u is sufficient to allow the above calculations to be rigorous. Since all of the eigenvalues are non-negative, we have used the notation α^2 for simplicity and will continue to do so. For a brief note, see [**Tay97b**] for a discussion of spherical harmonics. See [**Tay97b**], Chapter 8 for a detailed description of eigenfunctions for the Laplacian on the sphere.

REMARK 13.3.1. *Note that in the Section 1, we had to take advantage both of the frequency bounds on the solution as well as the operator equation itself in order to take advantage of the large negative parameter. However, due to the nature of the operator, the large negative parameter appeared in the commutator and all other terms could be viewed as lower order. Regularity is not an issue in the above analysis by an argument involving resolvents which is a variant of that for standard Schrödinger operators.*

In the commutator expression, the parameter that must dominate is α^4. Since f, g are smooth, exponentially decaying functions by assumption, all functions involving f, g and derivatives thereof are nicely bounded at 0 and exponentially decaying. Hence, for $0 \leq r \leq 1$, all of the functions above are easily controlled by $\frac{\alpha^4}{r^4}$ for α large. Similarly, for $r > R$, R large, the exponential decay of f, g and their derivatives imply that any function above is dominated again on this region by $\frac{\alpha^4}{r^4}$ for α large. In the intermediate region, again using the smoothness of the potential functions, we can find α^4 large

enough to bound the lower order terms. In order to determine α exactly, a careful analysis must be done involving all of the potential functions, but there is certainly an α_0 such that for all $\alpha \geq \alpha_0$, the operator when conjugated by the radial Mourre operator gives a negative definite system. □

REMARK 13.3.2. *Ideally, the potentials will be non-trivial only on a very small region, then the R and hence the α can be taken fairly small in all regions. There will definitely balancing that must be done in each special case. Hence, any embedded eigenvalue can be expressed purely as a finite sum of spherical harmonics with radial coefficients. Combining this with the result in Section 13.2 gives a limited number of calculations to do in order to determine whether or not an operator has embedded eigenvalues.*

spec:embres

13.4. Absence of Embedded Resonances

This result is developed in the earlier work of Erdogan-Schlag [**ES06**] and Agmon [**Agm75b**]. We present the argument here for the sake of completeness. Let

$$H_0 = -\Delta + \lambda,$$

$$\mathscr{H}_0 = \begin{bmatrix} H_0 & 0 \\ 0 & -H_0 \end{bmatrix},$$

and

$$V = \begin{bmatrix} -V_1 & -V_2 \\ V_2 & V_1 \end{bmatrix}.$$

Also, define the space

$$X_\sigma = L^{2,\sigma} \times L^{2,\sigma},$$

where

$$L^{2,\sigma} = \{f \,|\, |x|^\sigma f \in L^2\}.$$

Theorem 13.5
Let V_1, V_2 have sufficient decay at ∞. Then for any μ such that $|\mu| > \lambda^2$, $(\mathscr{H}_0 - (\mu \pm i0))^{-1} V : X_{-\frac{1}{2}-} \to X_{-\frac{1}{2}-}$ is a compact operator, and

$$I + (\mathscr{H}_0 - (\mu \pm i0))^{-1} V$$

is invertible on these spaces.

PROOF. The compactness follows from an argument similar to the one below for the square of $\mathscr{H}$. We refer the reader to this proof for the main idea. From Fredholm theory, we see that we must exclude solutions $\Psi = (\psi_1, \psi_2) \in X_{-\frac{1}{2}-}$ such that

$$[I + (\mathscr{H}_0 - (\mu \pm i0))^{-1} V]\Psi = 0,$$

of from standard results about the spectral theory of Schrödinger equations

spec:eqemres

$$0 = \psi_1 - R_0(\mu - \lambda + i0)(V_1\psi_1 + V_2\psi_2) \tag{13.16}$$

$$0 = \psi_2 - R_0(-\mu - \lambda)(V_2\psi_1 + V_1\psi_2), \tag{13.17}$$

where $R_0(z) = (-\Delta - z)^{-1}$ is the standard resolvent from the linear Schrödinger operator. Using the decay of V_j, $j = 1,2$ and the fact that $R_0(-\mu - \lambda) : L^2 \to L^2$, we see immediately that $\psi_2 \in L^2$.

By taking the inner products of (13.16) with

$$V_1\psi_1 + V_2\psi_2,\ V_2\psi_1, \text{ and } V_1\psi_2,$$

then combining necessary terms, we see

$$\mathrm{Im}\langle R_0(\mu-\lambda+i0)V_1\psi_1 + V_2\psi_2, V_1\psi_1 + V_2\psi_2\rangle = 0.$$

LEMMA 13.4.1 (Agmon). *Let $\lambda > 0$ and f sufficiently regular. Then,*

$$Im\langle R_0(\lambda \pm i0)f, f\rangle = \pm\frac{\pi}{2\sqrt{\lambda}}\int_{|\xi|=\sqrt{\lambda}} |(\tau\hat{f})(\xi)|^2 d\sigma,$$

where $\tau\hat{f}$ denotes the trace of $\hat{f}$ on the sphere $|\xi| = \sqrt{\lambda}$.

This is similar to a restriction theorem from harmonic analysis. The proof follows from a calculation using the the specific structure of R_0. This allows us to use the following bootstrapping theorem, also proved in Agmon [**Agm75b**].

Theorem 13.6 (Agmon)
Let $f \in H^s(\mathbb{R}^d)$ for some $s > \frac{1}{2}$. Suppose that $f(x) = 0$ for all x such that $|x| = k$ in the trace sense, and let $K^{-1} \le k \le K$, $K > 1$. For any multi-index α with $0 \le |\alpha| \le 2$, set

$$v_\alpha(x) = \frac{x^\alpha f(x)}{|x|^2 - k^2}.$$

Then, $v_\alpha \in H^s \cap L^1_{loc}$ and

$$\|v_\alpha\|_{H^{s-1}} \le C\|f\|_{H^s}.$$

From the Agmon bootstrapping theorem we have $\hat{\psi}_1 \in L^2$, hence $\psi_1 \in L^2$. Thus, any imbedded resonance is actually an embedded eigenvalue, which we have ruled out by assumption.

□

lin:dfb

13.5. Distorted Fourier Basis

Note that in the sequel, we take the convention that the soliton parameter is λ^2 instead of λ. This serves to remind the reader of the positivity of this parameter. The convention of λ slightly simplifies the variational formulation, but has no impact on the linear analysis presented in this Chapter.

We seek to understand the functions in the continuous spectrum of $\mathscr{H}$ by decomposing them using a Distorted Fourier Basis given by

eqn:dfbe

$$(-\Delta + \lambda^2 - V_1)(-\Delta + \lambda^2 - V_2)u_{\xi_0} = (\lambda^2 + |\xi_0|^2)^2 u_{\xi_0}, \tag{13.18}$$

where $u_{\xi_0} = e^{ix\xi_0} + g_{\xi_0}$ and g_{ξ_0} is yet to be determined.

From (13.18),

$$[(-\Delta+\lambda^2)^2 - (\lambda^2+|\xi_0|^2)^2]u_{\xi_0} \quad = \quad (-\Delta+\lambda^2)V_2 u_{\xi_0} + V_1(-\Delta+\lambda^2-V_2)u_{\xi_0}.$$

Hence,

eqn:dfbeq

$$[(-\Delta+\lambda^2)^2 - (\lambda^2+|\xi_0|^2)^2]g_{\xi_0} \quad = \quad F(x;\xi_0) + (-\Delta+\lambda^2)V_2 g_{\xi_0} + V_1(-\Delta+\lambda^2-V_2)g_{\xi_0} \tag{13.19}$$

$$\quad = \quad F_{\xi_0}(x)e^{ix\xi_0} + \tilde{V}(x,D)g_{\xi_0}, \tag{13.20}$$

where $F_{\xi_0}(x)$ is a Schwartz function. Then, taking the Fourier Transform, we have

$$[(|\xi|^2+\lambda^2)^2 - (|\xi_0|^2+\lambda^2)^2]\hat{g}_{\xi_0} = \hat{F}(\xi;\xi_0) + (\tilde{V}_{\mathscr{F}}\hat{g}_{\xi_0})(\xi),$$

where

$$\begin{aligned}(\tilde{V}_{\mathscr{F}} g)(\xi) &= \lambda^2(\hat{V}_2+\hat{V}_1)*(g)+(|\xi|^2\hat{V}_2)*(g)+(\hat{V}_2+\hat{V}_1)*(|\xi|^2 g)\\ &+ (\xi\hat{V}_2)*(\xi g)-(\widehat{V_1V_2})*(g).\end{aligned}$$

Given

$$\begin{aligned}L_{\xi_0} &= [(|\xi|^2+\lambda^2)^2-(|\xi_0|^2+\lambda^2)^2]\\ &= [(|\xi|+|\xi_0|)(|\xi|-|\xi_0|)(|\xi|^2+2\lambda^2+|\xi_0|^2)],\end{aligned}$$

we have

$$\begin{aligned}g_{\xi_0} &= \mathscr{F}^{-1}\left\{\{L^{\pm}_{\xi_0}\}^{-1}(\hat{F}+\tilde{V}_{\mathscr{F}}\hat{g}_{\xi_0})\right\}\\ &= K^{\pm}_{\xi_0}*F_{\xi_0}+K^{\pm}_{\xi_0}*(\tilde{V}(x,D)g_{\xi_0}),\end{aligned}$$

where

$$K^{\pm}_{\xi_0}(x)=(\mathscr{F}^{-1}\{L^{\pm}_{\xi_0}\}^{-1})(x)$$

and

$$L^{\pm}_{\xi_0}=[(|\xi|+|\xi_0|\pm i0)(|\xi|-|\xi_0|\mp i0)(|\xi|^2+2\lambda^2+|\xi_0|^2)].$$

Note that for simplicity we have omitted a small complex perturbation in the elliptic term $(|\xi|^2+2\lambda^2+|\xi_0|^2)$ since it does not effect the analysis.

To explore $K^{\pm}_{\xi_0}$ further, we see in $\mathbb{R}^3$

$$\int_{\xi}\frac{e^{i\xi\cdot x}}{(|\xi|+|\xi_0|\pm i0)(|\xi|-|\xi_0|\mp i0)(|\xi|^2+2\lambda^2+|\xi_0|^2)}d\xi=$$
$$\int_{\mathbb{R}^3}\frac{e^{i\xi_1|x|}}{(|\xi|+|\xi_0|\pm i0)(|\xi|-|\xi_0|\mp i0)(|\xi|+|\xi_0|)(|\xi|^2+2\lambda^2+|\xi_0|^2)}d\xi,$$

using the change of variables $\xi_1\rightarrow\frac{x}{|x|}$. Then, we have

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{e^{r\cos(\theta)|x|}}{(r+|\xi_0|\pm i0)(r-|\xi_0|\mp i0)(r^2+2\lambda^2+|\xi_0|^2)}r\sin(\theta)drd\theta d\phi.$$

Doing integration first in θ, then a contour integral, we have that

eqn:K

$$K^{\pm}_{\xi_0}=\hat{L^{-1}_{\xi_0}}=\frac{\pi^2}{|\xi_0|^2+\lambda^2}\left[\frac{e^{\pm i|x||\xi_0|}-e^{-|x|\sqrt{|\xi_0|^2+2\lambda^2}}}{|x|}\right]. \tag{13.21}$$

For simplicity, we take $K(x)=K^{+}_{\xi_0}(x)$ as the analysis for $K^{-}_{\xi_0}$ will be similar. Then, we want to use an iterative argument to show that for mid to high range frequencies, these distorted Fourier bases exist in L^4. It will become clear in the sequel why L^4 is chosen. Note that since near 0, K is bounded, we have $K\in L^{3+s}$ for any $s>0$. In particular we show the following:

lem:K

LEMMA 13.5.1. *For the operator $K^{\pm}$ defined in Equation* (13.21)*, we have*

$$K^{\pm}:L^{\frac{4}{3}}\rightarrow L^4\ (O((|\xi_0|^2+\lambda^2)^{-1}|\xi_0|^{-\frac{1}{2}})).$$

PROOF. We actually prove the result for

$$\tilde{K}^{a}_{\xi_0}(x) = \mathscr{F}^{-1}(\frac{1}{(|\xi|^2-(|\xi_0|+i0)^2)^a}).$$

The proof for K will be essentially the same.

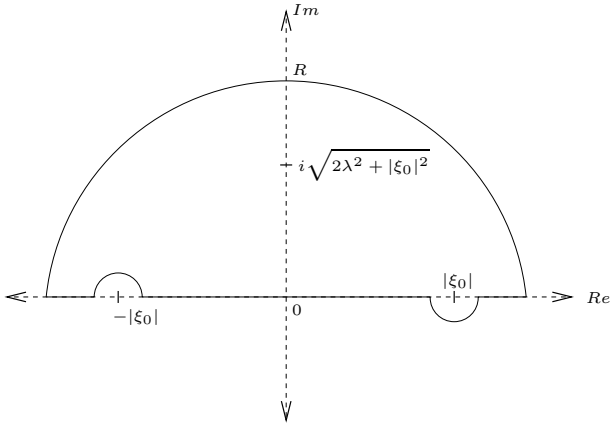

FIGURE 1. The contour for computing the behavior of the fundamental solution in the limiting case. fig:fund

Using distribution theory, we have

$$\begin{aligned}\tilde{K}^0_{\xi_0}(x) &= \delta(x),\\ \tilde{K}^1_{\xi_0}(x) &= \frac{4\pi^2}{|x|}[e^{i|x||\xi_0|}],\\ \tilde{K}^2_{\xi_0}(x) &= \frac{i2\pi^2 e^{i|x||\xi_0|}}{|\xi_0|}.\end{aligned}$$

As convolution operators,

$$\tilde{K}^0 : L^2 \to L^2\ (O(1))$$

and

$$\tilde{K}^2 : L^1 \to L^\infty\ (O(|\xi_0|^{-1})),$$

hence using complex interpolation

$$\tilde{K}^1 : L^{\frac{4}{3}} \to L^4\ (O(|\xi_0|^{-\frac{1}{2}})).$$

□

For simplicity, we from now on write $\tilde{K}$ instead of $\tilde{K}^1_{\xi_0}$. Now, we seek to analyze the equation

eqn:dfbu (13.22) $$g_{\xi_0} = K^\pm_{\xi_0} * F_{\xi_0} + K^\pm_{\xi_0} * (\tilde{V}(x,D)g_{\xi_0}),$$

In particular, we have the following:

thm:dfbu **Theorem 13.7** *Let $P(x,D)$ be a differential operator of the form*

$$P(x,D) = (-\Delta + \lambda^2 - V_1)(-\Delta + \lambda^2 - V_2),$$

where V_1, $V_2 \in \mathscr{S}$. Assuming that there are no eigenvalues embedded in the continuous spectrum $[\lambda^4, \infty)$, there exists $g^\pm_{\xi_0} \in L^4$ such that Equation (13.22) *is satisfied for $u_{\xi_0} = e^{ix\xi_0} + g^\pm_{\xi_0}(x)$. We have*

$$g^\pm_{\xi_0}(x) = K^\pm * [f_0(\cdot, \xi_0, |\xi_0|)],$$

where f_0 is smooth in x, ξ_0 and $|\xi_0|$,

$$|\langle x\rangle^N \partial_x^\alpha f_0| \lesssim 1.$$

Moreover, there exists a value M such that for $\xi_0 \geq M$,

$$f_0(x,\xi_0) = e^{i(x,\xi_0)} f(x,\xi_0),$$

where

eqn:symb

$$|\langle x\rangle^N \partial_{\xi_0}^\alpha \partial_x^\beta f(x,\xi_0)| \lesssim |\xi_0|^{2-|\alpha|}, \tag{13.23}$$

for any multi-indices α and β, $N > 0$.

PROOF. The solution to (13.22) will be solved differently for large and small values of ξ_0. In particular, we use a Fredholm theory approach for the small frequencies and an iterative approach for the large frequencies.

To begin, let us take $|\xi_0| > M$, where M will be determined in the exposition. Then, we solve Equation (13.22) using Picard iteration. For simplicity, let $g_{\xi_0} = v$. Setting $v^0 = 0$ and $Tu = \tilde{V}(K * u)$, we have

$$\begin{aligned}
v^1 &= K(x) * [F_{\xi_0}(x) e^{ix\cdot\xi_0}] \\
v^2 &= K(x) * [(F_{\xi_0}(x) e^{ix\cdot\xi_0}) + ((\tilde{V}(x,D)K(x) * (F_{\xi_0}(x) e^{ix\cdot\xi_0})) \\
&= K(x) * [(F_{\xi_0}(x) e^{ix\cdot\xi_0}) - (V_1+V_2)(\lambda^2 + |\xi_0|^2)K(x) * (F_{\xi_0}(x) e^{ix\cdot\xi_0}) \\
&+ (V_1+V_2)\tilde{K}(x) * (F_{\xi_0}(x) e^{ix\cdot\xi_0}) - (\nabla V_2 \cdot \nabla K(x) * (F_{\xi_0}(x) e^{ix\cdot\xi_0})) \\
&- (V_1(x)V_2(x) + \Delta V_2)K(x) * (F_{\xi_0}(x) e^{ix\cdot\xi_0})] \\
&\vdots \\
v^n &= K(x) * [F_{\xi_0}(x) e^{ix\cdot\xi_0} + \tilde{V}(x,D) v^{n-1}] \\
&= K(x) * [\sum_{m=0}^{n-1} T^m F_{\xi_0}(x) e^{ix\cdot\xi_0}] \\
&\vdots \quad .
\end{aligned}$$

We wish to show that this iteration converges in L^4. To see this, let $u \in L^4$. Note that

$$\|K * \tilde{V}(x,D)u\|_{L^4} \lesssim \|K * Vu\|_{L^4} + \|\nabla K * \bar{V}u\|_{L^4} + \|\Delta K * \bar{\bar{V}}u\|_{L^4},$$

where V, $\bar{V}$, $\bar{\bar{V}} \in \mathscr{S}$. Then,

$$\begin{aligned}
\|K * \tilde{V}(x,D)u\|_{L^4} &\lesssim \frac{1}{\xi_0^2+\lambda^2}\frac{1}{|\xi_0|^{\frac{1}{2}}}\|Vu\|_{L^{\frac{4}{3}}} \\
&+ \frac{|\xi_0|}{\xi_0^2+\lambda^2}\|(|y|^{-1}) * \bar{V}u\|_{L^4} + \frac{\xi_0^2}{\xi_0^2+\lambda^2}\|K * \bar{\bar{V}}u\|_{L^4},
\end{aligned}$$

so using the Hardy-Littlewood-Sobolev inequality and the bounds on K, we have

$$\begin{aligned}
\|K * \tilde{V}(x,D)u\|_{L^4} &\lesssim |\xi_0|^{-\frac{1}{2}}\|Vu\|_{L^{\frac{4}{3}}} \\
&\lesssim |\xi_0|^{-\frac{1}{2}}\|V\|_{L^2}\|u\|_{L^4},
\end{aligned}$$

for some $V \in \mathscr{S}$. As a result,

$$\|K * \tilde{V}(x,D)\|_{L^4 \to L^4} \leq C|\xi_0|^{-\frac{1}{2}},$$

where C is determined by V_1, V_2. If $|\xi_0| > C^2$, then

$$\|K * \tilde{V}(x,D)\|_{L^4 \to L^4} \leq 1,$$

and the existence of $g_\xi \in L^4$ for

$$(I - K * \tilde{V}(x,D)) g_\xi = g_\xi$$

follows from a contraction argument. In the notation from the theorem, we have $C^2 = M$.

Now, for the smaller frequencies, we wish to use Fredholm theory. This approach also works for large $|\xi_0|$, however the iterative approach gives us uniform bounds for all ξ_0 such that $|\xi_0| > M$. Once differentiability in ξ_0 has been obtained, we will then have uniform bounds for all ξ_0. However, we must be particularly careful near $\xi_0 = 0$ as K has a particularly challenging dependence upon $|\xi_0|$. We explore this shortly, but first let us finish the existence argument for low frequencies.

To begin, Equation (13.22) shows that

eqn:dfbg

$$g_{\xi_0} = K * (\tilde{V}(x,\xi_0) e^{ix\cdot\xi_0}) + K * (\tilde{V}(x,D) g_{\xi_0}), \tag{13.24}$$

where

$$\tilde{V}(x,D) = (-\Delta + \lambda^2 - V_1) V_2 + V_1 (-\Delta + \lambda^2)$$

is a second order operator.

Now, if $K * (\tilde{V}(x,D)\cdot)$ is a compact operator, we may use Fredholm Theory to say that either there is a unique solution to (13.24) or there exists a nontrivial $u \in L^4$ such that

$$(I - K * \tilde{V}) u = 0.$$

However, expanding the equation for u, we see this u is an embedded resonance and hence an embedded eigenvalue from Section 13.4. As our spectral assumptions preclude the existence of embedded eigenvalues, the solution to (13.24) is unique.

Let us now discuss the compactness. The operator itself is of the form

$$\begin{aligned} K * (\tilde{V} v) &= \int \pi^2 \frac{[e^{i|x-y||\xi_0|} - e^{-|x-y|\sqrt{|\xi_0|^2+2\lambda^2}}]}{|x-y|(|\xi_0|^2+\lambda^2)} \tilde{V}(y,D_y) v(y)\, dy \\ &= \int \pi^2 \frac{[e^{i|x-y||\xi_0|} - e^{-|x-y|\sqrt{|\xi_0|^2+2\lambda^2}}]}{|x-y|(|\xi_0|^2+\lambda^2)} \\ &\times \quad [(-\Delta_y + \lambda^2 - V_1(y)) V_2(y) + V_1(y)(-\Delta_y + \lambda^2)] v(y)\, dy. \end{aligned}$$

Hence, using integration by parts, we are concerned about the following two types of operators

$$\begin{aligned} (1)\ P_1 u &= \int K(x-y) V(y) u(y)\, dy \\ (2)\ P_2 u &= \int \tilde{K}(x-y) V(y) u(y)\, dy, \end{aligned}$$

where $V \in \mathscr{S}$. Of course, technically there will be terms with derivatives falling on K and V, however a brief calculation shows that these fall into the same class of operators as P_2. Note, to see this it is useful to remember that by construction

$$(-\Delta - |\xi_0|^2)\tilde{K} = 0$$

and

$$(-\Delta - |\xi_0|^2)K = \frac{4\pi^2}{|x|}[e^{-|x|\sqrt{|\xi_0|^2+2\lambda^2}}],$$

hence when all derivatives fall on K, simply by looking at $-\Delta - |\xi_0|^2 + |\xi_0|^2$, we get reduction back to P_1 or P_2 since K is a convolution kernel for an exact solution.

We now need to prove

$$P_i : L^4 \to L^4,$$

for $i = 1,2$.

Assume that $u_j \to^w 0$ in L^4. Since we are working in $\mathbb{R}^3$, using duality and the properties of V, we have

$$P_i u_j(x) \to 0 \text{ as } j \to \infty$$

for almost every x, where $i = 1,2$. By the uniform boundedness of weakly convergent sequences, the Hardy-Littlewood-Sobolev Inequality, and Hölder we have,

$$\begin{aligned} \|P_i u_j\|_{L^4} &\leq \|V\|_{L^{\frac{3}{2}}} \|u_j\|_{L^4} \\ &\leq C, \end{aligned}$$

for $i = 1,2$. Hence, there is a subsequence j_k such that $\|P_i u_{j_k}\|_{L^4}$ converges. Therefore, it must converge to 0. As a result, the operator $K * (\tilde{V}\cdot) : L^4 \to L^4$ is compact and there exists a unique g_{ξ_0} for all ξ_0. Note that $\tilde{V}K$ is compact from $L^{\frac{4}{3}} \to L^{\frac{4}{3}}$ using similar arguments.

To discuss the continuous dependence upon ξ_0, we need to study the functions g_{ξ_0} in more detail. From the expression for g_{ξ_0}, we know that

$$\begin{aligned} (I - K * (\tilde{V}_{\xi_0}\cdot))g_{\xi_0} &= (I - P_{\xi_0})g_{\xi_0} \\ &= K * \tilde{V}_{\xi_0} e^{ix\xi_0}, \end{aligned}$$

so

$$g_{\xi_0} = (I - P_{\xi_0})^{-1}(K * \tilde{V}_{\xi_0} e^{ix\xi_0}),$$

where

$$K(\xi) = [(-\Delta - \xi^2)(-\Delta + 2\lambda^2 + \xi^2)]^{-1}.$$

From Fredholm Theory and the spectral assumptions, $(I - P_{\xi_0})^{-1}$ is a resolvent which is uniquely defined. However, using the decay of $\tilde{V}$, we can write

$$\tilde{V} = \tilde{V}_1 \tilde{V}_2,$$

where $|e^{c|x|}\tilde{V}_1| \lesssim 1$, $|e^{c|x|}\tilde{V}_2 f| \lesssim \|f\|_{L^{2,\infty}}$ given $0 < c < c_0$. The constant c_0 is determined by the decay of $\tilde{V}$. Hence, we have

$$g_{\xi_0} = \tilde{V}_1(I - \tilde{V}_2 K \tilde{V}_1)^{-1}(\tilde{V}_2 e^{i(\cdot,\xi_0)}).$$

Using the decay properties of $\tilde{V}_i$ for $i = 1,2$ and the differentiability of K, for any ξ_0 we have $\tilde{V}_2 K \tilde{V}_1(z)$ is well-defined for $z \in \mathbb{C}$ in a small neighborhood of $|\xi_0|$. As a result,

$$(I - \tilde{V}_2 K \tilde{V}_1)^{-1}$$

is analytic with respect to z. Also, K is analytic with respect to $|\xi|$, $V_\xi e^{ix\xi}$ is analytic with respect to ξ and we see that g_{ξ_0} depends smoothly on $|\xi|$ and ξ. Using the resolvent identity

$$f_0(x,\xi) = \tilde{V} e^{ix\cdot\xi} + \tilde{V}(1 - K\tilde{V})^{-1} K * (\tilde{V} e^{ix\cdot\xi}),$$

the decay in x for f_0 follows.

For $|\xi_0| \geq M$, let us return to the iteration scheme

$$\begin{aligned} g^0_{\xi_0} &= K * [\tilde{V}(\cdot,\xi_0)e^{i(\cdot,\xi_0)}], \\ &\vdots \\ g^n_{\xi_0} &= K * [\tilde{V}(\cdot,\xi_0)e^{i(\cdot,\xi_0)} + \tilde{V}(\cdot,\xi_0)g^{n-1}_{\xi_0}], \end{aligned}$$

for $n \geq 1$. Assuming $g_\xi = e^{ix\cdot\xi_0} f_0(x,\xi_0,|\xi_0|)$, we have

$$f_0 = \tilde{V}(x,\xi_0) + e^{-ix\xi_0}\tilde{V}K * (e^{ix\xi_0} f_0),$$

where by the mapping properties of K, choosing M large enough, this expression is valid in $L^{\frac{4}{3}}_x$ for all $|\xi_0| \geq M$.

We would like to better understand the regularity in x and ξ. To begin, let $u = K * [e^{i(\cdot,\xi_0)}\phi(\cdot,\xi_0)]$. Then, we see

$$\begin{aligned} (\partial_x - i\xi_0)u(x) &= (\partial_x - i\xi_0)(K * [\phi(\cdot,\xi_0)e^{i(\cdot,\xi_0)}])(x) \\ &= i\xi_0 \int K(y)e^{i(x-y)\xi_0}\phi(x-y,\xi_0)dy - i\xi_0 \int K(y)e^{i(x-y)\xi_0}\phi(x-y,\xi_0)dy \\ &+ \int K(y)e^{i(x-y)\xi_0}\phi_x(x-y,\xi_0)dy \\ &= \int K(y)e^{i(x-y)\xi_0}\phi_x(x-y,\xi_0)dy. \end{aligned}$$

From here, recognizing that $e^{-ix\xi_0}$ cancels from

$$e^{-ix\xi_0}\tilde{V}K * (e^{ix\xi_0}\cdot)$$

and again using the mapping properties of K, we have

$$\|\partial^\alpha_x f_0\|_{L^{\frac{4}{3}}_x} \leq C_\alpha,$$

for all multi-indices α. Hence, $f_0 \in C^\infty_x \cap L^\infty_x$. Similarly,

$$\|\langle x\rangle^N \partial^\alpha_x f_0\|_{L^{\frac{4}{3}}_x} \leq C_{N,\alpha},$$

for any $N \geq 0$ using the decay in x of the operator $\tilde{V}$.

For the regularity in ξ, note that taking once again $u = K * [e^{i(\cdot,\xi_0)}\phi(\cdot,\xi_0)]$, we have

$$\begin{aligned}
(\partial_{\xi_0} - ix)u &= (\partial_{\xi_0} - ix)(K * [\phi(\cdot,\xi_0)e^{i(\cdot,\xi_0)}])(x) \\
&= \frac{4\pi^2}{(\xi_0^2+\lambda^2)}\left(i\frac{\xi_0}{|\xi_0|}\right)\int e^{i|x-y||\xi_0|}e^{i(y)\xi_0}\phi(y,\xi_0)dy \\
&+ i\frac{\xi_0}{\sqrt{\xi_0^2+2\lambda^2}}\int e^{-|x-y|\sqrt{\xi_0^2+2\lambda^2}}e^{i(y)\xi_0}\phi(y,\xi_0)dy \\
&- ix\int K(y)e^{i(x-y)\xi_0}\phi(x-y)dy + i\int K(y)e^{i(y)\xi_0}y\phi(y,\xi_0)dy \\
&+ \int K(y)e^{i(x-y)\xi_0}\phi(x-y,\xi_0)dy \\
&= \left(\frac{1}{\sqrt{\xi_0^2+2\lambda^2}} - \frac{1}{|\xi_0|}\right)\int K(y)e^{iy\xi_0}y\phi(y,\xi_0)dy \\
&+ \int K(y)e^{i(x-y)\xi_0}\phi_{\xi_0}(x-y,\xi_0)dy,
\end{aligned}$$

where we have used $i\xi_0 e^{iy\xi_0} = \partial_y e^{iy\xi_0}$ and integrated by parts. As a result,

$$\|\partial_{\xi_0}^{\beta} f_0\|_{L^{\frac{4}{3}}} \leq |\xi_0|^{2-|\beta|}C_\beta,$$

for any multi-index β, $|\beta| = 0,1,2,\ldots$. Combining the above results, we have

$$|\partial_\xi^\alpha \partial_x^\beta f_0(x,\xi)| \leq C_{\alpha,\beta}|\xi|^{2-|\alpha|},$$

or $f_0 \in S^2$, which gives (13.23).

For the spatial regularity result, we once again use that the distorted Fourier basis satisfies the equation

$$g_{\xi_0} = K * (Fe^{ix\cdot\xi_0}) + K * (\tilde{V}g_{\xi_0}).$$

We have existence for g_{ξ_0} in L^4, but we can take advantage of the structure of $K * P$ in order to show improved regularity. We have

$$\nabla g_{\xi_0} = (\nabla K) * (Fe^{ix\cdot\xi_0}) + (\nabla K) * (\tilde{V}g_{\xi_0}).$$

Hence, we must explore the nature of $(\nabla K) * (\tilde{V})$. Upon differentiating, we see

$$(\nabla K)(x-y) = O(|x-y|^{-1}),$$

which means by a similar approach to Section 13.5, we get

$$\|\nabla g_{\xi_0}\|_{L^4} \leq C(\|F\|_{L^{\frac{12}{11}}} + \|V\|_{L^{\frac{3}{2}}}\|g_{\xi_0}\|_{L^4}).$$

To see this, we first use the Hardy-Littlewood-Sobolev inequality with $\gamma = 1$ so

$$\frac{1}{p} = \frac{2}{3} + \frac{1}{4} = \frac{11}{12},$$

then Hölders inequality such that

$$\|Vg\|_{L^{\frac{12}{11}}} \leq \|V\|_{L^{\frac{3}{2}}}\|g\|_{L^4}.$$

Then, we can iterate this for all derivatives and using Sobolev embeddings, get continuity of all derivatives and hence smoothness.

To prove existence for $\partial_{\xi_0} g_{\xi_0}$ in Sobolev spaces, we must show that $\partial_{\xi_0} g_{\xi_0}$ is defined and bounded in some space of functions. In this direction, we look at

$$[(-\Delta + 2\lambda^2 + \xi_0^2)(-\Delta - \xi_0^2)] g_{\xi_0} \quad = \quad F_{\xi_0} e^{ix\xi_0} + \tilde{V} g_{\xi_0}$$

and

$$[(-\Delta + 2\lambda^2 + (\xi_0 + h_j)^2)(-\Delta - (\xi_0 + h_j)^2)] g_{\xi_0 + h_j} \quad = \quad F_{\xi_0 + h_j} e^{ix(\xi_0 + h_j)} + \tilde{V} g_{\xi_0 + h_j},$$

where $h_j = he_j$ and e_j is the unit vector in the j-th coordinate. Hence, if we define

$$v_h = g_{\xi_0 + h_j} - g_{\xi_0},$$

then we must solve

$$\begin{aligned} L_{\xi_0}(v_h) &= (F_{\xi_0 + h_j} e^{ix\cdot(\xi_0 + h_j)} - F_{\xi_0} e^{ix\xi_0}) + O(h) u_{\xi_0} + \tilde{V}(v_h) \\ &= O(h)(\tilde{F}_{\xi_0} + F_{\xi_0} + K * \tilde{V} u_{\xi_0}) + \tilde{V}(v_h). \end{aligned}$$

We can write this as

$$L_{\xi_0}[v_h - O(h) K * (K * (\tilde{V} g_{\xi_0}))] = O(h)(G) + \tilde{V}[v_h - O(h) K * (K * (\tilde{V} g_{\xi_0}))],$$

where we have

$$G = \tilde{F}_{\xi_0} + F_{\xi_0} - \tilde{V} K * (K * (\tilde{V} g_{\xi_0})).$$

To see that $G \in L^4$, we need only see that

$$\| \tilde{V} K * (K * (\tilde{V} g_{\xi_0})) \|_{L^4} < \infty$$

since the other terms are dealt with above in the spatial regularity analysis. However, we have

$$K * (K * \cdot) : L^1 \to L^\infty,$$

following analysis similar to the complex interpolation argument. Also, by moving all of the derivatives onto Pu, we see this is smooth. All we lack is nice decay, hence

$$\| \tilde{V} K * (K * (\tilde{V} g_{\xi_0})) \|_{L^4} < \| K * (K * (\tilde{V} g_{\xi_0})) \|_{L^\infty} \| V \|_{L^4},$$

for $V \in \mathscr{S}$ as given in the description of P. Hence, from the Fredholm Theory, we know

$$\| \frac{v_h}{h} - O(1) K * (K * (\tilde{V} g_{\xi_0})) \|_{L^4} \leq C,$$

for $C = C(\xi_0)$. However, given $w \in C_0^\infty \cup L^4$ a sufficiently decaying, smooth function, we have

$$\begin{aligned} \| w \frac{v_h}{h} \|_{L^4} &\leq C(1 + \| w K * (K * (\tilde{V} g_{\xi_0})) \|_{L^4}) \\ &\leq C, \end{aligned}$$

from Section 13.5, where C is independent of h. In this case, we have

$$K * (K * (\tilde{V} g_{\xi_0})) \in L^\infty$$

using Hölder's inequality, so we can take $w = \langle x \rangle^{-1}$. Thus, we can take the limit as $h \to 0$ to see that derivatives in ξ_0 are bounded in weighted L^4 spaces. Iterating this process involves taking stonger weight functions at each step of the iteration.

Hence, since $\tilde{V}$ has exponentially decaying terms in x and $\tilde{V} g_{\xi_0}$ is well-defined in L^4 from the spatial regularity, we have the desired regularity in ξ_0.

Now that we have differentiability with respect to ξ_0,

$$\partial_{(\xi_0)_j} \left([(-\Delta + 2\lambda^2 + \xi_0^2)(-\Delta - \xi_0^2)] g_{\xi_0} = F e^{ix\xi_0} + \tilde{V} g_{\xi_0} \right)$$

which implies

$$\begin{aligned} L_{\xi_0}\partial_{(\xi_0)_j} g_{\xi_0} &= \partial_{(\xi_0)_j}(Fe^{ix\xi_0}) + P\partial_{(\xi_0)_j} g_{\xi_0} \\ &- 2(\xi_0)_j(-\Delta - \xi_0^2) g_{\xi_0} - (\xi_0)(-\Delta + 2\lambda^2 + \xi_0^2) g_{\xi_0}. \end{aligned}$$

For higher derivatives in ξ_0, we iterate this procedure.

□

REMARK 13.5.2. *Note that the above analysis can also be done in the case where instead of L^4 we use $L^2(\langle x\rangle^{-s})$ as in [**Agm75b**]. To see this, note that*

$$\|\phi\|_{L^1} \lesssim \|\phi\|_{L^2(\langle x\rangle^s)},$$

where $s > d$, and

$$\|\phi\|_{L^2(\langle x\rangle^{-s})} \lesssim \|\phi\|_{L^\infty},$$

where $s > d$. Then, we can go to the Sobolev norms to apply Hardy-Littlewood-Sobolev and use Hölder's inequality in weighted spaces and the boundedness of V_1 and V_2 in weighted L^2 spaces to complete the argument.

rem:dfbasy

REMARK 13.5.3. *As $x \to \infty$, note that since V_1, $V_2 \in \mathscr{S}$, using Equation* (13.19)*, we have*

$$u_{\xi_0} \to \frac{\pi^2}{|\xi_0|^2 + \lambda^2}\left[\frac{e^{\pm i|x||\xi_0|} - e^{-|x|\sqrt{|\xi_0|^2+2\lambda^2}}}{|x|}\right],$$

*which explains the choice of spaces $L^{2,s}$ for $x > \frac{1}{2}$ in [**Agm75b**].*

13.6. Representation of the solution

lin:rep

We present here a slightly different approach to the distorted Fourier transform.

thm:rep

Theorem 13.8 *For $V \in \mathscr{S}$, there exists a distorted Fourier basis $\tilde{\phi}_\xi$ and correspondingly a distorted Fourier transform $\mathscr{G}$ for the nonselfadjoint operator $\mathscr{H}$, where*

$$\mathscr{G}_\pm f = \int \tilde{\phi}_\xi^\pm(x) f(x)\, dx.$$

Similarly, there exists an inverse Fourier basis $\tilde{\phi}_\xi^{-1}(x)$ and correspondingly an inverse Fourier transform $\mathscr{G}^{-1}$ for the nonselfadjoint operator $\mathscr{H}$, where

$$\mathscr{G}_\pm^{-1} f = \int \{\tilde{\phi}_\xi^\pm\}^{-1}(x) f(\xi)\, d\xi.$$

It follows that

$$\begin{aligned} \|\mathscr{G}_\pm\|_{L^2\to L^2} &\lesssim 1, \\ \|\mathscr{G}_\pm^{-1}\|_{L^2\to L^2} &\lesssim 1. \end{aligned}$$

These operators are not unitary, however

$$\|\mathscr{G}_\pm^{-1}\mathscr{G}\|_{L^2\to L^2} \lesssim 1$$

and

$$\mathscr{G}_\pm^{-1}\mathscr{G}_\pm \phi = P_c \phi.$$

Before we prove the theorem, look at the operator

$$\mathscr{H}^2 = \begin{bmatrix} L_-L_+ & 0 \\ 0 & L_+L_- \end{bmatrix},$$

for which we have the following self-adjoint realization

$$\tilde{\mathscr{H}} = \begin{bmatrix} L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} \end{bmatrix}.$$

Since

$$\begin{aligned} L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} &= (-\Delta + \lambda^2 - V_1)^{\frac{1}{2}} (-\Delta + \lambda^2 - V_1 - V_2)(-\Delta + \lambda^2 - V_1)^{\frac{1}{2}} \\ &= (-\Delta + \lambda^2 - V_1)^2 - (-\Delta + \lambda^2 - V_1)^{\frac{1}{2}} V_2 (-\Delta + \lambda^2 - V_1)^{\frac{1}{2}} \\ &= L_-^2 - L_-^{\frac{1}{2}} V_2 L_-^{\frac{1}{2}}. \end{aligned}$$

This is a 4th order constant coefficient operator with a lower order perturbation. However, the perturbation is no longer a differential operator. Ideally, by a similar analysis to that in [**Agm75b**], there exists a distorted Fourier basis, say $\tilde{u}_\xi$ such that

$$L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} \tilde{u}_\xi = (\lambda^2 + \xi^2)^2 \tilde{u}_\xi.$$

To prove this, we need to show $L_-^{\frac{1}{2}}$ is a pseudodifferential operator of strong enough class, which we explore in the sequel.

From Theorem 13.7, we have $u_\xi = e^{ix\xi} + f_\xi(x)$, $v_\xi = e^{ix\xi} + g_\xi(x)$ such that

$$\mathscr{H}^2 \begin{bmatrix} u_\xi \\ v_\xi \end{bmatrix} = (\lambda^2 + \xi^2)^2 \begin{bmatrix} u_\xi \\ v_\xi \end{bmatrix},$$

where $f_\xi(x)$, $g_\xi(x) \in L^4_x$, smooth in x and ξ, and

$$f_\xi, g_\xi \sim \frac{\pi^2}{|\xi_0|^2 + \lambda^2} \left[\frac{e^{\pm i|x||\xi_0|} - e^{-|x|\sqrt{|\xi_0|^2 + 2\lambda^2}}}{|x|} \right]$$

as $x \to \infty$.

Formally, we would like to say

$$\begin{bmatrix} L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} L_-^{-\frac{1}{2}} u_\xi \\ L_-^{\frac{1}{2}} v_\xi \end{bmatrix} = (\lambda^2 + \xi^2)^2 \begin{bmatrix} L_-^{-\frac{1}{2}} u_\xi \\ L_-^{\frac{1}{2}} v_\xi \end{bmatrix},$$

however as u_ξ, $v_\xi \notin L^2$, we must investige further.

Before we begin, let us analyze the connection between u_ξ and v_ξ. For instance,

$$\begin{aligned} L_+(L_-L_+u_\xi) &= L_+L_-(L_+u_\xi) \\ &= L_+(\lambda^2 + \xi^2)^2 u_\xi, \\ L_-(L_+L_-v_\xi) &= L_-L_+(L_-v_\xi) \\ &= L_-(\lambda^2 + \xi^2)^2 v_\xi. \end{aligned}$$

Hence

$$L_+ u_\xi = C v_\xi$$

and

$$L_- v_\xi = C u_\xi.$$

In particular, we are interested in

$$\begin{aligned} L_- v_\xi &= (-\Delta + \lambda^2 - V_1)(e^{ix\xi} + g_\xi) \\ &= (\xi^2 + \lambda^2)e^{ix\xi} + L_- g_\xi - V_1 e^{ix\xi}, \\ Cu_\xi &= C(e^{ix\xi} + f_\xi). \end{aligned}$$

Then, $C = (\lambda^2 + \xi^2)$, so

$$L_- v_\xi = (\lambda^2 + \xi^2) u_\xi,$$

$$L_-^{-1} u_\xi = (\lambda^2 + \xi^2)^{-1} v_\xi,$$

and

$$f_\xi = \frac{1}{\lambda^2 + \xi^2}(L_- g_\xi - V_1 e^{ix\xi}).$$

A similar calculation holds for $L_+ u_\xi = C v_\xi$.

Note also that if we look at the vector

$$\vec{\phi}_\xi = \begin{bmatrix} i u_\xi \\ v_\xi \end{bmatrix},$$

then we have

$$\mathscr{H}\vec{\phi}_\xi = (\lambda^2 + \xi^2)\vec{\phi}_\xi.$$

To be more precise, we say that the operator $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$ has a distorted Fourier basis given by $\tilde{u}_\xi$, then find an expression for the distorted Fourier transform of $\mathscr{H}P_c$. This distorted Fourier transform will be defined via a distorted Fourier basis that will give the relationship between $\tilde{u}_\xi$, u_ξ and v_ξ. The existence of $\tilde{u}_\xi$ must be proved since there is a lower order PDO perturbation instead of a differential operator. See [**Hör05**].

To prove $L_-^{\pm\frac{1}{2}}$ is a PDO, we must use a result similar to one from [**HÖ3b**], Chapter 29. To this end, we refer to the following theorem given in [**HÖ3b**]:

Theorem 13.9
Let X be a compact manifold, Ψ a space of pseudo-differential operators and $\Omega^{\frac{1}{2}}$ be the space of half-densities on X. Let $P \in \Psi^m_{phg}(X;\Omega^{\frac{1}{2}},\Omega^{\frac{1}{2}})$ be a positive, elliptic, symmetric operator. Then, P defines a positive, self-adjoint operator $\mathscr{P}$ in $L^2(X,\Omega^{\frac{1}{2}})$. If $m > 0$ and $a \in \mathbb{R}$, then $\mathscr{P}^a$ is also defined by a pseudodifferential operator in $\Psi^{am}_{phg}(X;\Omega^{\frac{1}{2}},\Omega^{\frac{1}{2}})$, with principal and subprincipal symbols p^a and $ap^{a-1}p^s$ if p and p^s are those for P.

We seek to prove a slightly different version here:

Theorem 13.10
Let P be a positive, symmetric, self-adjoint operator in $\Psi^{m,(2)}_{\rho,\delta}(\mathbb{R}^d)$. Then, P defines a postive, self-adjoint operator $\mathscr{P}$ in $L^2(\mathbb{R}^d,\mathbb{R}^d)$. If $m > 0$ and $a \in \mathbb{R}$, then $\mathscr{P}^a$ is also defined by a pseudodifferential operator in $\Psi^{am,(2)}_{\rho,\delta}(\mathbb{R}^d,\mathbb{R}^d)$, with principal and subprincipal symbols p^a and $ap^{a-1}p^s$ if p and p^s are those for P.

Note that since $R \in \mathscr{S}$, $F(R) \in \mathscr{S}$ by the properties of the nonlinearity. Hence, we have the following:

thm1:lemsr **LEMMA 13.6.1** *The perturbation V_1 is short-range.*

We seek to prove that given the operator,

$$L_- = -\Delta + \lambda^2 - V_1 \in S^2,$$

the new operator L_-^a is a pseudodifferential operator for $a \in \mathbb{R}$.

LEMMA 13.6.2
For an operator P, the resolvent $R(z) = (P-z)^{-1}$ exists and is analytic for all z except the eigenvalues of P. Also, $\|R(z)\|_{L^2 \to L^2}$ is bounded by the inverse of the distance from z to the nearest eigenvalue.

PROOF. This follows from basic facts from spectral theory, see [**HS96**]. □

thm:rep1 **Theorem 13.11** *The operator L_-^a is pseudodifferential operator in the class S^{2a} for $a \in \mathbb{R}$.*

Before we prove the theorem, let us prove the following lemma from [**Hö3a**].

LEMMA 13.6.3
Let $a \in S^m$. If

eqn:par (13.25) $$|a(x,\xi)| > c|\xi|^m$$

for $|\xi| > C$, then there exists $b \in S^{-m}$ such that

$$(i)\ a(x,\xi)b(x,\xi) - 1 \in S^{-1},$$
$$(ii)\ a(x,D)b(x,D) - I \in OpS^{-\infty},$$

and

$$(iii)\ b(x,D)a(x,D) - I \in OpS^{-\infty}.$$

PROOF OF LEMMA. First, let us prove that (13.25) implies (i). We can reduce this to the case where $m = 0$ by looking at $a(x,\xi)(1+|\xi|^2)^{-m/2}$ and $b(x,\xi)(1+|\xi|^2)^{m/2}$.

CLAIM 13.6.4. *If $a_1, a_2 \in S^0$ and $F \in C^\infty(\mathbb{C}^2)$, then $F(a_1,a_2) \in S^0$.*

PROOF. Since the $\mathfrak{Re}\, a_j$, $\mathfrak{Im}\, a_j \in S^0$ for $j = 1,2$, we may assume that a_j is real and $F \in C^\infty(\mathbb{R}^2)$. Then,

$$\begin{aligned} \frac{\partial F(a)}{\partial x_j} &= \sum_k \frac{\partial F}{\partial a_k} \partial_{x_j} a_k, \\ \frac{\partial F(a)}{\partial \xi_j} &= \sum_k \frac{\partial F}{\partial a_k} \partial_{\xi_j} a_k, \end{aligned}$$

where $\partial_{x_j} a_k \in S^0$, $\partial_{\xi_j} a_k \in S^{-1}$. Hence, it is clear the derivatives of $F(a)$ decay as necessary for $F(a)$ to be in S^0. □

Hence, for $m = 0$, choose $F \in C^\infty$ so that $F(z) = \frac{1}{z}$ for $|z| > c$. Set $b = F(a) \in S^0$ so $a(x,\xi)b(x,\xi) = 1$ for $|\xi| > C$. This proves (i).

Using (i), we have that

$$a(x,D)b(x,D) = I - r(x,D),\ r \in S^{-1}.$$

Set

$$b(x,D)r(x,D)^k = b_k(x,D),\ b_k \in S^{-m-k},$$

so we can iterate out the error. Let b' be the asymptotic sum of the b_k's, so

$$a(x,D)b'(x,D) - I = a(x,D)(b'(x,D) - \sum_{j<k} b_j(x,D)) - r(x,D)^k \in OpS^{-k},$$

for every k. Then, we have (ii) replacing b with b'. Similarly, we can find a b'' which satisfies (iii). Note also that

$$b' - b'' = b'(I - ab') + (b''a - I)b',$$

hence b' and b'' are equivalent modulo $S^{-\infty}$. □

PROOF OF THEOREM 13.11. Since L_- is self-adjoint, we have that $R(z)$ is defined and analytic for all z except at the eigenvalues of L_-. The L^2 norm of the resolvent can be estimated by the inverse of the distance to the set of eigenvalues. Now, since $a < 0$, we have by the spectral theorem

$$\tilde{L}^a u = -(2\pi i)^{-1} \int_{-i\infty}^{i\infty} z^a R(z) u dz,$$

where the contour is slightly deformed near the origin to avoid $z = 0$ and z^a is analytic in the right half plane and equal to 1 when $z = 1$. Since $L_-^{a+1} = L_-^1 L_-^a$, the distribution kernel of L_-^a is an entire analytic function of a.

To understand the behavior of the singularities, we construct a parametrix. Namely, since $|L_-(x,\xi)| > c|\xi|^2$ for $|\xi| > C$, we have the existence of an inverse modulo S^{-1}. Then, we can iterate that error, to find an inverse modulo $S^{-\infty}$.

In particular, we have B_z such that

$$(P - z)B_z = I - Q_z,$$

where $b_z = F(P(x,\xi) - z)$, $F(z) \sim 1/|z|$ for z large and $Q_z \in Op(S^{-1})$. Then, there is an E_z given by the asymptotic sum

$$\sum_{j=0}^{\infty} B_z(x,D)(Q_z(x,D))^j,$$

such that

$$(P - z)E_z = I - W_z,$$

where $W_z \in Op(S^{-\infty})$. So, we have

$$R(z) = E_z + R(z)W_z.$$

Then, for $a < 0$, we have

$$\tilde{L}^a = -(2\pi i)^{-1} \int_{-i\infty}^{i\infty} z^a E_z dz + T(a)u.$$

Here, $T(a)$ should be analytic in a for $a < 1$. In particular, this remainder will be a well-behaved pseudo-differential operator using Beals' Theorem. From the composition of pseudodifferential operators, we have that

$$Q_z = \sum_{\alpha>0} \partial_\xi^\alpha L_-(x,\xi)\partial_x^\alpha F(L_- - z)/\alpha!.$$

Hence, the terms of E_z outside of compact set in phase space look like

$$(P - z)^{-k-1} q$$

where $q \in S^{mk-\kappa}$ for some $\kappa \geq 0$.

Hence, there is a pseudodifferential operator representation of $L_-^{-\frac{1}{2}}$ and thus $L_-^{\frac{1}{2}}$ by multiplication by the operator. If p is the principal symbol of L_-, the principal symbol of L_-^a will be $F(p)$ where $F(z) = z^a$ for $|z| > C$.

□

LEMMA 13.6.5
The pseudodifferential operator $L_-V_1 + V_1(-\Delta + \lambda^2) + L_-^{\frac{1}{2}} V_2 L_-^{\frac{1}{2}}$ is a short range perturbation.

PROOF. This proof should be similar to that in Lemma 13.6.1. The argument for the differential operator $L_-V_1 + V_1(-\Delta + \lambda^2)$ follows precisely as above. Hence, we focus only on the compactness and iteration arguments for the pseudodifferential operator, $L_-^{\frac{1}{2}} V_2 L_-^{\frac{1}{2}}$. In what follows, let $Tu = L_-^{\frac{1}{2}} V L_-^{\frac{1}{2}} K * (u)$. In particular, we need to prove:

eqn:pdo1 (13.26) $$L_-^{\frac{1}{2}} V L_-^{\frac{1}{2}} e^{ix\cdot\xi_0} = e^{ix\cdot\xi_0} V_{\xi_0}, \text{ for } V_{\xi_0} \in \mathscr{S},\ \|V_{\xi_0}\|_L^\infty \sim |\xi_0|^2,$$

eqn:pdo2 (13.27) $$\|K * T^n(e^{ix\cdot\xi_0} V_{\xi_0})\|_{L^4} = O(|\xi_0|^{-\frac{n+1}{2}}),$$

eqn:pdo3 (13.28) $$K * (L_-^{\frac{1}{2}} V L_-^{\frac{1}{2}} \cdot) : W^{2,4} \hookrightarrow W^{2,4}.$$

For (13.26), we have in the sense of distributions that

$$\mathscr{F} e^{ix\xi_0} = \delta_{\xi_0}(\xi).$$

Hence, since $V \in \mathscr{S}$,

$$\begin{aligned} L_-^{\frac{1}{2}} V L_-^{\frac{1}{2}} &= \int P(x,\xi) e^{i(x-x_1)\xi} V(x_1) \int P(x_1,\xi_1) e^{ix_1\xi_1} \delta_{\xi_0}(\xi_1) d\xi_1 dx_1 d\xi \\ &= \int P(x,\xi) e^{i(x-x_1)\xi} V(x_1) \int P(x_1,\xi_0) e^{ix_1\xi_0} dx_1 d\xi \\ &= e^{ix\xi_0} \tilde{V}(x,\xi_0) + l.o.t., \end{aligned}$$

where $\tilde{V} \in \mathscr{S}(x)$ and $|\tilde{V}| \lesssim \xi_0^2$ precisely as in Section 13.5. This comes in particular from realizing that the principal symbol of $L_-^{\frac{1}{2}} V L_-^{\frac{1}{2}}$ is

$$(\xi^2 + \lambda^2 - V_1(x)) V_2(x).$$

The results (13.27) and (13.28) follow from the following theorem proved in [**Ste93a**], Chapter VI.

Theorem 13.12 (Stein)
Suppose T_a is a pseudo-differential operator whose symbol a belongs to S^m. If m is an integer and $k \geq m$, then T_a is a bounded mapping from $W^{k,p} \to W^{k-m,p}$, whenever $1 < p < \infty$.

Since $L_-^{\frac{1}{2}} \in S^1$ and $V \in S^0$, we have $L_-^{\frac{1}{2}} V L_-^{\frac{1}{2}} \in S^2$, hence

$$L_-^{\frac{1}{2}} V L_-^{\frac{1}{2}} : W^{2,4} \to L^4.$$

As $V \in \mathscr{S}$, we in fact have more than this. Define the symbol class

$$S_r^m = \{p(|\, p \in S^m,\ |x^\alpha \partial_x^\beta \partial_\xi^\gamma p(x,\xi)| \leq C_{\alpha,\beta} |\xi|^{m-\gamma}.$$

In other words, we have the standard symbol class S^m, where the symbol has rapid decay in x. Here, $V \in S^0_r$. Note that due to the properties of Schwarz class functions, we have for $p \in S^{m_1}$ and $q \in S^{m_2}$,

$$pq, qp \in S^{m_1+m_2}_r$$

and

$$qu : W^{m_2,p} \to L^q,$$

where $1 < p, q < \infty$.

For (13.27), from the analysis in Theorem 13.7 we have

$$(K * \cdot) : L^{\frac{4}{3}} \to W^{2,4}.$$

We have from (13.26)

$$\| \int K(x-y) e^{iy\cdot\xi_0} V_{\xi_0}(y) dy \|_{L^4} \lesssim |\xi_0|^{-\frac{1}{2}}.$$

Then,

$$\begin{aligned}
\| \int K(x-y) L^{\frac{1}{2}}_{-} V(y) L^{\frac{1}{2}}_{-} \int K(y-z) e^{iz\cdot\xi_0} V_{\xi_0}(z) dz dy \|_{L^4} \\
= |\xi_0|^{-\frac{1}{2}} \| L^{\frac{1}{2}}_{-} V(y) L^{\frac{1}{2}}_{-} \int K(y-z) e^{iz\cdot\xi_0} V_{\xi_0}(z) dz dy \|_{L^4 3} \\
\lesssim |\xi_0|^{-\frac{1}{2}} \| \int K(y-z) e^{iz\cdot\xi_0} V_{\xi_0}(z) dz dy \|_{W^{2,4}} \\
\lesssim |\xi_0|^{-1} \| e^{iz\cdot\xi_0} V_{\xi_0}(z) \|_{L^4 3},
\end{aligned}$$

using the fact that $V \in S^0_r$ and the mapping properties of K described in Theorem 13.7. Iterating this procedure, we get the result.

For (13.28), if $u \in W^{2,4}$,

$$\| L^{\frac{1}{2}}_{-} V L^{\frac{1}{2}}_{-} u \|_{L^{\frac{4}{3}}} \lesssim \| u \|_{W^{2,4}}.$$

By decay properties of V, we have

$$\| x L^{\frac{1}{2}}_{-} V L^{\frac{1}{2}}_{-} u \|_{L^{\frac{4}{3}}_x} \lesssim \| u \|_{W^{2,4}} + \| u \|_{W^{1,4}} \lesssim \| u \|_{W^{2,4}}.$$

The inherent integration by parts is justified as $V \in \mathscr{S}$. Hence, by iterating this procedure and properties of convolutions,

$$[K * (L^{\frac{1}{2}}_{-} V L^{\frac{1}{2}}_{-} \cdot)] : W^{2,4} \to W^{2,4}(\langle\cdot\rangle^N),$$

for any $N \in \mathbb{N}$. However, $W^{2,4}(\langle\cdot\rangle^N)$ is compactly embedded in $W^{2,4}$, so (13.28) holds.

□

LEMMA 13.6.6

There exists a distorted Fourier basis, $\tilde{u}_\xi$, for $L^{\frac{1}{2}}_{-} L_+ L^{\frac{1}{2}}_{-}$ with the aforementioned smoothness properties.

PROOF. Apply the techniques from the proof of Theorem 13.7, applying (13.26), (13.27), and (13.28) when necessary. Once the compactness is established, the standard

self-adjoint techniques are available to give

$$\begin{aligned} \|P_c\phi\|_{L^2}^2 &= (2\pi)^{-d}\int |\mathscr{F}_\pm\phi|^2 dx, \\ \mathscr{F}_\pm^{-1}P_0\mathscr{F}_\pm\phi &= P_c\phi, \end{aligned}$$

where $\mathscr{F}_\pm$ is the distorted Fourier transform associated to $\tilde{u}_{\xi_0}^\pm$ and $P_0(\xi_0) = (\xi_0^2+\lambda)^2$ is the symbol for the leading order constant coefficient operator.

□

Since

$$\begin{bmatrix} L_-^{\frac{1}{2}}L_+L_-^{\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}}L_+L_-^{\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} L_-^{-\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} L_-L_+ & 0 \\ 0 & L_+L_- \end{bmatrix} \begin{bmatrix} L_-^{\frac{1}{2}} & 0 \\ 0 & L_-^{-\frac{1}{2}} \end{bmatrix},$$

we have

$$\begin{bmatrix} L_-^{-\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}} \end{bmatrix} \mathscr{H}^2 \begin{bmatrix} L_-^{\frac{1}{2}} & 0 \\ 0 & L_-^{-\frac{1}{2}} \end{bmatrix} P_c f = \tilde{\mathscr{F}}_\pm^* |(\xi^2+\lambda^2)^2| \tilde{\mathscr{F}}_\pm f,$$

where $\tilde{\mathscr{F}}_\pm$ is the distorted Fourier transform with respect to $\tilde{u}_\xi$. Setting

$$f = \begin{bmatrix} L_-^{-\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}} \end{bmatrix} \tilde{f},$$

we see

$$\begin{bmatrix} L_-^{-\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}} \end{bmatrix} \mathscr{H}^2 \begin{bmatrix} L_-^{\frac{1}{2}} & 0 \\ 0 & L_-^{-\frac{1}{2}} \end{bmatrix} P_c f = \tilde{\mathscr{F}}_\pm^* |(\xi^2+\lambda^2)^2| \tilde{\mathscr{F}}_\pm \begin{bmatrix} L_-^{-\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}} \end{bmatrix} \tilde{f}.$$

Hence,

$$\mathscr{H}^2(P_c\tilde{f}) = \begin{bmatrix} L_-^{\frac{1}{2}} & 0 \\ 0 & L_-^{-\frac{1}{2}} \end{bmatrix} \tilde{\mathscr{F}}_\pm^* |(\xi^2+\lambda^2)^2| (\tilde{\mathscr{F}}_\pm \begin{bmatrix} L_-^{-\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}} \end{bmatrix}) \tilde{f},$$

or

$$\mathscr{H}^2(P_c\tilde{f})(x) = \begin{bmatrix} \int (L_-^{\frac{1}{2}}\tilde{u}_\xi)(x) |(\xi^2+\lambda^2)^2| \int \tilde{u}_\xi(y)(L_-^{-\frac{1}{2}}\tilde{f}_1)(y)dyd\xi \\ \int (L_-^{-\frac{1}{2}}\tilde{u}_\xi)(x) |(\xi^2+\lambda^2)^2| \int \tilde{u}_\xi(y)(L_-^{\frac{1}{2}}\tilde{f}_2)(y)dyd\xi. \end{bmatrix}$$

The inverse operations in these arguments are justified by the fact that

$$L^2 = \mathrm{Ker}(\mathscr{H}) \oplus \mathrm{Ker}(\mathscr{H}^*)^\perp.$$

We desire an oscillatory integral formulation for $\mathscr{H}P_c$. The continuous spectrum is spanned by the values $\pm(\lambda^2+\xi^2)$ for all $|\xi| \geq 0$. Hence, we seek a diagonalization of the form

$$\mathscr{H}P_c = Q^{-1} \begin{bmatrix} (\lambda^2+\xi^2) & 0 \\ 0 & -(\lambda^2+\xi^2) \end{bmatrix} Q.$$

Using the above analysis for $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$, we see that

$$\begin{aligned} Q &= \frac{1}{\sqrt{2}} \begin{bmatrix} i(\lambda^2+\xi^2)^{\frac{1}{2}} \mathscr{F} L_-^{-\frac{1}{2}} & (\lambda^2+\xi^2)^{-\frac{1}{2}} \mathscr{F} L_-^{\frac{1}{2}} \\ -i(\lambda^2+\xi^2)^{\frac{1}{2}} \mathscr{F} L_-^{-\frac{1}{2}} & (\lambda^2+\xi^2)^{-\frac{1}{2}} \mathscr{F} L_-^{\frac{1}{2}} \end{bmatrix} \\ Q^{-1} &= \frac{1}{\sqrt{2}} \begin{bmatrix} -iL_-^{\frac{1}{2}} \mathscr{F}^* (\lambda^2+\xi^2)^{-\frac{1}{2}} & iL_-^{\frac{1}{2}} \mathscr{F}^* (\lambda^2+\xi^2)^{-\frac{1}{2}} \\ L_-^{-\frac{1}{2}} \mathscr{F}^* (\lambda^2+\xi^2)^{\frac{1}{2}} & L_-^{-\frac{1}{2}} \mathscr{F}^* (\lambda^2+\xi^2)^{\frac{1}{2}} \end{bmatrix} \end{aligned}$$

Note that we have for $\vec{f} = P_c \vec{f}$

$$\mathscr{H} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} iL_- f_2 \\ -iL_+ f_1 \end{bmatrix},$$

which is exactly what results from the decomposition. The resulting integral equation is

$$\mathscr{H} P_c \vec{f} = \begin{bmatrix} -iL_-^{\frac{1}{2}} \mathscr{F}^* \mathscr{F} L_-^{\frac{1}{2}} f_2 \\ iL_-^{-\frac{1}{2}} \mathscr{F}^* (\lambda^2+\xi^2)^2 \mathscr{F} L_-^{-\frac{1}{2}} \end{bmatrix}.$$

So, since we have a pseudo-differential operator representation of $L_-^{\frac{1}{2}}$, we could write $\mathscr{H} P_c$ in terms of an oscillatory integral.

REMARK 13.6.7. *Note that we have now made precise the definition*

eqn:uvrep1 (13.29)
$$\tilde{\phi}_\xi \asymp \begin{bmatrix} iu_\xi & v_\xi \\ -iu_\xi & v_\xi \end{bmatrix}$$

eqn:uvrep2 (13.30)
$$= \begin{bmatrix} i(\xi^2+\lambda^2) L_-^{-\frac{1}{2}} \tilde{u}_\xi & (\xi^2+\lambda^2)^{-1} L_-^{\frac{1}{2}} \tilde{u}_\xi \\ -i(\xi^2+\lambda^2) L_-^{-\frac{1}{2}} \tilde{u}_\xi & (\xi^2+\lambda^2)^{-1} L_-^{\frac{1}{2}} \tilde{u}_\xi \end{bmatrix},$$

where using the pseudo-differential analysis above, $L_-^{\pm\frac{1}{2}} \tilde{u}_\xi$ is well-defined.

PROOF OF THEOREM 13.8. If

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \sigma_{ac}(\mathscr{H}),$$

then

$$Pf = \begin{bmatrix} L_-^{\frac{1}{2}} & 0 \\ 0 & L_-^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \sigma_{ac}(\tilde{\mathscr{H}}),$$

where

$$\tilde{\mathscr{H}} = \begin{bmatrix} L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} \end{bmatrix}.$$

Assume $f \in \mathscr{S}$, which we will relax later. Let ψ the PDO representation of P and $\Phi_\xi(x)$ is the vector where both elements are the distorted Fourier basis function ϕ_ξ for the self-adjoint operator $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$. Then, we have

$$\begin{aligned} (\mathscr{G}\psi f)(\xi) &= \langle \psi f, \Phi_\xi \rangle \\ &= \langle f, T\psi^* \Phi_\xi \rangle \\ &= \langle f, \tilde{\Phi}_\xi \rangle, \end{aligned}$$

where

$$T = \begin{bmatrix} i(\lambda^2+\xi^2)^{\frac{1}{2}} & (\lambda^2+\xi^2)^{-\frac{1}{2}} \\ -i(\lambda^2+\xi^2)^{\frac{1}{2}} & (\lambda^2+\xi^2)^{-\frac{1}{2}} \end{bmatrix},$$

and $\tilde{\Phi}_\xi$ is uniquely defined in the sense of distributions as

$$P(x,\xi)e^{ix\xi} + \tilde{u}_\xi(x,\xi),$$

where

$$\tilde{u} = P(x,D)u_\xi(x),$$

and $\tilde{u} \in \mathscr{S}$. Then,

$$(\mathscr{G}f)(\xi) = \int f\tilde{\Phi}_\xi dx.$$

Similarly, we have

$$(\mathscr{G}^{-1}f)(x) = \int f(\xi)\tilde{\Phi}_\xi^{-1} d\xi,$$

where

$$\tilde{\Phi}_\xi^{-1} = PT^*\Phi_\xi^*(x).$$

The modified Fourier transforms are in fact variations on the expansion involving the matrix Q. In fact, the Fourier transforms are arrived at by □

lin:cor COROLLARY 13.6.8. *As a result of the decomposition, we have a new proof of the fact that*

$$\|P_c e^{it\mathscr{H}} f\|_{L^2} \lesssim \|f\|_{L^2}.$$

PROOF. This follows simply from mapping properties of pseudodifferential operators and the fact that the self-adjoint distorted Fourier transform is an L^2 isometry. □

REMARK 13.6.9. *Note that for convenience in terms of defining the resolvent, our result has been proved here only in $\mathbb{R}^3$. However, using similar bounds developed in [**Agm75b**] for higher dimensional resolvents, we expect that a result similar to that of 13.8 holds in all dimensions and as a result similar estimates will follow below. The main difficulties presented would be a thorough discussion of the spectrum of $\mathscr{H}$ as some of the numerical techniques are unique to $\mathbb{R}^3$.*

13.7. Time Decay

lin:tdecay

Using our distorted Fourier basis, we have that a solution to the problem

eqn:intrep (13.31) $$e^{i\mathscr{H}t}P_c\phi = Q^{-1}e^{itW}Q\phi,$$

for

$$W = \begin{bmatrix} (\lambda^2+\xi^2) & 0 \\ 0 & -(\lambda^2+\xi^2) \end{bmatrix}.$$

The structure on Q allows us to do oscillatory integration in order to study the properties of $e^{i\mathscr{H}t}$. First of all, we show the following:

thm:lindec **Theorem 13.13***Let $\mathscr{H}$ be an admissible Hamiltonian as defined above. Then,*

eqn:lindec (13.32) $$\|e^{it\mathscr{H}}P_c\vec{\psi}\|_{L^\infty} \leq Ct^{-\frac{d}{2}}\|\vec{\psi}\|_{L^1}.$$

PROOF OF 13.13. Using matrix notation, we have

$$\{\mathcal{G}\vec{\psi}\}(\xi) = \int \tilde{\phi}_\xi(x)\vec{\psi}(x)dx,$$

where

$$\vec{\psi}(x) = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix},$$

and $\tilde{\phi}_\xi$ is given by (13.29).

Looking at the integral representation, we have

$$e^{it\mathcal{H}}P_c\vec{\psi}(x) = \int_\xi \tilde{\phi}_\xi^{-1}(x)e^{itW}\int_y \tilde{\phi}_\xi(y)\vec{\psi}(y)dyd\xi.$$

Let $\chi \in C_c^\infty$, be a smooth, cut-off function chosen such that the iteration techniques in Theorem 13.7 hold for $\xi \in \mathbb{R}^d \setminus \text{spt}(\chi)$. Then, take

eqn:stph1 (13.33) $$e^{it\mathcal{H}}P_c\vec{\psi}(x) = \int_\xi \chi(\xi)\tilde{\phi}_\xi^{-1}(x)e^{itW}\int_y \tilde{\phi}_\xi(y)\vec{\psi}(y)dyd\xi$$

eqn:stph2 (13.34) $$+ \int_\xi [1-\chi(\xi)]\tilde{\phi}_\xi^{-1}(x)e^{itW}\int_y \tilde{\phi}_\xi(y)\vec{\psi}(y)dyd\xi.$$

Hence, we must bound

$$\begin{aligned} I &= \int_\xi e^{ix\xi}e^{\pm it(\xi^2+\lambda^2)}\int_y e^{-iy\xi}\psi(y)dyd\xi, \\ II &= \int_\xi [\chi(\xi)+(1-\chi(\xi)]g_\xi^{-1}(x)e^{\pm it(\xi^2+\lambda^2)}\int_y e^{-iy\xi}\psi(y)dyd\xi, \\ III &= \int_\xi [\chi(\xi)+(1-\chi(\xi)]e^{ix\xi}e^{\pm it(\xi^2+\lambda^2)}\int_y g_\xi(y)\psi(y)dyd\xi, \\ IV &= \int_\xi [\chi(\xi)+(1-\chi(\xi)]g_\xi^{-1}(x)e^{\pm it(\xi^2+\lambda^2)}\int_y g_\xi(y)\psi(y)dyd\xi. \end{aligned}$$

From henceforward, we work only with the terms

$$\frac{e^{\pm i|x||\xi|}}{|x|}$$

from g_ξ, as the analysis for the exponentially decaying term will follow using simpler versions of the methods for this case. Many of the techniques used are developed from the presentation in [**Sch09**]. The challenge lies mostly in that $\partial_\xi^\alpha|\xi|$ is not bounded near 0 for $|\alpha| \geq 2$. Thus, we must be careful near the origin using stationary phase arguments since error terms require a minimum of two derivatives. A discussion of stationary phase complete with proofs is given in [**EZ06**]. Take the integral,

$$\mathcal{I} = \int g(x)e^{i\tau P(x)}dx,$$

where $g(x) \in C_c^\infty$, $P(x) \in C^\infty$. Assume that $\partial_x P(0) = 0$ and $\partial_x^2 P(0) \neq 0$. Then, the principle of stationary phase gives

$$\mathcal{I} \sim \tau^{-\frac{d}{2}}\sum_{j=0}^\infty a_j\tau^{-j},$$

where the asymptotic terms in the stationary phase expansion are given by

$$a_j = L^j g(0),$$

for L^j an order $2j$ differential operator.

Equation I is bounded using standard techniques of contour integration from the Linear Schrödinger equation. In particular, we have

$$\|I\|_{L^\infty} \lesssim t^{-\frac{d}{2}} \|\vec{\psi}\|_{L^1}.$$

Before we investigate further, we recall some properties of the functions $\partial^\alpha_{\xi_0} g_{\xi_0}$. From the expression for g_{ξ_0}, we know that

$$\begin{aligned} [I - K * (V_{\xi_0} \cdot)] g_{\xi_0} &= [I - P_{\xi_0})] g_{\xi_0} \\ &= K * (V_{\xi_0} e^{ix\xi_0}), \end{aligned}$$

so

$$g_{\xi_0} = [I - P_{\xi_0}]^{-1} (K * V_{\xi_0} e^{ix\xi_0}),$$

where

$$K(\xi) = [(-\Delta - \xi^2)(-\Delta + 2\lambda^2 + \xi^2)]^{-1}.$$

From Fredholm Theory and the spectral assumptions on $\mathscr{H}$, $(I - P_{\xi_0})^{-1}$ is well-defined, hence we can show that g_{ξ_0} is smooth in $|\xi_0|$ and ξ_0. Also, K is smooth with respect to the variables $|\xi|$, $V_\xi e^{ix\xi}$ is smooth with respect to ξ. As a result, g_{ξ_0} depends smoothly on $|\xi|$ and ξ. Therefore, for ξ near 0, we can take up to 3 derivatives before we lose integrability in g_ξ. For ξ_0 large enough, from Theorem 13.7, we have

$$g_{\xi_0} = K * f,$$

where

$$f = e^{ix\cdot\xi_0} f_0(x, \xi_0),$$

where $f_0(x, \xi_0)$ behaves like a symbol in S^2.

For (13.34), we use the principle of nonstationary phase and the principle of stationary phase in different regions. We have

$$\int_\xi [1 - \chi(\xi)] \tilde{\phi}^{-1}_\xi(x) e^{itW} \int_y \tilde{\phi}_\xi(y) \vec{\psi}(y) dy d\xi,$$

where $1 - \chi$ is supported away from 0. In particular, we have integrals of the type

$$\int_\xi [1 - \chi(\xi)] (e^{ix\xi} + \tilde{g}_\xi(x)) e^{it(\xi^2 + \lambda^2)} \int_y (e^{iy\xi} + g_\xi(y)) \psi(y) dy d\xi,$$

where g and $\tilde{g}$ are of the same form described above. Hence, we must bound the following

$$\begin{aligned} II &= \int_\xi [1 - \chi(\xi)] g_\xi(x) e^{it(\xi^2 + \lambda^2)} e^{-iy\xi} \psi(y) d\xi, \\ III &= \int_\xi [1 - \chi(\xi)] e^{ix\xi} e^{it(\xi^2 + \lambda^2)} g_\xi(y) \psi(y) d\xi, \\ IV &= \int_\xi [1 - \chi(\xi)] g_\xi(x) e^{it(\xi^2 + \lambda^2)} g_\xi(y) \psi(y) d\xi. \end{aligned}$$

For integrals of type II and III, we have oscillatory integrals of the form

eqn:dec2 (13.35)
$$\int e^{i(|z||\xi_0| + (x-z)\xi_0 + t\xi_0^2 - y\xi_0)} \frac{f_0(x - z, \xi_0)}{|z|} d\xi_0.$$

Looking at the phase function, we have

$$\begin{aligned}\phi(\xi_0) &= |z||\xi_0| + (x-z)\xi_0 + t\xi_0^2 - y\xi_0,\\ \nabla_{\xi_0}\phi(\xi_0) &= 2t\xi_0 + (x-z-y) + |z|\frac{\xi_0}{|\xi_0|},\\ \nabla^2_{\xi_0}\phi(\xi_0) &= 2tI_d + \frac{|z|}{|\xi_0|}(I_d - \frac{\xi_0\otimes\xi_0}{|\xi_0|^2}).\end{aligned}$$

If we restrict ξ_0 to a region such that

$$|\xi_0| \geq \frac{\max|z+y-x|-|z|,0}{2t} + 1,$$

then $\phi(\xi_0)$ has no critical points. As a result, we can use the principle of non-stationary phase on this region with the decay properties of the function f_0 to see we have decay like t^{-N} for any N.

Let us hence assume that we are restricted a region

$$|\xi_0| \leq \frac{\max|z+y-x|-|z|,0}{2t} + 1,$$

so ϕ has at least one critical point. In fact, the critical point occurs where

eqn:cp

$$\xi_0\left(1+\frac{|z|}{|\xi_0|}\right) = z+y-x. \tag{13.36}$$

Also, for $|z+y-x|-|z| \neq 0$, we have

eqn:cpabs

$$|\xi_0| = \frac{|z+y-x|-|z|}{2t}. \tag{13.37}$$

As a result, all critical points occur on the same sphere. Using (13.37), we have that if z, y and x are such that a critical point exists, that critical point is unique. Hence, we can define a cut-off function $\chi_{x,y,z} \in C_c^\infty(\mathbb{R}^d)$ such that

$$\chi_{x,y,z}(\xi) = \begin{cases} 1 \text{ for } |\xi_0| \leq \frac{\max|z+y-x|-|z|,0}{2t} + \frac{1}{2},\\ 0 \text{ for } |\xi_0| \geq \frac{\max|z+y-x|-|z|,0}{2t} + 1.\end{cases}$$

Let us assume that a critical point exists, say ξ_0^c. If $|\xi_0^c| < \frac{|z|}{2t}$, the Hessian matrix is at least of rank 1 as $\xi\otimes\xi$ is a rank 1 matrix. So, there is at least one nondegenerate direction for ξ. After making an orthogonal change of coordinates bringing that nondegenerate direction to ξ_1, using stationary phase on $\mathbb{R}$, we have decay of the form

$$\|(13.35)\|_{L^\infty} \lesssim t^{-\frac{1}{2}}.$$

However, in the integral, we have $\frac{1}{|z|} < \frac{1}{|\xi_0|t}$, so using the decay of f_0 in z, the overall decay is once again

$$\|(13.35)\|_{L^\infty} \lesssim t^{-\frac{d}{2}},$$

where the integral is bounded in the remaining directions. For $|\xi_0^c| > \frac{|z|}{2t}$, the Hessian is nondegenerate. We can thus apply stationary phase in ξ to get decay of the form

$$\|(13.35)\|_{L^\infty} \lesssim t^{-\frac{d}{2}},$$

where we have once again used the regularity of f_0 is x and ξ. Then, given the uniform decay of f_0 and boundedness in y and x, we have uniform boundedness with decay of type $t^{-\frac{d}{2}}$.

The analysis for oscillatory integrals of type IV is similar in that the phase function becomes

$$\begin{aligned}\phi(\xi_0) &= |z||\xi_0| + (x-z)\xi_0 + t\xi_0^2 - |z_0||\xi_0| + (y-z_0)\xi_0,\\ \nabla_{\xi_0}\phi(\xi_0) &= 2t\xi_0 + (x-z) + (y-z_0) + (|z|-|z_0|)\frac{\xi_0}{|\xi_0|},\\ \nabla^2_{\xi_0}\phi(\xi_0) &= 2tI_d + \frac{|z|-|z_0|}{|\xi_0|}(I_d - \frac{\xi_0\otimes\xi_0}{|\xi_0|^2}).\end{aligned}$$

Hence, where critical points exist, we split up the regions of integration into $|\xi_0| > \frac{|z|-|z_0|}{2t}$ and $|\xi_0| < \frac{|z|-|z_0|}{2t}$. Once again, we have stationary phase in full on the first region and stationary phase in at least one direction, coupled with the fact that $\frac{1}{|z|} < \frac{1}{2|\xi_0|t}$. Away from the critical points, we once again apply non-stationary phase.

Let us now analyze (13.33). In particular, we have integrals of the type

$$\int_\xi [\chi(\xi)](e^{ix\xi} + g_\xi(x))e^{it(\xi^2+\lambda^2)}\int_y (e^{iy\xi} + g_\xi(y))\psi(y)dyd\xi.$$

Thus, we have to bound

$$\begin{aligned}II &= \int_\xi [\chi(\xi)]g_\xi^{-1}(x)e^{\pm it(\xi^2+\lambda^2)}e^{-iy\xi}d\xi,\\ III &= \int_\xi [\chi(\xi)]e^{ix\xi}e^{\pm it(\xi^2+\lambda^2)}g_\xi(y)d\xi,\\ IV &= \int_\xi [\chi(\xi)]g_\xi^{-1}(x)e^{\pm it(\xi^2+\lambda^2)}g_\xi(y)d\xi.\end{aligned}$$

For integrals of type II and III, we have an oscillatory integral of the form

eqn:dec1 (13.38) $$\int e^{i(|x||\xi_0|+t\xi_0^2-y\xi_0)}\frac{f_0(x-z)}{|x|}d\xi_0.$$

The phase function is

$$\begin{aligned}\phi(\xi_0) &= |x||\xi_0| + t\xi_0^2 - y\xi_0,\\ \nabla_{\xi_0}\phi(\xi_0) &= 2t\xi_0 - y + |x|\frac{\xi_0}{|\xi_0|},\\ \nabla^2_{\xi_0}\phi(\xi_0) &= 2tI_d + \frac{|x|}{|\xi_0|}(I_d - \frac{\xi_0\otimes\xi_0}{|\xi_0|^2}).\end{aligned}$$

Let us begin with an integral of type II. After making the orthogonal change of coordinates $\xi_1 \rightarrow \frac{y}{|y|}$ and moving to polar coordinates in ξ, we need to bound

$$\begin{aligned}\boxed{\|II\|_{L^1}} \lesssim\ & \|\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\int \chi(r)\frac{e^{-ir|z_0|}}{|z_0|}e^{itr^2}e^{-ir|y|\cos(\theta)}\\ \times\ & f_0(x-z_0, r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta), r)r^2\sin(\theta)drd\theta d\phi\|_{L^\infty}.\end{aligned}$$

If we Taylor expand f_0 in terms of $(r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta), r)$, then we can integrate in ϕ. In which case, all terms in the expansion with odd powers of $\cos(\phi)$ or $\sin(\phi)$ vanish under integrate out, leaving us with a function of the form

$$\tilde{f}_0(r^2\cos^2(\theta), r).$$

Integrating by parts in θ, we have

$$\begin{aligned}
\|II\|_{L^\infty} &\lesssim \|\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\chi(r)\frac{e^{-ir|z_0|}}{|z_0|}e^{itr^2}e^{-ir|y|\cos(\theta)}\\
&\times \tilde{f}_0(x-z_0, r\cos^(\theta), r)r^2\sin(\theta)drd\theta d\phi\|_{L^\infty}\\
&\lesssim \|\int_0^{\infty}\chi(r)\frac{e^{-ir|z_0|}}{|z_0|}e^{itr^2}\frac{\sin(r|y|)}{|y|}\\
&\times \tilde{f}_0(x-z_0, r)rdr\|_{L^\infty}\\
&+ \|\int_0^{\pi}\int_0^{\infty}\chi(r)\frac{e^{-ir|z_0|}}{|z_0|}e^{itr^2}\frac{e^{-ir|y|\cos(\theta)}}{|y|}\\
&\times \partial_\theta\tilde{f}_0(x-z_0, r\cos^2(\theta), r)rdydz_0drd\theta\|_{L^\infty}\\
&= \|\int_0^{\infty}\chi(r)\frac{e^{-ir|z_0|}}{|z_0|}e^{itr^2}\frac{\sin(r|y|)}{|y|}\\
&\times \tilde{f}_0(x-z_0, r)rdr\|_{L^\infty}\\
&+ \|\int_0^{\pi}\int_0^{\infty}\chi(r)\frac{e^{-ir|z_0|}}{|z_0|}e^{itr^2}\frac{\sin(-ir|y|\cos(\theta))}{|y|}\\
&\times \partial_\theta\tilde{f}_0(x-z_0, r\cos^2(\theta), r)rdrd\theta\|_{L^\infty},
\end{aligned}$$

since for n odd,

$$\int_{-1}^{1}e^{i\mu x}x^n dx = i\int_{-1}^{1}\sin(\mu x)x^n dx.$$

Note that the boundedness in y and θ is hence maintained after the integration by parts.

Let us extend the region of integration in r to $\mathbb{R}$. Due to the nature of the oscillatory functions involved, we experience no loss in doing so. Then, using the linear Schrödinger equation dispersion, we have

$$\begin{aligned}
\|II\|_{L^\infty} &\lesssim \frac{1}{t}\int_{-\infty}^{\infty}e^{itr^2}\\
&\times \partial_r\left[\frac{e^{-ir|z_0|}}{|z_0|}\frac{e^{-ir|y|}-e^{ir|y|}}{|y|}\chi(r)f_0(x-z_0,r)r\right]drd\theta d\phi\|_{L^\infty}\\
&\lesssim \frac{1}{t^{\frac{3}{2}}}\int_{-\infty}^{\infty}|\int\frac{1}{|z_0||y|}\\
&\times [\mathscr{F}^{-1}\left[\chi(r)f_0(x-z_0,r)r\right](u+|z_0|+|y|)(x-z_0)\\
&- \mathscr{F}^{-1}\left[\chi(r)f_0(x-z_0,r)r\right](u+|z_0|-|y|)(x-z_0)|du\|_{L^\infty}.
\end{aligned}$$

From the estimate

$$\|\hat{u}\|_{L^1}\lesssim\sup_{|\alpha|\le d+1}\|\partial^\alpha u\|_{L^1},$$

coupled with the facts that $\chi\in C_0^\infty$, $f_0\in C_r^\infty$, and f_0 is rapidly decaying in x, we have

$$\|II\|_{L^\infty}\lesssim\frac{1}{t^{\frac{3}{2}}}\|f\|_{L^1}.$$

For the integrals of type III, we immediately apply the linear Schrödinger estimate to get

$$\begin{aligned}\|III\|_{L^\infty} &\lesssim \frac{1}{t^{\frac{3}{2}}}\int |\int \chi(\xi)\frac{e^{i|y-x_1||\xi|}}{|y-x_1|}e^{iy\xi}f_0(x_1,\xi,|\xi|)f(y)dx_1d\xi|dy\\ &\lesssim \frac{1}{t^{\frac{3}{2}}},\end{aligned}$$

using once again the smoothness and decay of χ, f_0.

The analysis for oscillatory integrals of type IV is similar to that for type II, except now we have no θ dependence in the phase. Thus, we have phase functions of the form

$$\begin{aligned}\phi(\xi_0) &= |x||\xi_0| + t\xi_0^2 - |y||\xi_0|,\\ \nabla_{\xi_0}\phi(\xi_0) &= 2t\xi_0 + (|x|-|y|)\frac{\xi_0}{|\xi_0|},\\ \nabla^2_{\xi_0}\phi(\xi_0) &= 2tI_d + \frac{|x|-|y|}{|\xi_0|}(I_d - \frac{\xi_0\otimes\xi_0}{|\xi_0|^2}).\end{aligned}$$

At this point, it becomes convenient to move to polar coordinates in ξ. As a result, we have

$$\begin{aligned}\|IV\|_{L^\infty} &\lesssim \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\chi(r)\frac{e^{-ir|z_0|}}{|z_0|}e^{itr^2}\frac{e^{-ir|z_1|}}{|z_1|}\\ &\times f_0(y-z_0, r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta), r)\\ &\times \tilde{f}_0(x-z_1, r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta), r)\\ &\times r^2\sin(\theta)drd\theta d\phi.\end{aligned}$$

Hence, we can first extend the interval of integration in r to $\mathbb{R}$, then immediately integrate by parts in r to gain a factor of $\frac{1}{t}$. We once again apply the linear Schrödinger dispersive estimate to get

$$\|IV\|_{L^\infty} \lesssim \frac{1}{t^{\frac{3}{2}}}\|f\|_{L^1}.$$

Combining the above results, we have

$$\|(13.33)\|_{L^\infty} \leq t^{-\frac{d}{2}}\|\psi\|_{L^1}$$

and

$$\|(13.34)\|_{L^\infty} \leq t^{-\frac{d}{2}}\|\psi\|_{L^1}.$$

Hence, the theorem follows.

□

Before we begin, let us define the space

$$L^{1,M} = \{f\in L^1 | \|\langle\cdot\rangle^N f(\cdot)\|_{L^1}\leq\infty,\ N=0,1,\dots,2M\},$$

with norm $\|\cdot\|_{L^{1,M}}$ defined standardly. As necessary for the contraction argument, we now show:

thm:wlindec **Theorem 13.14** *Let $\mathscr{H}$ be an admissible Hamiltonian as defined above. Assume $\vec{\psi}\in L^{1,M}$ and*

eqn:mom1 (13.39) $$\partial_\xi^\alpha\partial_{|\xi|}^\beta\vec{\Psi}(0) = 0,$$

for multi-indices α, β such that $|\alpha|+|\beta| = 0,1,2,\dots,2M$, where

$$\vec{\Psi}(\xi) = \int_y \tilde{\phi}_\xi(y)\vec{\psi}(y)dy.$$

Then,

eqn:lindecw

$$\|e^{-c|x|}e^{it\mathscr{H}}P_c\vec{\psi}\|_{L^\infty} \leq Ct^{-\frac{d}{2}-M}\|\vec{\psi}\|_{L^{1,M}}, \tag{13.40}$$

for any $c > 0$.

PROOF OF 13.14. We proceed similarly to the proof of Theorem 13.13, except now we must bound the following:

$$\begin{aligned} I &= |e^{-c|x|}\int_\xi \chi(\xi)\phi_\xi^{-1}(x)e^{\pm it(\xi^2+\lambda^2)}\int_y \phi_\xi(y)\psi(y)dyd\xi|, \\ II &= |e^{-c|x|}\int_\xi [1-\chi(\xi)]e^{ix\xi}e^{\pm it(\xi^2+\lambda^2)}\int_y g_\xi(y)\psi(y)dyd\xi|. \end{aligned}$$

For II, we look at oscillatory integrals of the form

$$\int_\xi [1-\chi(\xi)](e^{ix\cdot\xi}+g_\xi^{-1}(x))e^{it(\xi^2+\lambda^2)}\int_y (e^{iy\cdot\xi}+g_\xi(y))\psi(y)dyd\xi,$$

considering the phase function as $\phi(\xi) = it\xi^2$. By applying the principle of non-stationary phase (see [**EZ06**]), then using that

$$\begin{aligned} |\partial_\xi g_\xi| &\leq |\int \frac{\xi}{|\xi|}e^{i|x-y||\xi|}e^{iy\xi}f_0(y,\xi)dy| \\ &+ |\int \frac{\xi}{\sqrt{\xi^2+\lambda^2}}e^{-|x-y|\sqrt{\xi^2+\lambda^2}}e^{iy\xi}f_0(y,\xi)dy| \\ &+ |\int \frac{e^{i|x-y||\xi|}-e^{-|x-y|\sqrt{\xi^2+\lambda^2}}}{|x-y|}e^{iy\cdot\xi}yf_0(y,\xi)dy| \\ &+ |\int \frac{e^{i|x-y||\xi|}-e^{-|x-y|\sqrt{\xi^2+\lambda^2}}}{|x-y|}e^{iy\cdot\xi}\partial_\xi f_0(y,\xi)dy| \\ &\lesssim \int (\langle x\rangle+\langle y\rangle)|f_0(y,\xi)|dy + \int |\partial_\xi f_0(y,\xi)|dy \end{aligned}$$

and the regularity of f_0 in y and ξ_0, we have

$$\|e^{-c|x|}\int_\xi (1-\chi(\xi))\tilde{\phi}_\xi^{-1}(x)e^{itW}\int_y \tilde{\phi}_\xi(y)\vec{\psi}(y)dyd\xi\|_{L^\infty} \lesssim t^{-\frac{d}{2}-M}\|\psi\|_{L^{1,M}}.$$

Now, for I, we need to bound

$$\int_\xi [\chi(\xi)](e^{ix\cdot\xi}+g_\xi^{-1}(x))e^{it(\xi^2+\lambda^2)}(e^{iy\cdot\xi}+g_\xi(y))d\xi,$$

we use stationary phase with the phase function

$$\phi(\xi) = t\xi^2.$$

As a result, we arrive at the moments condition (13.39). This condition serves a dual purpose. First of all, we have a gain in time decay using the stationary phase argument

$$L^j\vec{g}(0) = 0,$$

for $j = 1, 2, \ldots, 2M$ is well-defined, where

$$\vec{g}(\xi) = \chi(\xi)\tilde{\phi}_\xi^{-1}(x)\int_y \tilde{\phi}_\xi(y)\vec{f}(y)dy, \tag{13.41}$$

and L_j is the order $2j$ differential operator resulting from stationary phase. Second of all, we have control near $\xi_0 = 0$ for the error bound from stationary phase

$$\sum_{|\alpha|\leq 2M+2} |D_\xi^\alpha \vec{u}| \lesssim 1,$$

where $\xi \in \text{spt}(\chi)$ and

$$u = [\chi(\xi)](e^{ix\cdot\xi} + g_\xi^{-1}(x))(e^{iy\cdot\xi} + g_\xi(y)).$$

In turn, (13.39) becomes our moments condition for the function space $\mathscr{P}_2^A$ (REFERENCE), where N is chosen to give a gain of t^{-A} in time decay. Specifically, $A = 1$ implies $N = 1$ and $A = 2$ implies $N = 4$. Note that to gain t^{-A}, we must have $N = 2A$ at least. Such decay comes at the cost of polynomial growth in x and y from when the ∂_ξ terms fall on either $e^{i(\cdot,\xi)}$ or $g_\xi(\cdot)$. Hence, under our assumptions, we have

$$\|e^{-c|x|}\int_\xi \chi(\xi)\tilde{\phi}_\xi^{-1}(x)e^{itW}\int_y \tilde{\phi}_\xi(y)\vec{f}(y)dyd\xi\|_{L^\infty} \lesssim t^{-\frac{d}{2}-M}.$$

□

13.8. Dispersive Estimates

lin:disp

We have $H^1 = M \otimes S$ where M is $2d+4$ dimensional set of functions that span the 4th order generalized null space at 0 and S is the continuous spectrum.

Since M is spanned by functions with exponential decay, we have for $\phi \in M$

$$\|e^{it\mathscr{H}}\phi\|_{H^1} \leq C(1+|t|^3)\int e^{-c|x|}|\phi(x)|dx,$$

where c is determined by the exponential decay of all functions in M.

Now, from [**ES06**] and 13.6 we have for $\phi \in S$,

eqn6:1

$$\|e^{it\mathscr{H}}\phi\|_{L^2} \leq C\|\phi\|_{L^2}. \tag{13.42}$$

LEMMA 13.8.1. *Given Equation* (13.42)*, we have*

$$\|e^{it\mathscr{H}}\phi\|_{H^1} \leq C\|\phi\|_{H^1}.$$

PROOF. For $\phi \in S$, we have

$$\begin{aligned}
\|e^{it\mathscr{H}}\phi\|_{H^2} &\leq \|\mathscr{H}e^{it\mathscr{H}}\phi\|_{L^2} + C\|e^{it\mathscr{H}}\phi\|_{L^2} \\
&\leq \|e^{it\mathscr{H}}\mathscr{H}\phi\|_{L^2} + C\|e^{it\mathscr{H}}\phi\|_{L^2} \\
&\leq \|\mathscr{H}\phi\|_{L^2} + C\|e^{it\mathscr{H}}\phi\|_{L^2} \\
&\leq \|\phi\|_{H^2} + C\|\phi\|_{L^2} \\
&\leq C\|\phi\|_{H^2}.
\end{aligned}$$

Hence, the result follows from interpolation. □

In order to push through the contraction argument, we need various dispersive estimates from [**BW98**]. We present the proofs here.

thm:disp1 **Theorem 13.15 (Schlag-Erdogan,Bourgain)** *Let P_c and P_d be projections onto the continuous and discrete spectrum of $\mathcal{H}$ respectively. Then,*

$$\begin{aligned}
(i)\ \|e^{it\mathcal{H}}P_c\phi\|_{H^1} &\leq C\|\phi\|_{H^1}\\
(ii)\ \|e^{it\mathcal{H}}(P_c\phi)\|_{H^s} &\leq C\|\phi\|_{H^s}\\
(iii)\ \|e^{it\mathcal{H}}(P_d\phi)\|_{H^s} &\leq C(1+|t|^3)\int e^{-c|x|}|\phi(x)|dx\\
(iv)\ \||x|^\alpha e^{it\mathcal{H}}(P_c\phi)\|_{L^2} &\leq C(\||x|^\alpha\phi\|_{L^2}+(1+|t|^\alpha)\|\phi\|_{H^\alpha}\\
(v)\ \||x|^\alpha e^{it\mathcal{H}}(P_d\phi)\|_{L^2} &\leq C(1+|t|^3)\int |\phi|e^{-c|x|}dx.
\end{aligned}$$

PROOF. Estimate (iii) follows from the discrete spectral decomposition into a 4 dimensional generalized null space. The exponential decay is apparent from the properties of the eigenfunctions. Estimate (v) follows similarly.

For $\phi\in\sigma_{ac}(\mathcal{H})$, we have from [**ES06**] or the analysis in [**Mar10**] that

$$\|e^{it\mathcal{H}}P_c\phi\|_{L^2} \leq C\|\phi\|_{L^2}.$$

For $\phi\in\sigma_{ac}(\mathcal{H})$, we have

$$\begin{aligned}
\|e^{it\mathcal{H}}\phi\|_{H^2} &\leq \|\mathcal{H}e^{it\mathcal{H}}\phi\|_{L^2}+C\|e^{it\mathcal{H}}\phi\|_{L^2}\\
&\leq \|e^{it\mathcal{H}}\mathcal{H}\phi\|_{L^2}+C\|e^{it\mathcal{H}}\phi\|_{L^2}\\
&\leq \|\mathcal{H}\phi\|_{L^2}+C\|e^{it\mathcal{H}}\phi\|_{L^2}\\
&\leq \|\phi\|_{H^2}+C\|\phi\|_{L^2}\\
&\leq C\|\phi\|_{H^2}.
\end{aligned}$$

This gives (i). A similar argument shows

$$\|e^{it\mathcal{H}}\phi\|_{H^{2s+1}}\lesssim\|\phi\|_{H^{2s+1}}+\|e^{it\mathcal{H}}\phi\|_{H^{2s-1}}.$$

Thus, by induction, we have (ii) for all positive integers s and hence by interpolation all $s>0$.

Let $\phi\in\sigma_{ac}(\mathcal{H})$ and $u=e^{it\mathcal{H}}\phi$. Then, since

$$iv_t-\mathcal{H}v=0,$$

then

$$\begin{aligned}
\frac{d}{dt}\int|x|^{2\alpha|}|v(x,t)|^2dx &= 2\mathfrak{Re}\langle|x|^{2\alpha}v,v_t\rangle\\
&= 2\mathfrak{Im}\langle|x|^{2\alpha}v,\mathcal{H}v\rangle\\
&= 2\mathfrak{Im}\langle|x|^{2\alpha}v,\Delta v\rangle+O\left(\int|v|^2e^{-c|x|}\right)\\
&\lesssim \int|x|^{2\alpha-1}|v||\nabla v|dx+\|v\|_2^2.
\end{aligned}$$

Using the following interpolation inequality

$$\||x|^{\alpha-\gamma}|D^\gamma v|\|_{L^2}\leq\||x|^\alpha v\|_{L^2}^{1-\frac{\gamma}{\alpha}}\|v\|_{H^\alpha}^{\frac{\gamma}{\alpha}},$$

we have

$$\begin{aligned}
\int|x|^{2\alpha-1}|v||\nabla v|dx &\leq \||x|^\alpha|v|\|_{L^2}\||x|^{\alpha-1}|\nabla v|\|_{L^2}\\
&\leq \||x|^\alpha|v|\|_{L^2}^{2-\frac{1}{\alpha}}\|v\|_{H^\alpha}^{\frac{1}{\alpha}}.
\end{aligned}$$

Hence, using (ii)

$$\frac{d}{dt}[\||x|^\alpha v(t)\|_2^2] \lesssim \||x|^\alpha v\|_{L^2}^{2-\frac{1}{\alpha}} \|\phi\|_{H^\alpha}^{\frac{1}{\alpha}} + \|v\|_{L^2}^2.$$

Integrating, we have

$$\begin{aligned}
\||x|^\alpha v\|^2_{L^2L^\infty([0,t])} &\lesssim \||x|^\alpha|\phi\|_2^2 + \int_0^t [\||x|^\alpha v(s)\|_{L^2}^{2-\frac{1}{\alpha}} \|\phi\|_{H^\alpha}^{\frac{1}{\alpha}} + \|v(s)\|_{L^2}^2] ds \\
&\lesssim \||x|^\alpha|\phi\|_2^2 + \||x|^\alpha v\|_{L^2L^\infty([0,t])}^{2-\frac{1}{\alpha}} \int_0^t [\|\phi\|_{H^\alpha}^{\frac{1}{\alpha}} + \|\phi\|_{L^2}^2] ds \\
&\lesssim \||x|^\alpha|\phi\|_2^2 + \epsilon\||x|^\alpha v\|^2_{L^2L^\infty([0,t])} + C(\epsilon)(t^{2\alpha}+t)\||x|^\alpha|\phi\|_2^2.
\end{aligned}$$

Hence, estimate (iv) follows. □

The following result proving Strichartz estimates is from [**Sch09**].

thm:strich2 **Theorem 13.16 (Schlag)** *For every $\phi \in L^2$ and every admissible pair (q,r), the function $t \to e^{i\mathscr{H}t}\phi$ belongs to $L^q(\mathbb{R}, L^r(\mathbb{R}^d)) \cap C(\mathbb{R}, L^2(\mathbb{R}^d))$, and there exists a constant C depending only on q such that*

eqn:strich2 (13.43) $$\|e^{i\mathscr{H}t}\phi\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))} \le C\|\phi\|_{L^2}.$$

PROOF. We must use the Christ-Kiselev Lemma [**CK03**].

LEMMA 13.8.2. *Let X, Y be Banach spaces and let $K(t,s)$ be the kernel of the operator*

$$K : L^p([0,T];X) \to L^q([0,T];Y).$$

Denote by $\|K\|$ the operator norm of K. Define the lower diagonal operator

$$\tilde{K} : L^p([0,T];X) \to L^q([0,T];Y)$$

to be

$$\tilde{K}f(t) = \int_0^t K(t,s)f(s)ds.$$

Then, the operator $\tilde{K}$ is bounded from $L^p([0,T];X) \to L^q([0,T];Y)$ and it norm $\|\tilde{K}\| \le c\|K\|$, provided $p<q$.

A perturbative approach originated by Kato is used. Define

$$(SF)(t,x) = \int_0^t (e^{-i(t-s)\mathscr{H}} P_c F(s,\cdot))(x) ds.$$

Then,

$$\|SF\|_{L_t^\infty L_x^2} \lesssim \|F\|_{L_t^1 L_x^2}.$$

Using the fractional integration argument from the unitary case, we have

$$\|SF\|_{L_t^r L_s^p} \lesssim \|F\|_{L_t^{r'} L_x^{q'}},$$

where (r,p) is admissible. By Duhamel, we have

$$e^{-it\mathscr{H}}P_c = e^{-it\mathscr{H}_0}P_c - i\int_0^t e^{-i(t-s)\mathscr{H}_0} V e^{-is\mathscr{H}} P_c ds.$$

Set $V = \tilde{M}\tilde{M}^{-1}V$, where

$$\tilde{M} = \begin{bmatrix} \langle x\rangle^{-1-} & 0 \\ 0 & \langle x\rangle^{-1-} \end{bmatrix}.$$

Then,

$$\left\| \int_0^\infty e^{-i(t-s)\mathscr{H}_0} \tilde{M} g(s) ds \right\|_{L_t^r L_x^p} \lesssim \left\| \int_0^\infty e^{is\mathscr{H}_0} \tilde{M} g(s) \right\|_{L^2} \lesssim \|g\|_{L_t^2 L_x^2},$$

where the last inequality follows from local smoothing. Applying the Christ-Kiselev lemma, for any Strichartz pair (r,p), we have

$$\left\| \int_0^t e^{-i(t-s)\mathscr{H}_0} \tilde{M} g(s) ds \right\|_{L_t^r L_x^p} \lesssim \|g\|_{L_t^2 L_x^2}.$$

Then,

$$\left\| e^{-it\mathscr{H}} P_c f \right\|_{L_t^r L_x^p} \lesssim \|f\|_{L^2} + \left\| \tilde{M}^{-1} V e^{-is\mathscr{H}} P_c f \right\|_{L_s^2 L_x^2},$$

so we need

$$\left\| \tilde{M}^{-1} V e^{-is\mathscr{H}} P_c f \right\|_{L_s^2 L_x^2} \lesssim \|f\|_{L^2}.$$

Taking a Fourier transform in s gives

$$\int_{-\infty}^{\infty} \|\tilde{M}^{-1} V [P_c(\mathscr{H} - \lambda - i0) P_c]^{-1} P_c f\|_{L^2}^2 d\lambda \lesssim \|f\|_{L^2}^2.$$

However, this follows from the smoothing estimate on $\mathscr{H}_0$, plus the standard resolvent identity under the spectral assumptions on $\mathscr{H}$. Hence,

$$\|e^{-it\mathscr{H}} P_c f\|_{L_t^r L_x^p} \lesssim \|f\|_{L^2}.$$

□

We prove here that $L_-^{\pm\frac{1}{2}}$ is a PDO for X a compact manifold as in Hörmander [**Hö3b**].

Theorem 13.17
Let X be a compact manifold, Ψ a space of pseudo-differential operators and $\Omega^{\frac{1}{2}}$ be the space of half-densities on X. Let $P \in \Psi_{phg}^m(X;\Omega^{\frac{1}{2}},\Omega^{\frac{1}{2}})$ be a positive, elliptic, symmetric operator. Then, P defines a positive, self-adjoint operator $\mathscr{P}$ in $L^2(X,\Omega^{\frac{1}{2}})$. If $m>0$ and $a \in \mathbb{R}$, then $\mathscr{P}^a$ is also defined by a pseudodifferential operator in $\Psi_{phg}^{am}(X;\Omega^{\frac{1}{2}},\Omega^{\frac{1}{2}})$, with principal and subprincipal symbols p^a and $ap^{a-1}p^s$ if p and p^s are those for P.

PROOF. $\mathscr{P}$ is bounded if $m \le 0$, hence self-adjoint. If $m>0$, then $\mathscr{P}$ is the restriction of P to all $u \in L^2$ with $Pu \in L^2$. This implies $u \in H_{(m)}$ and C^∞ is dense in $H_{(m)}$, hence $\mathscr{P}$ is self-adjoint. The resolvent,

$$R(z) = (\mathscr{P} - z)^{-1},$$

is defined and analytic in z except at eigenvalues of $\mathscr{P}$ on $\mathbb{R}_+$, and the L^2 operator norm can be estimated by the reciprocal of the distance to the eigenvalues. If $a<0$, it follows from the spectral theorem that with absolute convergence in L^2,

$$\mathscr{P}^a u = -(2\pi i)^{-1} \int_{-i\infty}^{i\infty} z^a R(z) u dz,\ u \in L^2,$$

where z^a is analytic in the right half plane and equal to 1 when $z=1$. Since $\mathscr{P}^{a+1}u = \mathscr{P}^a \mathscr{P} u$ when u is in the domain of $\mathscr{P}$ and $\mathfrak{Re}(a) \le 0$, it follows that the distribution kernel of $\mathscr{P}^a$ is an entire analytic function of a. By approximating R with a parametrix

for $P-z$, we shall determine the singularities for $a<0$ and by analytic continuation for $a\geq 0$ as well.

Let $Y\subset X$ be a local coordinate patch identified with an open set in $\mathbb{R}^n$. In the local coordinates, the symbol for $P-z$ is uniformly bounded with respect to z in

$$S((1+|z|+|\xi|^m),g),\ g=|dx|^2+\frac{|d\xi|^2}{1+|\xi|^2},$$

where $u\in S(m,g)$ if $u\in C^\infty$ and for every $k\geq 0$, we have

$$\sup_k \frac{|u|_k^g(x)}{m(x)}<\infty,$$

where

$$|u|_k^g=\sup_{t_j\in X}\frac{|u^{(k)}(x;t_1,\dots,t_k)|}{\Pi_{m=1}^k G(t_m)^{\frac{1}{2}}},$$

and $G=g_x$ is a quadratic form. For fixed k, this is equivalent to the maximum of the derivatives of order k with respect to a G orthogonal coordinate system.

Then, $(P-z)^{-1}$ is uniformly bounded in $S((1+|z|+|\xi|^m)^{-1},g)$ if $\mathfrak{Re}(z)=0$, which we assume henceforward. Note that we may assume $\mathfrak{Re}(P(x,\xi))>0$ everywhere for $p>0$ since P is strongly elliptic. Hence,

$$(P(x,D)-z)(P-z)^{-1}(x,D)=I-Q_z(x,D),$$

where Q_z is uniformly bounded in $S((1+|\xi|)^{m-1}(1+|z|+|\xi|^m)^{-1},g)$ by standard bounds on inverses for PDO's. Specifically, we have

$$Q_z(x,\xi)\sim\sum_{\alpha\neq 0}\frac{P^{(\alpha)}(x,\xi)D_x^\alpha(P(x,\xi)-z)^{-1}}{\alpha!}.$$

Let E_z be the asymptotic sum of the symbols of

$$(P-z)^{-1}(x,D)(Q_z(x,D))^N,\ N=0,1,\dots.$$

Then,

$$(P(x,D)-z))E_z(x,D)=I-W_z(x,D),$$

where W_z is uniformly bounded in $S((1+|z|)^{-1}(1+|\xi|)^{-N},g)$ for any N.

Up to now, we have worked only within coordinate patch Y. Choose ψ, $\Psi\in C_0^\infty(Y)$ with $\Psi=1$ on $\mathrm{spt}(\psi)$. Then, $\Psi E_z(x,D)\psi u$ is defined in X for every u in $\mathscr{D}'(X)$, and

$$(P-z)\Psi E_z(x,D)\psi u=\psi u-W_z(x,D)u$$

for another W_z such that $(1+|z|)W_z$ is uniformly bounded in $\Psi^{-\infty}$. Covering X by coordinate patches Y_j and selecting corresponding $\psi_j,\Psi_j\in C_0^\infty(Y_j)$ such that $\Psi_j=1$ on $\mathrm{spt}(\psi_j)$ and $\sum_j\psi_j=1$, we have a parametrix E_z satisfying

$$(P(x,D)-z))E_z(x,D)=I-W_z(x,D),$$

globally in X. Then, we have

$$R(z)=E_z+R(z)W_z,$$

where $(1+|z|)W_z$ is uniformly bounded from $H_{(s)}$ to $H_{(t)}$ for arbitrary s, t and

$$\|R(z)\|_{H_{(t)}\to H_{(t)}}\leq C/(1+|z|$$

since $\|u\|_{(t)} \equiv \|\mathscr{P}^{\frac{t}{m}} u\|_{(0)}$, at least if t is a positive multiple of m. Thus, we have for $a < 0$

$$\mathscr{P}^a = -(2\pi i)^{-1} \int_{-i\infty}^{i\infty} z^a E_z dz + T(a)u,$$

where $T(a)$ is an operator with kernel analytic in a when $\mathfrak{Re}(a) < 1$, with values in $C^\infty(X \times X; \Omega^{\frac{1}{2}}(X \times X))$. Every term in the symbol of E_z is of the form $-(z-P)^{-k-1} q dz$ where $q \in S^{mk-\kappa}$ for some $\kappa \geq 0$. We have

$$-(2\pi i)^{-1} \int_{-i\infty}^{i\infty} z^a (z-P)^{-k-1} = a(a-1)\dots(a-k+1) P^{a-k} q/k!,$$

which is a symbol in S^{am-k} and depends analytically on a. There are a finite number of terms for fixed κ. Terminating the series for E_z after sufficiently many terms adds an extra error to $T(a)$, yet this error will as many derivatives as we need for $\mathfrak{Re}(a) < 1$. Hence, $\mathscr{P}^a$ is a pseudo-differential operator for $a < 1$. Calculating the terms with $\kappa = 0$ or $\kappa = 1$, we find mod S^{am-2}, the symbol is equal to

$$P^a + \frac{a(a-1)}{2} i \sum p^{(j)} p_{(j)} p^{a-2}.$$

Since P is congruent to $p + p^s - \frac{1}{2} i \sum p^{(j)}_{(j)}$, it follows that P^a is congruent to

$$p^a + a p^{a-1} (p^s - \frac{1}{2} i \sum p^{(j)}_{(j)}).$$

To obtain the sum of the principal and subprincipal symbols for $\mathscr{P}^a$, we add $\frac{1}{2} i \sum \frac{\partial^2 p^a}{\partial x_j \partial \xi_j}$ to the symbol of $\mathscr{P}^a$, which proves the statement on the subprincipal symbol when $a < 1$. It follows immediately from the Weyl calculus that if $\mathscr{P}^a$ is defined by a pseudo-differential operator, this the same holds for $\mathscr{P}^{2a}$, with the principal and subprincipal symbols obtained by multiplication of those of $\mathscr{P}^a$. Hence, the result extends through to all $a \in \mathscr{R}$. □

13.9. Exercises

CHAPTER 14

Modulational Stability

14.1. Asymptotic Completeness

14.2. Scattering

14.3. U^p, V^p spaces

14.4. Set-up and Properties of the U^p, V^p Spaces for the Linear KdV Equation

app:u2v2

To define the function spaces U^2, V^2, we summarize Section 2 of Hadac-Herr-Koch [**?**], where we suggest the reader look for further details. Let $\mathcal{Z}$ be the set of finite partitions $-\infty < t_0 < t_1 < \ldots < t_K = \infty$. In the following, we consider functions taking values in $L^2 := L^2(\mathbb{R}^d;\mathbb{C})$, but in the general part of this section L^2 may be replaced by an arbitrary Hilbert space.

def:u

DEFINITION 14.4.1. *Let $1 \le p < \infty$. For $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset L^2$ with $\sum_{k=0}^{K-1} \|\phi_k\|_{L^2}^p = 1$ we call the function $a : \mathbb{R} \to L^2$ given by*

$$a = \sum_{k=1}^{K} \chi_{[t_{k-1},t_k)} \phi_{k-1}$$

a U^p-atom where χ_I is the standard cut-off function to interval I. Furthermore, we define the atomic space

$$U^p := \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j \;\middle|\; a_j \text{ a } U^p\text{-atom}, \ \lambda_j \in \mathbb{C} \text{ s.th. } \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\}$$

with norm

eq:norm_u

$$\|u\|_{U^p} := \inf\left\{ \sum_{j=1}^{\infty} |\lambda_j| \;\middle|\; u = \sum_{j=1}^{\infty} \lambda_j a_j, \ \lambda_j \in \mathbb{C}, \ a_j \text{ a } U^p\text{-atom} \right\}. \tag{14.1}$$

Atoms are bounded in the supremum norm, and hence every convergence here implies uniform convergence.

prop:u

PROPOSITION 14.4.2. *Let $1 \le p < q < \infty$.*

it:u_banach
(1) *The expression $\|.\|_{U^p}$ is a norm. The space U^p is complete and hence a Banach space.*

it:u_emb
(2) *The embeddings $U^p \subset U^q$ have norm 1.*

it:u_right_cont
(3) *For $u \in U^p$ all one sided limits exist, including at $\pm\infty$, u is continuous from the right, and the limit at $-\infty$ is zero.*

it:u_c
(4) *The subspace of continuous functions U_c^p is closed.*

def:v

DEFINITION 14.4.3. *Let $1 \le p < \infty$. We define V^p as the normed space of all functions $v : \mathbb{R} \to L^2$ for which the norm*

eq:hom_norm_v

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left(\sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{L^2}^p \right)^{\frac{1}{p}} \tag{14.2}$$

is finite. Here we understand $v(\infty)$ as zero. Let V_-^p denote the subspace of all right continuous functions with limit 0 at $-\infty$.

Taking the partition $\{t, \infty\}$ one sees that the supremum norm is not larger than the V^p norm.

prop:v

PROPOSITION 14.4.4. *Let $1 \le p < q < \infty$.*

it:v_limits
it:v_spaces
it:v_emb1
it:v_emb2

(1) *The expression $\|.\|_{V^p}$ is a norm and V^p is complete.*
(2) *For $v \in V^p$ all one sided limits including at $\pm\infty$ exist.*
(3) *The subspace V_-^p is closed.*
(4) *The embedding $U^p \subset V_-^p$ is continuous and $\|u\|_{V^p} \le 2^{1/p}\|u\|_{U^p}$.*
(5) *The embeddings $V^p \subset V^q$ are continuous and $\|v\|_{V^q} \le \|v\|_{V^p}$.*

From the proof of Proposition 2.17 of Hadac-Herr-Koch [**?**], we have the following

lem:UVlem

LEMMA 14.4.5. *Let $f \in V_-^p$, $q > p$. Then, given $\delta > 0$ and $m > 1$, there exist $f_1 \in U^p$ and $f_2 \in U^q$ such that $f = f_1 + f_2$ and*

$$m^{-1}\|f_1\|_{U^p} + e^{\delta m}\|f_2\|_{U^q} \lesssim \|f\|_{V^p}.$$

The following corollary is obvious.

COROLLARY 14.4.6. *The space V_-^p is continuously embedded in U^q for $q > p$.*

There is a bilinear map, B, which for $1/p + 1/q = 1$, $1 < p, q < \infty$ can formally be written as

$$B(f, g) = -\int f_t g \, dt,$$

for $f \in V^p$, $g \in U^q$. It satisfies

$$|B(f, g)| \le \|f\|_{V^p}\|g\|_{U^q},$$

which is natural if we replace g by an atom. The map

$$V^p \ni f \to (g \to B(f, g)) \in (U^q)^*$$

is an isometric bijection. Moreover,

$$\|u\|_{U^p} = \sup\{B(u, v) : v \in C(\mathbb{R}), \|v\|_{V^p} = 1\}.$$

If $v \in V_-^p$, then

$$\|v\|_{V^q} = \sup\{B(u, v) : u \in C(\mathbb{R}), \|u\|_{U^p} = 1\}.$$

If the distributional derivative of u is in L^1 and $v \in V^p$, then

$$B(u, v) = -\int u_t v \, dt.$$

Given $f \in L^1$, then $F(t) = \int_{-\infty}^t f \, ds \in V^p$ for all $p \ge 1$, and hence in $F \in U^p$. Moreover, $\|f\|_{DU^p} := \|F\|_{U^p} \le \|f\|_{L^1}$. We denote by DU^p the metric completion of L^1 in the norm given by the duality pairing. Similarly we define DV^q.

There is a close relation to Besov spaces, namely

embed

$$B_1^{\frac{1}{p},p} \subset U^p \subset V^p \subset B_\infty^{\frac{1}{p},p} \tag{14.3}$$

with continuous embeddings. These embeddings clarify the relation to $X^{s,b}$ spaces below.

We claim that the convolution with an L^1 function η defines a bounded operator on U^p and V^p with norm $\leq \|\eta\|_{L^1}$. Because of the duality statement it suffices to verify boundedness on U^p. We approximate the characteristic function by a sum of Dirac measures. The convolution with an atom clearly has norm at most 1. Convergence in U^1 to the convolution with the characteristic function is immediate. The full statement is an immediate consequence, as well as the boundedness of the convolution by a Schwarz function on U^p and V^p. In particular smooth projections on high and low frequencies are bounded.

Following Bourgain's strategy for the Fourier restriction spaces we define the adapted function spaces

$$\begin{aligned} U^p_{KdV} &= S(-t)U^p, \\ V^p_{KdV} &= S(-t)V^p \end{aligned}$$

and similarly DU^p and DV^p.

Again, we define a bilinear map B_{KdV} such that for $u \in V^p_{KdV}$, $v \in U^q_{KdV}$, we have for function u with $(\partial_t + \partial_x^3)u \in L^1L^2$

$$B_{KdV}(u,v) = -\int \langle (\partial_t + \partial_x^3)u, v\rangle \, dt.$$

Note, this bilinear map is well-defined and gives a duality relation. Hence,

$$\|u\|_{DV^p_{KdV}} = \sup_{\|f\|_{U^q_{KdV}} \leq 1} \int ufdxdt,$$

$$\|u\|_{DU^p_{KdV}} = \sup_{\|f\|_{V^q_{KdV}} \leq 1} \int ufdxdt.$$

Moreover, we may restrict f to suitable subspaces. More details on how the construction of such atomic spaces allows us to put u_t in the dual space are included in Hadac-Herr-Koch [**?**].

By the construction of our spaces we obtain for a solution u of the linear KdV equation

lin:kdv (14.4)
$$\begin{cases} u_t + u_{xxx} = f, \\ u(0,x) = u_0(x), \end{cases}$$

the estimates

eqn:linkdv1 (14.5)
$$\|u\|_{V^2_{KdV}} \lesssim \|u_0\|_{L^2} + \|f\|_{DV^2_{KdV}}$$

and

eqn:linkdv2 (14.6)
$$\|u\|_{U^2_{KdV}} \lesssim \|u_0\|_{L^2} + \|f\|_{DU^2_{KdV}},$$

which follow trivially from the construction of the V^2_{KdV}, DV^2_{KdV} and U^2_{KdV}, DU^2_{KdV} spaces.

Spatial Fourier multipliers act on U^p, V^q, DU^p, DV^q in the obvious way and their operator norm is bounded by the supremum of the multiplier.

Let (p,q) be a Strichartz pair. Then,

$$\|u\|_{L^pL^q} \leq c\||D|^{-\frac{1}{p}}u\|_{U^p}$$

and the dual estimate

$$\|f\|_{DV^{p'}} \leq c\||D|^{-\frac{1}{p}} f\|_{L^{p'}L^{q'}}$$

hold. The first estimate is not hard to check on atoms. Since convergence in U^p and in L^pL^q both imply pointwise convergence for subsequences we obtain the full estimate. The second estimate follows by duality.

Similarly the local smoothing estimates carry over to U^p spaces and to DV^q. Let $c(t)$ and $y(t)$ satisfy (**??**) Then

$$\|u\|_{L^2H^1_{\sqrt{\gamma'}})} \leq c\|u\|_{U^2}$$

and

$$\|f\|_{DV^2} \leq c\|f\|_{L^2\sqrt{\gamma'}H^{-1}}.$$

In the same fashion the bilinear estimates for solutions to the free equation imply bilinear estimates for functions in U^2.

The smooth decomposition into high and low modulation (i.e. the smooth projection of the frequencies to $\tau - \xi^3$ large respectively small) is bounded in U^2 and V^2, and the L^2 norm of the high modulation part gains the inverse of square root of the truncation as factor by the embeddings (14.3).

14.5. Exercises

CHAPTER 15

Bielefeld Lecture 10

Most of the work presented here spans the works [**MS11a**, **HMZ07a**, **MRS10**].

15.1. Numerical Bound States

We now explore computation methods for studying nonlinear bound states.

15.2. Shooting Methods

In [**McC93**], uniqueness of the soliton is proved for a large class of nonlinearities. To prove this, they work in radial coordinates and hence prove that for solutions of the resulting Ordinary Differential Equation (ODE), there is a unique initial value at the origin which leads to a decreasing, positive solution by shooting. If this value is shifted downward, there are an infinite number of extrema and the solution is always positive. If the value is shifted upward, the solution becomes negative and is in fact, monotone decreasing. Hence, we use this analysis to numerically find the unique soliton.

Specifically, we use the method of spectral deferred correction coupled with a Linearly Implicit Euler (LIE) scheme on the radial ODE. It is possible for solitons of NLS equations with monomial nonlinearities to be found using a spectral scheme described in [**DS05**], however, the convergence of the method depends heavily upon a scaling factor. Developing such a sophisticated method for saturated nonlinearities will be a topic of further research. We specifically choose linearly implicit Euler because it allows us to avoid the singularity at the origin in radial coordinates. We use deferred correction to get an incredibly accurate method without having a large number of grid points.

To begin, we compute a first approximation u^0 to

$$\partial_t u = f(t,u),$$

at m stretched Legendre points

$$s_1 = T + H(1+t_1)/(1+t_m), \ldots, s_m = T + H(1+t_m)(1+t_m) = T + H,$$

where t_i is the i-th root of the m-th Legendre polynomial. Then, we have

$$(I - h_j Df(s_{j+1}, u_j^0))(u_{j+1}^0 - u_j^0) = h_j f(s_{j+1}, u_j^0),$$

for $j = 0, 1, \ldots, m-1$, $h_j = s_{j+1} - s_j$. Then, we need to estimate the error caused by the numerical approximation by a repeated correction step taking $u_k = (u_1^k, ..., u_m^{k+1})$ to $u^{k+1} = (u_1^{k+1}, ..., u_m^{k+1})$.

We have the itegral formulation given by

$$u(t) = u_0 + \int_0^t f(s, u(s))ds,$$

and define the residual to be

$$R(t) = u_0 + \int_0^t f(s, u(s))ds - u(t).$$

An order m accurate appoximate solution for R at the Legendre points is given by

$$R_n^k = u_0 + \sum_{j=1}^{m} S_{nj} f(s_j, u_j^k) - u_n^k,$$

where S_{nj} are elements of the spectral integration matrix derived from Gaussian quadrature. If y is the exact solution, the error E can be defined as

$$\begin{aligned} E(t) &= y(t) - u(t) \\ &= u_0 + \int_0^t f(s, y(s))ds - (u_0 + \int_0^t f(s, u(s))ds - R(t)) \\ &= R(t) + \int_0^t (f(s, u(s) + E(s)) - f(s, u(s)))ds \\ &= R(t) + \int_0^t G(s, E(s); u(s))ds, \end{aligned}$$

where $G(s, E; u) = f(s, u + E) - f(s)$. Then using linearly implicit Euler, we have

$$(I - hDf(s_{n+1}, u_{n+1}^k))E_{n+1}^k = E_n^k + R_{n+1}^k - R_n^k.$$

Hence, we set

$$u_n^{k+1} = u_n^k + E_n^k,$$

for $n = 1, 2, ..., m$. Each time we correct, we gain one order of accuracy, up to order m. Hence, we have a highly accurate method with a small number of relative grid points.

sec:sinc_disc

15.2.1. Sinc Discretization. See [**MRS10**] The problem of finding a soliton solution of (1.1) is a nonlinear boundary value problem posed on $\mathbb{R}$. We respect this description in our discretization by approximating functions with the Sinc spectral method. This technique is thoroughly explained in [**LB92**, **Ste93b**, **Ste81**, **Ste00**]. In the Sinc discretization, the problem remains posed on $\mathbb{R}$ and the boundary conditions, that the solution vanish at $\pm\infty$, are naturally incorporated.

Given a function $u(x) : \mathbb{R} \to \mathbb{R}$, u is approximated using a superposition of shifted and scaled Sinc functions:

eq:sinc_approx

$$C_{M,N}(u, h)(x) \equiv \sum_{k=-M}^{N} u_k \mathrm{Sinc}\left(\frac{x - x_k}{h}\right) = \sum_{k=-M}^{N} u_k S(k, h)(x), \tag{15.1}$$

where $x_k = kh$ for $k = -M, \ldots, N$ are the *nodes* and $h > 0$. There are *three* parameters in this discretization, h, M, and N, determining the number of and spacing of lattice points. This is common to numerical methods posed on unbounded domains; see [**Boy01**].

A useful and important feature of this spectral method is that, when evaluated at a node,

eq:disc_delta

$$C_{M,N}(u, h)(x_k) = u_k. \tag{15.2}$$

Additionally, the convergence is rapid both in practice and theoretically, see literature.

Since the soliton is an even function, we may take $N = M$. We will thus write

$$C_M(u, h)(x) \equiv C_{M,M}(u, h)(x). \tag{15.3}$$

The symmetry implies $u_{-k} = u_k$ for $k = -M, \ldots M$. We take advantage of this constraint in our computations. In addition, we slave h to M in accordance with standard convergence analysis techniques.

To compute a discrete sinc approximation of the ground state, we frame the soliton equation as a *nonlinear collocation problem.* Approximating $\phi(x)$ as in (15.1), we seek coefficients $\{R_k\}$ such that

eq:collocation_eqs

$$\begin{aligned}\partial_x^2 C_M(\phi,h)(x_k) - \lambda C_M(\phi,h)(x_k) \\ + g(|C_M(\phi,h)(x_k)|^2) C_M(\phi,h)(x_k) = 0, \quad \text{for } k = -M,\ldots,M.\end{aligned} \tag{15.4}$$

By satisfying (15.4), the discrete approximation solves the soliton equation in the *strong* sense at the nodes, also known as *collocation points.* This is in contrast to a Galerkin formulation, which solves the equation in the weak sense. However, for the one dimensional under consideration, sinc-Galerkin and sinc-collocation lead to the same algebraic system.

(15.4) yields a system of nonlinear algebraic equations. Let $\vec{\phi}$ be the column vector associated with the discrete approximation of ϕ:

$$C_M(\phi,h)(x_k) \mapsto \vec{\phi} = \begin{pmatrix} \phi_{-M} \\ \phi_{-M+1} \\ \vdots \\ \phi_M \end{pmatrix}. \tag{15.5}$$

Differentiation of a Sinc approximated function that is evaluated at the collocation points corresponds to matrix multiplication:

$$\partial_x^2 C_M(\phi,h)(x_k) \mapsto D^{(2)}\vec{\phi}. \tag{15.6}$$

Explicitly, $D^{(2)}$ is

eq:matrix_d2

$$D^{(2)}_{jk} = \frac{d^2}{dx^2} S(j,h)(x)|_{x=x_k} = \begin{cases} \frac{1}{h^2}\frac{-\pi^2}{3} & j = k \\ \frac{1}{h^2}\frac{-2(-1)^{k-j}}{(k-j)^2} & j \neq k \end{cases}. \tag{15.7}$$

Using 15.2,

$$g(|C_M(\phi,h)(x_k)|^2) C_M(\phi,h)(x_k) = C_M(g(|\phi|^2)\phi,h)(x_k) = g(|\phi_k|^2)\phi_k.$$

Thus

$$g(|C_M(\phi,h)(x_k)|^2) C_M(\phi,h)(x_k) \mapsto g(|\vec{\phi}|^2)\vec{\phi} \equiv \begin{pmatrix} g(|\phi_{-M}|^2)\phi_{-M} \\ \vdots \\ g(|\phi_M|^2)\phi_M \end{pmatrix}.$$

With these relations, the discrete system is

eq:disc_solition_eq

$$D^{(2)}\vec{\phi} - \omega\vec{\phi} + g(|\vec{\phi}|^2)\vec{\phi} = 0. \tag{15.8}$$

It is this equation to which we apply a nonlinear solver, subject to an appropriate guess. We discuss an important subtlety in Section 15.2.3.

int:sat

15.2.2. Saturated Nonlinear Schrödinger Equations. In this section, we develop the techniques used to prove stability of solitons for a focusing nonlinear Schrödinger equation (NLS) in $\mathbb{R}\times\mathbb{R}^d$:

$$\begin{aligned} i u_t + \Delta u + \beta(|u|^2)u &= 0 \\ u(0,x) &= u_0(x), \end{aligned}$$

for $\beta : \mathbb{R} \to \mathbb{R}$, $\beta(s) \geq 0$ for all $s \in \mathbb{R}$.

As a source of interesting calculations, we will need the following definitions.

def:non1

DEFINITION 15.2.1. *Saturated nonlinearities of type* 1 *are of the form*

sat:eqn1

$$\beta(s) = s^{\frac{q}{2}} \frac{s^{\frac{p-q}{2}}}{1+s^{\frac{p-q}{2}}}, \tag{15.9}$$

where $p > 2 + \frac{4}{d}$ *and* $\frac{4}{d} > q > 0$ *for* $d \geq 3$ *and* $\infty > p > 2 + \frac{4}{d} > \frac{4}{d} > q > 0$ *for* $d < 3$.

def:non2

DEFINITION 15.2.2. *Saturated nonlinearities of type* 2 *are of the form*

sat:eqn2

$$\beta(s) = \frac{s}{(1+s)^{\frac{2-q}{2}}}, \tag{15.10}$$

where $\frac{4}{d} > q > 0$, $d > 2$.

REMARK 15.2.3. *In both cases, for* $|u|$ *large, the behavior is* L^2 *subcritical and for* $|u|$ *small, the behavior is* L^2 *supercritical. For Definition 15.2.1,* p *is chosen much larger than the* L^2 *critical exponent,* $\frac{4}{d}$ *in order to allow sufficient regularity in linearizing the equation.*

Now, that we have established the method, we begin with a range of values and based on the analysis of [**McC93**], we bisect the interval until we have a very accurate approximation to the soliton. A plot of a known soliton for (NLS) on $\mathbb{R}$ compared to the computed soliton is shown in Figure 2. While it is not a particularly useful exercise, we show a plot of a soliton for a saturated nonlinearity in Figure 1. Using such a method, we can also approximately track the quantities $Q(\lambda)$ and $E(\lambda)$, conservation of mass and energy. We call the plots of Q and E with respect to λ the *soliton curve* for a particular nonlinearity. A sketch of the soliton curves for various nonlinearities is given in Figure 3. Notice that for the saturated nonlinearities, Q has a unique minimal mass attained at a unique λ.

sec:continuation

15.2.3. Numerical Continuation. As discussed in Section 15.2.1, the discrete system approximating (1.1) is

eq:discrete_satnls

$$\vec{F}(\vec{\phi}) = D^{(2)}\vec{\phi} - \omega\vec{\phi} + g(\vec{\phi})\vec{\phi} = 0. \tag{15.11}$$

The multiplication in $g(\vec{\phi})\vec{\phi}$ is performed elementwise. In order to solve this discrete system, we need a good starting point for our nonlinear solver. We produce this guess by numerical continuation.

Define the function

$$\hat{g}(x;\tau) = \frac{x^3}{1+\tau x^2}.$$

Note that $\hat{g}(x,0)$ is 7-th order NLS and $\hat{g}(x,1) = g(x)$, saturated NLS. We now solve

eq:discrete_satnls_mod

$$\vec{G}(\vec{\phi};\tau) = D^{(2)}\vec{\phi} - \lambda\vec{\phi} + \hat{g}(\vec{\phi};\tau)\vec{\phi} = 0. \tag{15.12}$$

At $\tau = 0$, the analytic NLS soliton serves as the initial guess for computing $\vec{\phi}_{\tau=0}$. $\vec{\phi}_{\tau=0}$ is then the initial guess for solving (15.12) at $\tau = \Delta\tau$. We iterate in τ until we reach $\tau = 1$. This is numerical continuation in the artificial parameter τ, [**AG90**]. This process succeeds with relatively few steps of $\Delta\tau$; in fact only $O(10)$ steps are required.

sec:other_methods

15.2.4. Comparisons with Other Methods. We can benchmark our Sinc algorithm against several other methods. Available algorithms include numerical quadrature along with more recent approaches such as spectral renormalization, also called the Petviashvili method [**Pet76**, **AM05**, **LY07**], the imaginary time method [**BD04**, **YL08**], and the squared operator method [**YL07**].

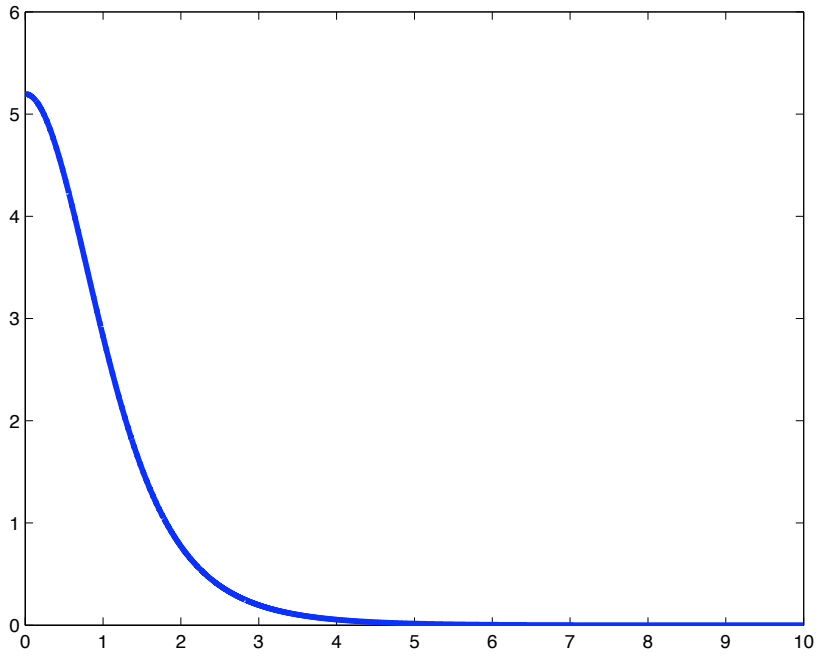

FIGURE 1. The plot of a numerically found soliton for a saturated NLS of type 2 in $\mathbb{R}^3$ with $q = 1$.

f:sol

TABLE 1. The convergence of the sinc discretization to the minimal mass soliton.

tab:mass_conv

M	$h\|\vec{\phi}\|^2$	ω^*
20	3.820771417633398	0.177000229690401
40	3.821145471868853	0.177576993694258
60	3.821148930202135	0.177587655985074
80	3.821149018422933	0.177588043323139
100	3.821149022493814	0.177588063805561
200	3.821149022780618	0.177588065432740
300	3.821149022780439	0.177588065432795
400	3.821149022780896	0.177588065433095
500	3.821149022780275	0.177588065432928

sec:quad_root

15.2.4.1. *Quadrature Methods.* The soliton equation may be integrated once to get

eq:first_integral

$$\frac{1}{2}(\partial_x\phi)^2 - \frac{1}{2}\lambda\phi^2 + \frac{1}{4}\left[\phi^4 - \log\left(1+\phi^4\right)\right] = 0. \tag{15.13}$$

Equation (15.13) yields an implicit algebraic expression for the amplitude, $\phi(0)$,

eq:amp

$$-\frac{1}{2}\lambda\phi(0)^2 + \frac{1}{4}\left[\phi(0)^4 - \log\left(1+\phi(0)^4\right)\right] = 0. \tag{15.14}$$

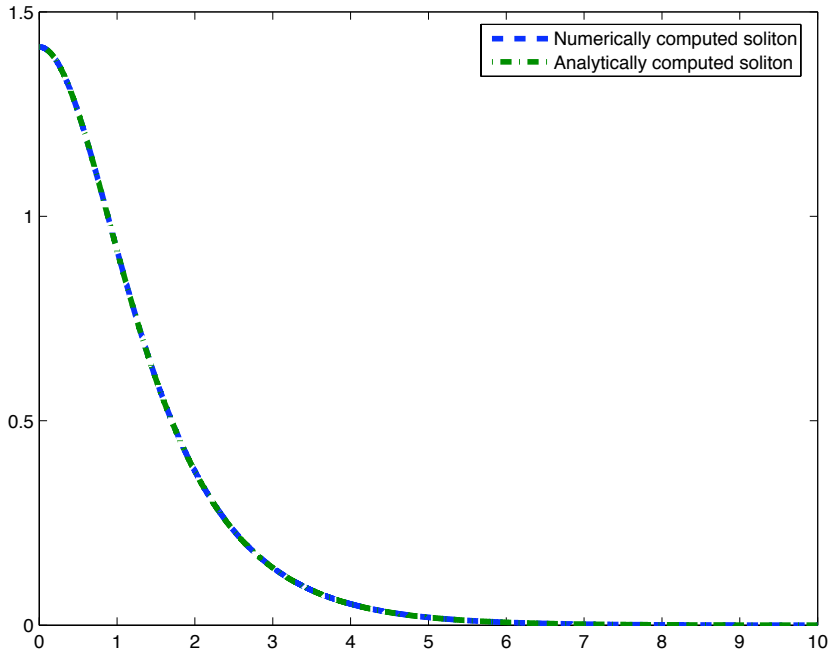

FIGURE 2. The plot of a numerically found soliton for cubic NLS in $\mathbb{R}$ vs. the known analytic soliton $R_1(x) = \sqrt{2}\mathrm{sech}(x)$.

f:solcomp

Using (15.13) and (15.14), we can express the mass as

$$\|\phi\|_{L^2}^2 = \int_{-\infty}^{\infty} \phi(x)^2 dx = 2\int_0^{\infty} \phi(x)^2 dx = 2\int_0^{\phi(0)} \rho^2 \left\{\lambda\rho^2 - \frac{1}{2}\left[\rho^4 - \log\left(1+\rho^4\right)\right]\right\}^{-1/2} d\rho.$$

Thus, the mass of the soliton with parameter ω is

eq:mass_integral

$$\|\phi_\omega\|_{L^2}^2 = 2\int_0^{\phi(0;\omega)} \rho^2 \left\{\omega\rho^2 - \frac{1}{2}\left[\rho^4 - \log\left(1+\rho^4\right)\right]\right\}^{-1/2} d\rho. \tag{15.15}$$

Equations (15.14) and (15.15) can be used to approximate ω^* by numerically minimizing (15.15). To compute the amplitude of the soliton, we solve (15.14) using Brent's method with a tolerance of 1.0e-14. We use the singular integral integrator QAGS from QUADPACK, which for this problem is, unfortunately, limited to a relative error of 5.0e-12 and an absolute error of 1.0e-15. Trying different routines from the optimization module of SciPy, [**JOP**$^+$], we summarize our results in Table 2, which contains data from our Sinc computations. There is a spread of O(1e-12) amongst the computed minimal masses and a spread of O(1e-7) amongst the ω^*. These differences are consistent with the prescribed relative error of the quadrature, suggesting the precision of this approach to computing the minimal mass and associated ω is limited by the quadrature algorithm. Note that we do not compute the solitary wave with this technique; we merely identify the minimal mass soliton parameter and the mass of that soliton.

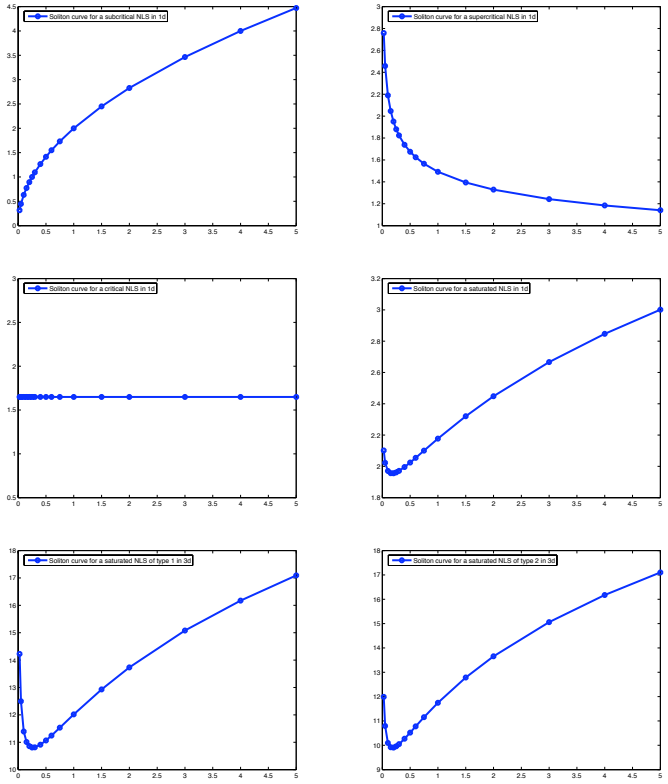

FIGURE 3. Plots of the soliton curves ($Q(\lambda)$ with respect to λ) for a subcritical nonlinearity ($d = 1,\ p = 3$), supercritical nonlinearity ($d = 3,\ p = 3$), critical nonlinearity ($d = 1,\ p = 5$), saturated nonlinearity of type 1 ($p = 7,\ q = 3$) in $\mathbb{R}$, saturated nonlinearity of type 1 in $3d$ ($p = 4,\ q = 2$), saturated nonlinearity of type 2 in $\mathbb{R}^3$ ($q = 2$). The curves for the monomial nonlinearities are found analytically, while the curves for the saturated nonlinearities are found numerically. f:solcurves

sec:spec_re

15.2.4.2. *Spectral Renormalization Methods.* A Fourier transform may be applied to the soliton equation to get

eq:soleq_ft

$$\hat{\phi_\omega}(k) = \frac{\mathcal{F}(g(\phi_\omega^2)\phi_\omega)(k)}{k^2+\omega} \tag{15.16}$$

where k is the the wave number. Let us introduce variable $w(x)$, with $\phi_\omega(x) = \theta\, w(x)$,where θ is an unknown, nonzero, constant. Introducing this into (15.16), we have

eq:weq

$$\hat{w}(k) = \frac{\mathcal{F}(g(\theta^2 w^2)\, w)(k)}{k^2+\omega} \equiv \mathcal{Q}_\theta[\hat{w}](k). \tag{15.17}$$

Multiplying (15.17) by $\hat{w}(k)^*$, the complex conjugate, and integrating over all k, we compute

$$G(\theta;\hat{w}) \equiv \int |\hat{w}(k)|^2 dk - \int \hat{w}(k)^* \mathcal{Q}_\theta[\hat{w}](k)\, dk = 0. \tag{15.18}$$

table:other_results

TABLE 2. The soliton parameter and mass of the minimal mass soliton computed by both quadrature and spectral renormalization. Also included is some of the data for the Sinc method appearing in Table 1.

Algorithm	ω^*	$\int \lvert\phi\rvert^2 dx$
fminbound	0.177588368745261	3.821149022780204
Brent	0.177587963826864	3.821149022778472
golden	0.177587925853761	3.821149022776717
Spec. Re.	0.177588064106709	3.821149022780361
Sinc with $M = 100$	0.177588063805561	3.821149022493814
Sinc with $M = 200$	0.177588065432740	3.821149022780618
Sinc with $M = 400$	0.177588065433095	3.821149022780896

This may be interpreted as a constraint on θ. This motivates the iteration described in [**AM05**]. Suppose we know $w_m(x)$ and θ_m, a pair of approximations of the true values. To get the next approximation, we compute

$$\hat{w}_{m+1}(k) = \mathcal{Q}_{\theta_m}[\hat{w}_m](k) \tag{15.19}$$

and then solve

eq:theta_iterate

$$G(\theta; \hat{w}_{m+1}) = 0 \tag{15.20}$$

for θ_{m+1}. We repeat this until the sequence $\{w_m\}$ satisfies our convergence criteria. From this we then recover ϕ_ω. The advantage of this is it can use the fast Fourier transform. Using spectral renormalization, we then minimize the approximate mass numerically. This is readily implemented in MATLAB, using `fzero` to solve (15.20) for a given value of ω and `fminbnd` to find the minimal mass value of ω. We iterate in w until either $\|w_{m+1} - w_m\|_{l^2} <$ Abs. Tol. or $\|w_{m+1} - w_m\|_{l^2} / \|w_{m+1}\|_{l^2} <$ Rel. Tol.. This is performed with a relative and absolute tolerances of 1e-15 in the spectral renormalization component, and a tolerance of 1e-15 in both `fminbnd` `fzero`. Our spatial domain is $[-200, 200)$ with 2^{12} grid points. As seen in Table 2, this is in good agreement with both the Sinc method and the quadrature method.

CHAPTER 16

Bielefeld Lecture 11

In this note, we introduce a convergent iterative ODE solver and prove it convergence, see [**DS05**, **PS04**].

16.1. Modified Petviashvili Method

Such a method is explicitly designed to numerically solve

eqn:usol

$$-\Delta u + u - u^{2\beta+1} = 0, \tag{16.1}$$

by defining an iteration scheme

$$u_{n+1} = \lambda_n^{\gamma}(I-\Delta)^{-1}(|u_n|^{2\beta}u_n) + \delta\sum_{j=1}^{d}\rho_{n,j}\frac{\partial u}{\partial x_j},$$

for a choice of λ_n, γ, δ and $\vec{\rho}_n$ with u_0 chosen to be something nicely localized, say a Gaussian. The parameters for convergence are defined as

$$\begin{aligned} M_n &= \frac{\int(1+|\xi|^2)(\hat{u}_n)^2 d\xi}{\int \hat{u}_n(\widehat{|u_n^{2\beta}|u_n})d\xi}, \\ \rho_{n,j} &= \frac{\int(1+|\xi|^2)(\hat{u}_n)(\overline{\partial_j u_n})d\xi}{\int \partial_j\hat{u}_n(\widehat{\partial_j(|u_n^{2\beta}|u_n)})d\xi} \\ \gamma &= \frac{2\beta+1}{2\beta}, \\ \delta &= -\frac{1}{2}. \end{aligned}$$

The choice of parameters above will become clear in the proof of convergence. The slight correction that differs strongly from Spectral Renormalization method will arise naturally as a means of enforcing orthogonality conditions that are related to those from the orbital stability section.

16.2. Proof of Convergence

CLAIM 16.2.1. *Given u the unique radial, decreasing solution to* (16.1)*, we have*

$$\|u_{n+1}-u\|_{H^1} \le (1-c)\|u_n-u\|_{H^1}$$

in an open neighborhood of u

PROOF. Starting with the case $\delta = 0$ and defining $w_n = u_n - u$, we have

$$\hat{w}_{n+1} = -2\beta\gamma a_n\hat{u} + (2\beta+1)\frac{\widehat{u^{1+2\beta}} * \hat{w}_n}{1+|\xi|^2}$$

with

$$a_n = \frac{\int w_n u^{1+2\beta}}{\int u^{2+2\beta}}.$$

Defining $A = (I-\Delta)^{-1}L_+$, we see using the properties of L_+ defined above that since this is a bounded, self-adjoint operator on H^1 with respect to the inner product

$$(f,g) = \int f(I-\Delta)g\,dx.$$

Decomposing onto the spectrum of A, we note that since $(I-\Delta)^{\frac{1}{2}}$ is well defined, we have that $\dim(\mathrm{Ker}(A)) = \dim(\mathrm{Ker}(L_+)) = d$ and that A, like L_+ has one negative eigenvalue. Indeed, the $d+1$ lowest eigenvalues of A are u with eigenvalue $-2\beta < 0$ and $\partial_j u$ with eigenvalue 0 by a direct computation using (16.1) to say

$$(I-\Delta)^{-1}u^{1+2\beta} = u.$$

Hence, expanding $w_n = a_n u + q_n$ by projecting onto the non-negative eigenspace of A using that w_n is always radial and hence ignoring the projections onto $\partial_j u$. Now, q_n lives in the orthogonal subspace

$$Y = \{f \in L^2 | \langle f, \phi^{2\beta+1}\rangle = \langle f, \partial_j u^{2\beta+1}\rangle = 0\}.$$

To see this, simply calculate that the adjoint of u with respect to the H^1 inner product is u^p. There is a similar calculation for $\partial_j u$ since ∂_j commutes with $I-\Delta$ for all ∂_j. Hence, we can see from the positivity of u that that operator A is such that

$$0 < \lambda_{d+2} \leq \inf_{f\in Y}\frac{(f,Af)}{(f,f)} < \sup_{f\in L^2}\frac{(f,Af)}{(f,f)} = 1 - p\inf_{f\in L^2}\frac{(f,u^{2\beta}f)}{(f,f)} = 1$$

by taking f to be functions supported further and further from the essential support of u. Computing by plugging into the iteration that

$$\begin{aligned} a_{n+1} &= (2\beta(1-\gamma)+1)a_n, \\ q_{n+1} &= (I-A)q_n. \end{aligned}$$

Hence, we have

$$w_{n+1} = G(w_n),$$

with

$$G'(u)w = (I-A)P_Y w,$$

but $I - A \leq 1 - \lambda_{d+1} < 1$, hence a contraction argument completes the proof.

For $\delta \neq 0$, the Petviashvili correction projects the initial condition away from $\partial_j u$ and gives the natural choice $\delta = -\frac{1}{2}$. Specifically, given

$$w_n = a_n u + b_{n,j}\partial_j u + q_n$$

the recurrence relation gives

$$b_{n+1,j} = (1+2\delta)b_{n,j}.$$

□

16.3. Exercises

CHAPTER 17

Bielefeld Lecture 12

17.1. Stability

The stability of solitons is a natural property to investigate numerically, but first we must clarify what kind of stability we seek. Some of the first results on NLS concerned the *orbital* stability of the solitons. For NLS, the soliton is said to be orbitally stable if the perturbation remains small the H^1 norm, modulo the group of symmetries associated with the equation, *i.e.* translation in space. In [**Wei85b**, **Wei86**, **GSS90**], Weinstein and Grillakis, Shatah & Strauss determined that the qustion of orbital stability of NLS solitons could be reduced to computing the sign of

e:slope_condition

$$\frac{d}{d\lambda}\int |R(\mathbf{x};\lambda)|^2 d\mathbf{x}. \tag{17.1}$$

When this is positive, the solitons are orbitally stable; when it is negative, they are unstable.

Though these orbital stability results are quite elegant, they do not tell us what happens as $t \to \infty$. In particular, orbital stability does not imply that the perturbation diminishes; the solution may perpetually oscillate about the soliton. For information on long time behavior, we seek *asymptotic* stability, which generally states that a perturbation of the soliton converges to a (possibly different) bound state and radiation. In this work, we study certain details necessary for asymptotic stability.

Asymptotic stability is usually proven perturbatively. Writing $\psi = e^{i\lambda t}(R(\cdot;\lambda)+\phi)$, the evolution of the perturbation, ϕ, is governed by

e:perturbation_cc

$$\partial_t \begin{pmatrix}\phi\\ \phi^*\end{pmatrix} = i\mathscr{H}\begin{pmatrix}\phi\\ \phi^*\end{pmatrix} + \mathbf{F}(\phi,\phi^*) = i\begin{pmatrix}-\Delta+\lambda-V_1 & -V_2\\ V_2 & \Delta-\lambda+V_1\end{pmatrix}\begin{pmatrix}\phi\\ \phi^*\end{pmatrix} + \mathbf{F}(\phi,\phi^*). \tag{17.2}$$

The operator $\mathscr{H}$ is the linearized (about a soliton) operator of NLS, and $V_1 = g(R^2) + g'(R^2)R^2$ and $V_2 = g'(R^2)R^2$; this is the *matrix Hamiltonian*. The perturbation $\mathbf{F}$ contains the nonlinear interactions. We could have also decomposed the perturbation into real and imaginary parts, $\phi = u + iv$, to get the evolution equation

e:perturbation_ri

$$\partial_t \begin{pmatrix}u\\ v\end{pmatrix} = JL\begin{pmatrix}u\\ v\end{pmatrix} + \mathbf{G}(u,v) = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}L_+ & 0\\ 0 & L_-\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} + \mathbf{G}(u,v) \tag{17.3}$$

where $L_\pm = -\Delta + \lambda - V_\pm$ and $\mathbf{G}$ contains the nonlinear terms. The potentials are $V_+ = V_1 + V_2$ and $V_- = V_1 - V_2$. The isometry between these two formulations permits us to go back and forth; we shall do this frequently.

Proofs of asymptotic stability typically require Strichartz estimates on the evolution operator of the form

e:strichartz

$$\|e^{i\mathscr{H}t}f\|_{L_t^p L_x^q} \lesssim \|f\|_{L_x^r}. \tag{17.4}$$

With such estimates, (17.2) is shown to be dominated by the linear flow, permitting the nonlinear term to be treated as a perturbation. This strategy for NLS was initiated by

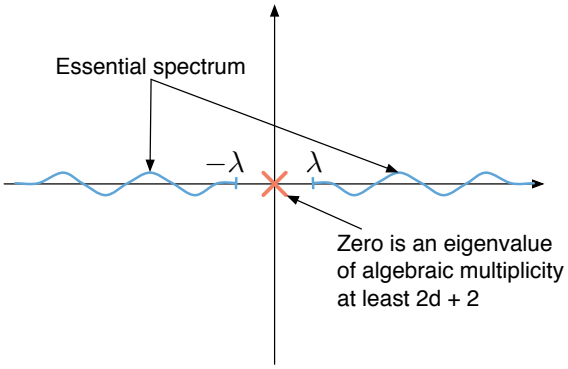

FIGURE 1. The generic spectrum of $\mathscr{H}$. Endpoint resonances might be found at $\pm\lambda$, and embedded eigenvalues may be located in $(-\infty,-\lambda]\cup[\lambda,\infty)$. Gap eigenvalues may be found in $(-\lambda,\lambda)$. fig:spec_decomp

Buslaev & Perelman, [**BP95**], who studied the stability of a single soliton known to be orbitally stable. There approach has been followed in many works with various restrictions on the perturbations and assumptions on the nonlinearity. Additional assumptions on the spectrum of the linearized operator are also often necessary. As examples, we point the reader to Buslaev & Sulem, [**BS03**], which generalizes the earlier work, and to Cuccagna and Rodnianski, Schlag, & Soffer, [**Cuc03**, **RSS03a**], who studied the stability of multiple soliton solutions. Schlag and Krieger & Schlag, [**Sch09**, **KS06b**], studied the stability on an orbitally *unstable* soliton, subject to constraints that mitigate the linear instability.

17.1.1. The Spectrum. Dispersive estimates on $e^{i\mathscr{H}t}$ require careful considerations of the spectrum of $\mathscr{H}$, $\sigma(\mathscr{H})$. Since the spectrum of $\mathscr{H}$ and JL are related by $\sigma(\mathscr{H}) = i\sigma(JL)$, we can also study that operator. Generically, the spectrum of $\mathscr{H}$ includes the points highlighted in Figure 1. Using well known results on relatively compact perturbations of differential operators, we can assert that the essential spectrum lies in $(-\infty,-\lambda]\cup[\lambda,\infty)$, [**HS96**, **RS78b**]. See [**ES06**] for a nice description of the essential spectrum for matrix Hamiltonian. By direct computation on JL, we can find a $2d+2$ dimensional kernel; details are given below in Section 17.1.1.1. This implies the $2d+2$ dimensional kernel of $\mathscr{H}$, see [**MS11a**].

Bound states corresponding to eigenvalues of $\mathscr{H}$, along with resonances, can obstruct the necessary dispersive estimates. In many of the results on asymptotic stability, the authors explicitly assume:

(1) $\mathscr{H}$ has no eigenvalues in the *essential* spectrum. Eigenvalues contained in the essential spectrum are called *embedded* eigenvalues. These will be the focus of this work.
(2) The only real eigenvalue in the *spectral gap*, $[-\lambda,\lambda]$, is zero.
(3) The values $\pm\lambda$, the endpoints of the essential spectrum, are not resonances.

See [**MS11b**] for a discussion related to these conditions. These properties, which are often assumed, motivate two problems:

p:embedded **Open Problem 17.1.1** *Prove that $\mathscr{H}$ has no embedded eigenvalues.*

and

p:endpoint **Open Problem 17.1.2** *Prove that $\mathscr{H}$ has no endpoint resonances.*

One application we can study numerically is the spectrum of $\mathscr{H}$, and resolve the questions of embedded eigenvalues and endpoint resonances for some cases of (12.1). We believe our approach may provide a framework applicable to other problems. The question of gap eigenvalues is more subtle, as they may exist in subcritical problems where the soliton is orbitally stable; we will not address this here.

REMARK 17.1.3. *Krieger & Schlag, [**KS06b**], resolved these questions for 1D supercritical power nonlinearities using scattering theory ideas similar to those of Perelman, [**Per01**]. Interestingly, our algorithm only works for a subset of supercritical monomial nonlinearities in 1D. The existence of both embedded eigenvalues and embedded resonances for operators of the form $\mathscr{H}$ requires precise asymptotic matching with direct algebraic dependence upon the forms of the potentials. They are thus believed to be rare. However, in higher dimensions, neither can be ruled out analytically, even under strong assumptions on the symmetry, regularity and decay of the potentials. See Schlag's discussion in [**Sch09**].*

REMARK 17.1.4. *Linearizing about the ground state soliton for the cubic nonlinear Schrödinger equation in 1D, which is L^2 subcritical, does actually result in endpoint resonances, see [**CGNT08**]. Despite this, asymptotic stability still holds due to the celebrated inverse scattering theory. In higher dimensions, as seen by [**ES06**], endpoint resonances result in weaker dispersive estimates that make proving asymptotic stability rather challenging.*

REMARK 17.1.5. *In Cuccagna & Pelinovksy and Cuccagna, Pelinovsky, & Vougalter [**CP05**, **CPV05**], it is shown that embedded eigenvalues of positive Krein signature in 1D can be dispersion managed via a Fermi Golden Rule approach, and that this is a generic assumption. However, it is unknown if this can be extended to higher dimensions and embedded eigenvalues still complicate the analysis. Hence, it may be simpler to prove their absence.*

specnum:disc 17.1.1.1. *Results on the Discrete Spectrum of $\mathscr{H}$.* We wish to show that the spectrum of $\mathscr{H}$ when linearized about the ground state has the discrete spectral decomposition Figure 1. Namely, the following result holds

Theorem 17.1
The only discrete eigenvalue for $\mathscr{H}$ in the interval $[-\lambda, \lambda]$ is 0.

REMARK 17.1.6. *For the 3D monomial NLS equation, the structure of the discrete spectrum away from the essential spectrum has been verified numerically in [**DS05**] for a range of supercritical exponents.*

REMARK 17.1.7. *In [**Sch09**], using arguments derived from [**Per01**], it is shown that the discrete spectrum for the Hamiltonian $\mathscr{H}$, as in* (17.2)*, associated with supercritical monomial nonlinearities is determined by the discrete spectrum of L_+. A slightly stronger version of the theorem, with minimal changes to the proof, applies to linearizations about a minimal mass soliton, $R = R_{\min}$, of saturated nonlinearities.*

Let us review the generalized kernel of a Hamiltonian resulting from linearizing about a soliton. Studying JL and following [**Wei85b**], we see by direct calculation that the vectors

$$\begin{pmatrix} 0 \\ R \end{pmatrix}, \quad \begin{pmatrix} R_j \\ 0 \end{pmatrix} \tag{17.5}$$

for all $j = 1, \ldots, d$ are contained in $\mathrm{Ker}(JL)$. Differentiating (1.1) with respect to λ, we have by a simple calculation that $L_+ \partial_\lambda R = -R$. Differentiating with respect to $\mathbf{x}$, $L_-(\mathbf{x}R) = -2\nabla R$. Hence, the vectors

$$\begin{pmatrix} 0 \\ x_j R \end{pmatrix}, \quad \begin{pmatrix} (\partial_\lambda R)_{\lambda_0} \\ 0 \end{pmatrix} \tag{17.6}$$

lie in the generalized null space. So far we have constructed at $2d+2$ dimensional generalized null space. Since we know the null spaces of L_- and L_+ exactly, these are unique. For power nonlinearities, $g(s) = s^\sigma$,

$$(\partial_\lambda R)_{\lambda_0 = 1} = \frac{1}{2}\left(\frac{1}{\sigma} R + \mathbf{x} \cdot \nabla R\right). \tag{17.7}$$

We use this explicit form in our calculations. See in [**Sch09**, **KS06b**, **Wei85b**] for a nice analytic description of the discrete spectrum.

sec:verification

17.1.2. Natural Orthogonality Conditions. Given that L_+, L_- have kernels, we must establish any stability/spectral results on a subspace. This subspace will be defined as the orthogonal complement to the span of a set of vectors. If this collection of vectors is not chosen properly, we may find that the spectral property holds though the operator still has embedded eigenvalues. Thus the constraints on the set of vectors whose orthogonal complement will define $\mathscr{U}$ are:

(1) They must be orthogonal to eigenstates of embedded eigenvalues,
(2) Orthogonality with respect to them should induce positivity of $\mathscr{L}$ on $\mathscr{U}$.

A way of satisfying both requirements is to use the discrete spectrum of the adjoint problem To that end, we rely on the following simple results.

LEMMA 17.1.8. *If $(\lambda, \vec{u})$ is an eigenvalue, eigenvector pair for JL and $(\sigma, \vec{v})$ is an eigenvalue, eigenvector pair for $(JL)^*$, then*

$$(\lambda - \sigma^*) \langle \vec{u}, \vec{v} \rangle = 0.$$

Thus, if $\lambda - \sigma^* \neq 0$, the states are orthogonal.

COROLLARY 17.1.9. *An eigenstate of JL associated with a purely imaginary eigenvalue, $i\tau \neq 0$, is orthogonal to $Ker_g((JL)^*)$.*

COROLLARY 17.1.10. *Let $(i\tau \neq 0, \vec{\psi})$ and $(\lambda > 0, \vec{\phi})$ be eigenvalue, eigenvector pairs of JL. Then*

$$\langle \psi_1, \phi_2 \rangle = 0,$$
$$\langle \psi_2, \phi_1 \rangle = 0.$$

PROOF. By the Hamiltonian symmetry of the problem, $-\lambda\,(\phi_2, \phi_1)^T$ and λ, $(-\phi_2, -\phi_1)^T$ are eigenvalue pairs of the adjoint, $(JL)^*$. Therefore,

$$\langle \psi_1, \phi_2 \rangle - \langle \psi_2, \phi_1 \rangle = 0,$$
$$-\langle \psi_1, \phi_2 \rangle - \langle \psi_2, \phi_1 \rangle = 0.$$

Adding and subtracting these equations gives the result. □

These observations motivate using the known spectrum of the adjoint system in constructing the orthogonal subspace. For the 3D cubic problem, we can thus use the eigenstates at the origin and the two off axis real eigenvalues.

17.2. Exercises

CHAPTER 18

Bielefeld Lecture 13

num:NLS

18.1. Solving NLS numerically

We outline numerical methods for one dimension used to produce the results described at the end of this section. Note that these methods generalize to radial problems in higher dimensions quite easily.

We discretize our equation,

$$i u_t + u_{xx} + \beta(|u|^2)u = 0,$$
$$u(0,x) = u_0,$$

using a finite element scheme in space and the standard Crank-Nicholson scheme in time (at first order, this is the midpoint rule). Namely, our ODE time stepping scheme is

$$u_t = f(t,u),$$

and time step using the algorithm

$$u_{n+1} = u_n + h_t f(u_{n+\frac{1}{2}}, t_{n+\frac{1}{2}}),$$

where we take $t_{n+\frac{1}{2}} = (n+\frac{1}{2})h_t$ and

$$u_{n+\frac{1}{2}} = \frac{u_n + u_{n+1}}{2}.$$

Just as the equation itself this method is L^2 conservative. A similar scheme was implemented in [**ADKM03**], where the blow-up for NLS in several dimensions was analyzed. This method was applied by the author with Justin Holmer and Maciej Zworski in [**HMZ07b**] and [**HMZ07a**] to study soliton interactions with delta potentials.

Note that we require the spatial grid to be large enough to insure negligible interaction with the boundary. The convergence of such methods without potentials was studied in [**ADKM91**], [**ADKM97**].

We select a symmetric region about the origin, $[-R,R]$, upon which we place a mesh of N elements. The standard hat function basis is used in the Galerkin approximation. We allow for a finer grid in a neighbourhood of length 1 centered at the origin to better study the effects of the interaction with the delta potential. In terms of the hat basis the problem becomes:

$$\langle u_t, v\rangle + i\langle u_x, v_x\rangle/2 - i\langle\beta(|u|^2)u, v\rangle = 0,$$
$$u(0,x) = u_0\,,\ \ u(t,x) = \textstyle\sum_v c_v(t)\,v\,,$$

where $\langle\cdot,\cdot\rangle$ is the standard L^2 inner product, v is a basis function and u, u_0 are linear combinations of the v's. We remark that since v's are continuous the pairing of uv with the delta function is justified.

Since v's are hat functions, we have created a tridiagonal linear system with one contribution to the central element resulting from the delta function. Let $h_t > 0$ be a uniform time step,

$$u_n = \sum_v c_v(nh_t)v\,,$$

be the approximate solution at the nth time step. Implementing the midpoint rule in time, the system becomes:

$$\begin{aligned}&\langle u_{n+1}-u_n, v\rangle + ih_t\langle((u_{n+1}+u_n)/2)_x, v_x\rangle \\ &\quad = ih_t\left\langle\beta((u_{n+1}+u_n)/2)(u_{n+1}+u_n)/2, v\right\rangle, \quad u_0 = \sum_v \alpha_v v,\end{aligned}$$

By defining

$$y_n = (u_{n+1}+u_n)/2\,,$$

we have simplified our system to:

$$\langle y_n, v\rangle + i\frac{h_t}{4}\langle(y_n)_x, v_x\rangle = i\frac{h_t}{2}\langle\beta(|y_n|^2)y_n, v\rangle + \langle u_n, v\rangle.$$

An iteration method from [**ADKM03**] is now used to solve this nonlinear system of equations. To wit,

$$\langle y_n^{k+1}, v\rangle + i\frac{h_t}{4}\langle(y_n^{k+1})_x, v_x\rangle = i\frac{h_t}{2}\langle\beta(|y_n^k|)y_n^k, v\rangle + \langle u_n, v\rangle.$$

We take $y_n^0 = u_n$ and perform three iterations in order to obtain an approximate solution.

In order to take this radial problems in higher dimensions, we refine our inner product to be

$$\langle u, v\rangle = \int_0^\infty u(r)v(r)r^{d-1}dr.$$

The remainder of the analysis follows trivially. Note that a finite difference argument can also be applied here by linearizing the problem and discretizing derivatives directly.

18.2. Crank-Nicolson is Mass Conservative

We show here that the 2nd order Crank-Nicolson scheme above is mass conservative. The idea is to simply apply summation by parts on the discretized equation. The key observation is that

$$(u_{n+1}-u_n)(\bar{u}_{n+1}+\bar{u}_n)/2 + (\bar{u}_{n+1}-\bar{u}_n)(u_{n+1}+u_n)/2 = |u_{n+1}|^2 - |u_n|^2.$$

Use

$$\begin{aligned}\langle u_{n+1}-u_n, v\rangle \;&+\; ih_t\langle((u_{n+1}+u_n)/2)_x, v_x\rangle \\ &= \quad ih_t\left\langle\beta(|u_{n+1}+u_n)/2|^2)(u_{n+1}+u_n)/2, v\right\rangle,\end{aligned}$$

with $v = (\bar{u}_{n+1}+\bar{u}_n)/2$ and

$$\begin{aligned}\langle \bar{u}_{n+1}-\bar{u}_n, v\rangle \;&-\; ih_t\langle((\bar{u}_{n+1}+\bar{u}_n)/2)_x, v_x\rangle \\ &= \quad -ih_t\left\langle\beta(|u_{n+1}+u_n)/2|^2)(\bar{u}_{n+1}+\bar{u}_n)/2, v\right\rangle,\end{aligned}$$

with $v = (u_{n+1}+u_n)/2$.

18.3. A Spectral Method

Another very nice method for solving NLS involves a robust spectral method originally derived by Kassam-Trefethen [**KT05a**]. This turns out to be a broadly applicable, nicely stable means of studying several dissipative/dispersive equations or systems thereof. For a nice application to soliton dynamics in NLS, see [**Pot09**].

Let us focus on finding a solution $u(x,t)$ to (12.1), but now put the problem on a periodic domain. Doing so means that we can take a Fourier series decomposition to discretize in space such that

$$\partial_t \hat{u}_k + ik^2 \hat{u}_k - i(\widehat{(|u|^p - V(x))u})_k = 0, \hat{u}_k = (\hat{u_0})_k.$$

The general idea for the time integration is to implement standard PDE techniques by observing that such dynamics are described by decomposing the evolution into linear/nonlinear components, i.e.

$$u_t = L(u) + N(u),$$

where

$$\begin{aligned} \widehat{(L(u))}_k &= -ik^2 \hat{u}_k, \\ \widehat{(N(u))}_k &= i(\widehat{(|u|^p - V(x))u})_k. \end{aligned}$$

Let us take uniform time steps, h_t and set $u_n \approx u(t_n)$ for $t_n = nh_t$. Then, as in ODE solvers, we see

$$u_{n+1} = e^{h_t L} u_n + \int_0^{h_t} e^{(h_t - s)L} N(u(t_n + s), t_n + s) ds,$$

Note, given that the Laplacian is a local operator on each Fourier mode, implementing this algorithm in the given spatial decomposition means that L, $e^{\tau L}$ are diagonal matrices for any $\tau \in \mathbb{R}$. Hence, the computational difficulty comes down to choosing the quadrature method for evaluating

$$\int_0^{h_t} e^{(h_t - s)L} N(u(t_n + s), t_n + s) ds.$$

Defining $v = e^{-Lt}u$, one observes that

$$v_t = e^{-Lt} N(e^{Lt} v).$$

To now solve this implicity nonlinear system of ODEs, in [**KT05a**] the use a 4th order Runge-Kutta method

$$\begin{aligned} y_1 &= h_t f(v_n, t_n), \\ y_2 &= h_t f(v_n + y_1/2, t_n + h_t/2), \\ y_3 &= h_t f(v_n + y_2/2, t_n + h_t/2), \\ y_4 &= h_t f(v_n + y_3, t_{n+1}), \\ v_{n+1} &= v_n + \frac{1}{6}(y_1 + 2y_2 + 2y_3 + y_4). \end{aligned}$$

In the remainder of this chapter, we use the finite element method to display soliton stability/instability for the stable/unstable parts of the soliton curve with saturated nonlinearities, dispersion for initial data with L^2 mass below the minimal mass for a saturated nonlinearity and interactions between stable solitons and close to minimal mass solitons for saturated nonlinearities in $\mathbb{R}$.

18.4. Strang-Splitting Methods

In what follows, we review some joint work with Jianfeng Lu.

Consider a general quasi-linear Schrödinger equation

eq:qls
$$i u_t = -\Delta u + u f(|u|^2) + u g'(|u|^2)\Delta g(|u|^2), \tag{18.1}$$

which can be written

eqn:quasiquad
$$\begin{cases} i u_t + g^{jk}(u)\partial_j\partial_k u = F(u,\nabla u), \quad u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}^m \\ \\ u(0,x) = u_0(x) \end{cases} \tag{18.2}$$

with small initial data in a space with relatively low Sobolev regularity but with some extra decay assumptions. Here

$$g : \mathbb{C}^m \times (\mathbb{C}^m)^d \to \mathbb{R}^{d\times d}, \qquad F : \mathbb{C}^m \times (\mathbb{C}^m)^d \to \mathbb{C}^m$$

are smooth functions which we will assume satisfy

$$g(y,z) = I_d + O(|y|^2 + |z|^2), \qquad F(y,z) = O(|y|^3 + |z|^3) \text{ near } (y,z) = (0,0).$$

Quasilinear equations of this form have arisen in several models. See [**Pop01**] for a thorough list, but we mention here works related to the superfluid thin-film equation [**Kur81**] and modeling ultra-short pulse lasers [**dBHS97**, **dBHNS99**]. The model we will consider here numerically equates to setting $g(s) = f(s) = s$, and hence

eqn:model
$$i u_t = -\Delta u + |u|^2 u + u\Delta(|u|^2). \tag{18.3}$$

This is a pseudo-attractive version of the superfluid thin-film equation, which is given by

eqn:thinfilm
$$i u_t = -\Delta u + |u|^2 u - u\Delta(|u|^2) \tag{18.4}$$

and can be seen as a leading order contribution to the ultra-short pulse laser models from [**dBHS97**, **dBHNS99**]. In addition, existence of solutions to quasilinear equations have been studied analytically in several case, see [**dBHS97**, **dBHNS99**, **KPV04**, **KPRV06**, **KPRV05**, **MMT12**, **MMT13**, **Pop01**] and many others. The reason we choose this particular model is that it while it is guaranteed to have small data local well-posedness from [**MMT13**] and hence can be used to verify our numerical convergence results for general quasilinear models, it will also allow interesting singularities to form in the evolution for large enough initial data.

Let us consider the nonlinear part of the equation

eq:nonlinear
$$i v_t = v f(|v|^2) + v g'(|v|^2)\Delta g(|v|^2). \tag{18.5}$$

Taking the complex conjugate, we have

$$-i\bar{v}_t = \bar{v} f(|v|^2) + \bar{v} g'(|v|^2)\Delta g(|v|^2).$$

We calculate

eqn:nonlinampcon
$$\begin{aligned} i\partial_t|v|^2 &= i\bar{v}\partial_t v + i v\partial_t\bar{v} \\ &= |v|^2 f(|v|^2) + |v|^2 g'(|v|^2)\Delta g(|v|^2) \\ &\quad - |v|^2 f(|v|^2) - |v|^2 g'(|v|^2)\Delta g(|v|^2) \\ &= 0, \end{aligned} \tag{18.6}$$

and hence under the evolution (18.5) the amplitude is conserved. This will be a key property used to develop the numerical scheme.

We consider a Strang splitting method for the quasilinear Schrödinger equation, which is a composition of the exact flows of the differential equations

$$i\partial_t u = -\Delta u \tag{18.7}$$

and

eq:splittingB

$$i\partial_t u = uf(|u|^2) + ug'(|u|^2)\Delta g(|u|^2). \tag{18.8}$$

More concretely, we approximate $u(t_n)$ with $t_n = n\tau$ for a step size $\tau > 0$ by u_n via

eqn:splitting

$$\begin{aligned} u^-_{n+1/2} &= e^{\frac{i}{2}\tau\Delta} u_n; \\ u^+_{n+1/2} &= u^-_{n+1/2} e^{\left(-i\tau(f(|u^-_{n+1/2}|^2)+g'(|u^-_{n+1/2}|^2)\Delta g(|u^-_{n+1/2}|^2))\right)}; \\ u_{n+1} &= e^{\frac{i}{2}\tau\Delta} u^+_{n+1/2}. \end{aligned} \tag{18.9}$$

We note that the scheme is explicit and reversible, thanks to the amplitude preserving property (18.6) of (18.5). One can use a Fourier pseudo-spectral method for the spatial discretization, and hence the flow $e^{(\frac{i}{2}\tau\Delta)}$ can be efficiently calculated using fast Fourier transform (FFT), and the flow (18.8) amounts to changing the phase of the solution on each mesh point.

Due to the advantage of being structure-preserving, the Strang splitting [**Str68**] and higher order splitting schemes (e.g. [**Suz93**, **Yos90**]) have been widely applied to nonlinear Schrödinger equations, mainly semilinear Schrödinger equations, see for example [**ABB13**, **AG**, **BC12**, **BJM03a**, **BJM03b**, **BMS03**, **BS05**, **Chi07**, **FGL09**, **HT73**, **MQ02**, **PM90**, **PGL03**, **SW13**, **WH86**]. While we focus on the Strang splitting scheme for quasilinear Schrödinger equations, let us also mention that many other time discretization approaches to solve non-linear evolution equations have been developed, including Crank-Nicholson type schemes (see e.g. [**SS84**] and also [**ADKM91**] and [**HMZ07b**] for applications in studying numerical blow-ups and nonlinear scattering), Magnus expansion approaches ([**Mag54**] and also the recent review article [**BCOR09**]), exponential time-differencing schemes (see e.g. [**CM02**, **KT05b**]), implicit-explicit methods (see e.g. [**ARW95**]), the comparison study in [**TA84**], and many others. Our study is also motivated by numerical approaches towards time-dependent density functional theory computations as discussed in the documentation of the software package Octopus [1] and also [**CMR04**] and references therein.

The convergence of splitting schemes for semilinear Schrödinger equation was analyzed in [**DT12**, **Gau11**, **JL00**, **Lub08**, **SW13**, **Tha08**]. In the present work, we extend the previous works to quasi-linear Schrödinger equation. We will mainly focus on the convergence of the time-splitting method to the original evolution for the superfluid thin-film equation. The analysis follows the ideas in the seminal contribution by Lubich in [**Lub08**], where the main tools are the calculus of Lie derivatives. We will emphasize on the regularity of the time flow, for which the behavior of the quasilinear Schrödinger equation is different from the semilinear ones.

We explore here the relevant Lie theory results and necessary multilinear estimates in Section 18.5. In Section 18.6, we discuss the regularity of the time flow of the quasilinear Schrödinger equation and the time-splitting scheme. In order to establish the differentiability of the numerical solution with respect to time to sufficiently high accuracy, we rely on bounds in a much stronger topology in space.

[1] http://www.tddft.org/programs/octopus/wiki/index.php/Main_Page

sec:Lie

18.5. Convergence of time-splitting scheme

We follow here the idea of Lubich [**Lub08**] relying on small data local existence in H^s for cubic quasilinear nonlinear terms in Marzuola-Metcalfe-Tataru [**MMT12**, **MMT13**] (See also the works of Poppenberg [**Pop01**], Kenig-Ponce-Vega and Kenig-Ponce-Rolvung-Vega [**KPV93**, **KPV98**, **KPV04**, **KPRV06**, **KPRV05**]). Let us focus on the simple case of equation (18.3) in arbitrary dimension, which equates to setting $g(s) = f(s) = s$, hence allowing us to use that for small initial data, there exists a local in time solution in H^s as long as $s > \frac{d+5}{2}$ provided u_0 is sufficiently small. See for instance Theorem 1, [**MMT13**] given that the nonlinearity is cubic by nature.

Before we begin let us take

$$m_k = \max_{0\leq t\leq T} \|u(t)\|_{H^k},\ k \leq \max(7, \frac{d+5}{2} + \epsilon), \tag{18.10}$$

for and $\epsilon > 0$ such that u the solution to (18.1) with small initial data can be defined in H^k using [**MMT12**, **MMT13**]. We can approximate the solution through the continuous time generators of the split step equations:

eqn:Deltadot
$$i\dot{\psi} = -\Delta\psi \tag{18.11}$$

eqn:Vdot
$$i\dot{\psi} = V[\psi]\psi \tag{18.12}$$

where

eqn:sftfq
$$V[\psi] = |\psi|^2 - \Delta(|\psi|^2). \tag{18.13}$$

The generators of the split step method can thus be described as exponential maps of the vector fields given by

eqn:hatT
$$\hat{T}(\psi) = i\Delta\psi, \tag{18.14}$$

eqn:hatV
$$\hat{V}(\psi) = -iV[\psi]\psi = -i[|\psi|^2 + \Delta(|\psi|^2)]\psi. \tag{18.15}$$

The key estimates we will require are of the type

eqn:tri1
$$\|\Delta(uv)w\|_{H^s} \leq C\|u\|_{H^{s+\frac{6+d}{3}}}\|v\|_{H^{s+\frac{6+d}{3}}}\|w\|_{H^{s+\frac{d}{3}}} \tag{18.16}$$

using $L^6 \times L^6 \times L^6 \to L^2$ Hölder's inequality and the Sobolev embedding for L^6, as well as

eqn:tri2
$$\|\Delta(uv)w\|_{H^s} \leq C\|u\|_{H^{s+\frac{4+d}{2}+\epsilon}}\|v\|_{H^{s+\frac{4+d}{2}+\epsilon}}\|w\|_{H^s} \tag{18.17}$$

using the $L^\infty \times L^\infty \times L^2 \to L^2$ Hölder's inequality and the Sobolev embedding for L^∞.

Before computing Lie Derivatives, we want to understand the stability of the evolution generated by $\hat{V}$. To do this, we study

eqn:Vode
$$i\dot{\nu} = V[\psi]\nu,\ \ \nu(0) = \psi. \tag{18.18}$$

For ψ sufficiently regular, it is possible to show that the evolution varies continuously with the choice of initial data in a weak topology. In particular, we can show that given

$$i\dot{\nu} = V[\psi]\nu,\ \ \nu(0) = \psi,$$
$$i\dot{\mu} = V[\phi]\mu,\ \ \mu(0) = \phi,$$

then by looking at the difference of these two evolutions, expanding $V[\phi]\mu - V[\psi]\nu = (V[\phi] - V[\psi])\mu - V[\psi](\nu - \mu)$ and applying (18.17) we have

eqn:Liecont
$$\|\mu(t) - \nu(t)\|_{H^s} \leq \|\psi - \phi\|_{H^s} + C_1 t\|\psi - \phi\|_{H^{s+\frac{d+4}{2}+\epsilon}} + C_2\int_0^t \|\mu(s) - \nu(s)\|_{H^s}, \tag{18.19}$$

for any $\epsilon > 0$ and $s > \frac{d+4}{2}$ chosen sufficiently large to control the evolution, where C_1, C_2 both depend upon $M_s = \max_{0\le t\le T}\{\|\phi\|_{H^s}, \|\psi\|_{H^s}\}$. As a result, a Gronwall type argument shows

eqn:gronwall1 (18.20) $$\|\mu(t) - \nu(t)\|_{H^s} \le e^{C_0\tau}\|\psi - \phi\|_{H^{s+\frac{d+5}{2}+\epsilon}}$$

where C_0 depends upon m_k.

Now, to compare the full evolution to the split-step method, we must compute the Lie commutators between generating vector fields:

eqn:Lie (18.21) $$\begin{aligned}[\hat{T}, \hat{V}] &= \Delta\left(|\psi|^2\psi - \Delta(|\psi|^2)\psi\right)\\ &\quad - \left[2\Delta\psi(\bar{\psi}\psi - \psi^2\overline{\Delta\psi})\right]\\ &\quad - \left[\Delta(\Delta\psi\bar{\psi})\psi - \Delta(\psi\overline{\Delta\psi})\psi + \Delta(|\psi|^2)\Delta\psi\right].\end{aligned}$$

Hence, we observe

eqn:commutbd1 (18.22) $$\|[\hat{T}, \hat{V}](\psi)\|_{H^1} \le C\|\psi\|^3_{H^{\max(5, \frac{d}{2}+)}}.$$

In addition, we then can easily compute

eqn:commutbd2 (18.23) $$\|[\hat{T}, [\hat{T}, \hat{V}]](\psi)\|_{H^1} = C\|\psi\|^3_{H^{\max(7, \frac{d}{2}+)}}.$$

Setting the vector field $\hat{H} = \hat{T} + \hat{V}$, the underlying idea is to compare the evolution of the full QLS given by the exact evolution

eqn:duhamel (18.24) $$\psi(\tau) = e^{(\tau D_H)}\,\mathrm{Id}(\psi_0)$$

when well defined can be compared through a double Duhamel expansion to the split-step generator

eqn:dduhamel (18.25) $$\psi_{SS}(\tau) = e^{(\frac{1}{2}\tau D_T)}e^{(\tau D_V)}e^{(\frac{1}{2}\tau D_T)}\,\mathrm{Id}(\psi_0),$$

the error terms of which can be written using the Lie Commutators. Indeed, the error estimates come from successive application of the quadrature first order error formula

$$\tau f(\frac{1}{2}\tau) - \int_0^\tau f(s)\,\mathrm{d}s = \tau^2\int_0^1 \kappa_1(\theta) f'(\theta\tau)\,\mathrm{d}\theta$$

and the second-order error formula

$$\tau f(\frac{1}{2}\tau) - \int_0^\tau f(s)\,\mathrm{d}s = \tau^3\int_0^1 \kappa_2(\theta) f''(\theta\tau)\,\mathrm{d}\theta$$

where $\kappa_1(\theta)$ and $\kappa_2(\theta)$ are the Peano kernels for the midpoint rule and

$$f(s) = e^{(\tau - s)D_T)}D_V e^{(sD_T)}\,\mathrm{Id}(\psi_0)$$

and hence

$$\begin{aligned}f'(s) &= e^{is\Delta}[\hat{T}, \hat{V}]e^{i(\tau-s)\Delta}\psi_0,\\ f''(s) &= e^{is\Delta}[\hat{T}, [\hat{T}, \hat{V}]]e^{i(\tau-s)\Delta}\psi_0.\end{aligned}$$

Note, the Peano kernels are defined as the integral kernels of the linear transformation $L : C^{k+1}[0, T] \to \mathbb{R}$ such that

$$L(f) = f - \sum_{j=0}^{k}\frac{f^{(j)}(0)}{j!}x^j = \frac{1}{k!}\int_0^T \kappa_k(s) f^{(k+1)}(s)\,ds.$$

Hence, it is essential that for the below we can prove that for our approximation we have $f(s) \in C^3$, which very much relates to the analyticity of the linear Schrödinger

evolution kernel in the Strang-Splitting scheme as in particular a generic quasilinear Schrödinger flow cannot be shown to be more than C^0 by the purely dispersive techniques in [**MMT12**, **MMT13**]. We will go back to this in more details in the next section.

Applying (18.24), (18.25), (18.22), (18.23) and (18.19) in succession as in [**Lub08**] gives

eqn:quadconv

$$\|\phi_n - \phi(t_n)\|_{H^1} \leq C(m_{K_0}, T)\tau^2 \tag{18.26}$$

for $t_n = n\tau \leq T$ and $K_0 = \max(7, \frac{d+5}{2} + \epsilon)$. In order to obtain the τ^2 convergence here, it is important to compute the double commutator bound leading to (18.25) in order to expand out to 3rd order in the Lie derivatives. However, we note the same quadratic convergence would hold in L^2 with only $K_0 = \max(5, \frac{d+5}{2} + \epsilon)$ as then the double Duhamel commutator would not be required.

So far we have considered the convergence of the time-splitting flow to the flow of the original PDE. We further discretize the spatial degree of freedom using a Fourier pseudo-spectral method. The convergence of the fully discretized scheme follows if we can show that the fully discretized scheme converges to the time-splitting flow. This follows from standard theory for pseudo-spectral methods, in particular, the analysis in [**Gau11**, **SW13**] for semilinear Schrödinger equations. Actually, as the time flow is regularized after the time discretization, there is no essential difference between the analysis for the quasilinear case and the semilinear case. Hence we will omit the details here.

Remark 18.5.1
Note, we have chosen a model example that demonstrates the techniques quite nicely in our choice of (18.11) *and* (18.12)*, but the flow of the metric term and the nonlinear term in* (18.2) *are both well defined and have well defined Lie derivatives provided one can prove the existence of solutions. Hence, the splitting process can be done in complete generality using the method*

eqn:Deltametricdot

$$i\dot{\psi} = \partial_i g^{ij}[\psi]\partial_j \psi \tag{18.27}$$

eqn:Fdot

$$i\dot{\psi} = V[\psi]\psi. \tag{18.28}$$

*Each evolution would require time-dependent kinetic energy terms and multilinear estimates akin to those in [**MMT12**, **MMT13**] however, this gives a nonlinear dependence in the Hamiltonian related to the evolution of* (18.27)*. The nonlinearity causes troubles in designing structure preserving schemes and also difficulty associated with the regularity of the solution map with respect to time in order to approximate using Lie derivatives (as will be further discussed in the next section).*

18.6. Regularity of the Time Evolution

sec:Reg

In the analysis of the convergence of the Strang splitting scheme, we have used the analyticity of the linear Schrödinger evolution. This is in general however not true for the quasilinear Schrödinger evolution. The Strang splitting scheme actually regularizes the time flow of the original PDE. In this section, we give some further discussion for the regularity of the time evolution.

The continuity of the solution map partially hinges upon the proof of uniqueness for the evolution of (18.1). In particular, take two solutions to (18.1), say u_1 and u_2.

Setting $v = u_1 - u_2$ and linearizing (18.1), we have an equation of the form

$$\begin{cases} i v_t + g^{jk}(u)\partial_j\partial_k v + V\nabla v + W v = 0, \\ v(0,x) = u_1(0) - u_2(0) \end{cases}$$

with

$$V = V(u_1, \nabla u_1, u_2, \nabla u_2),\ W = h(u_1, \nabla u_1, u_2, \nabla u_2) + g(u_1, u_2)\nabla^2 u_1$$

for functions V, h and g related to Taylor expanding the metric and the nonlinearity. Then, to solve this linear equation, we use [**MMT13**], Proposition 5.1 (see also [**MMT12**], Proposition 5.2) to show that the weak Lipschitz bound

weak-lip

$$\|v\|_{L^\infty H^\sigma} \lesssim \|v(0)\|_{H^\sigma} \tag{18.29}$$

holds for any $0 \le \sigma \le s-1$ via energy estimates on the linearized equation, where we recall that the initial condition lies in H^s for $s > \frac{d+5}{2}$. In the well-posedness result for the linearized version of (18.1), see Proposition 5.1 of [**MMT13**] for instance, we have at most continuity of the solution map with respect to time in the H^s norm however.

The key ideas to the proof follow from the theory of frequency envelopes as discussed in both [**MMT12**, **MMT13**], Sections 2 and 5, where it is proven that the size of a dyadic frequency component of the solution to (18.1) in a natural energy space is bounded by a uniform constant times the corresponding dyadic frequency component of the initial data in H^s. To be more precise, we shall use a Littlewood-Paley decomposition of the spatial frequencies,

$$\sum_{i=0}^{\infty} S_i(D) = 1,$$

where S_i localizes to frequency $[2^{i-1}, 2^{i+1}]$ for $i > 0$ and to frequencies $|\xi| \le 2$ for $i = 0$. By a frequency envelope, we recall from [**MMT12**], Section 2.4 we mean that given a translation invariant space U such that

$$\|u\|_U^2 \sim \sum_{k=0}^{\infty} \|S_k u\|_U^2,$$

a frequency envelope for u in U is a positive sequence a_j so that

$$\|S_j u\|_U \le a_j \|u\|_U, \qquad \sum a_j^2 \approx 1. \tag{18.30}$$

We say that a frequency envelope is admissible if $a_0 \approx 1$ and it is slowly varying,

$$a_j \le 2^{\delta|j-k|} a_k, \qquad j,k \ge 0, \qquad 0 < \delta \ll 1.$$

An admissible frequency envelope always exists, say by

freqEnv

$$a_j = 2^{-\delta j} + \|u\|_U^{-1} \max_k 2^{-\delta|j-k|} \|S_k u\|_U. \tag{18.31}$$

Abusing notation and avoiding for simplicity the atomic space formulations in [**MMT12**, **MMT13**], we rely upon a uniform bound over the evolution such that effectively

mainb

$$\|u\|_{L^\infty H^s} \lesssim \|u_0\|_{H^s}. \tag{18.32}$$

We note that the $L^\infty H^s$ norm appears the estimate here is due to the cubic interactions in the nonlinearity and the compactness of our domain, otherwise one must enforce further summability as in [**MMT12**]. A key estimate is the following proposition.

envelopes

PROPOSITION 18.6.1 (Proposition 5.3, [**MMT12**]; Proposition 5.4, [**MMT13**]). *Let u be a small data solution to* (18.1)*, which satisfies* (18.32)*. Let $\{a_j\}$ be an admissible frequency envelope for the initial data u_0 in H^s. Then $\{a_j\}$ is also a frequency envelope for u in $L^\infty H^s$.*

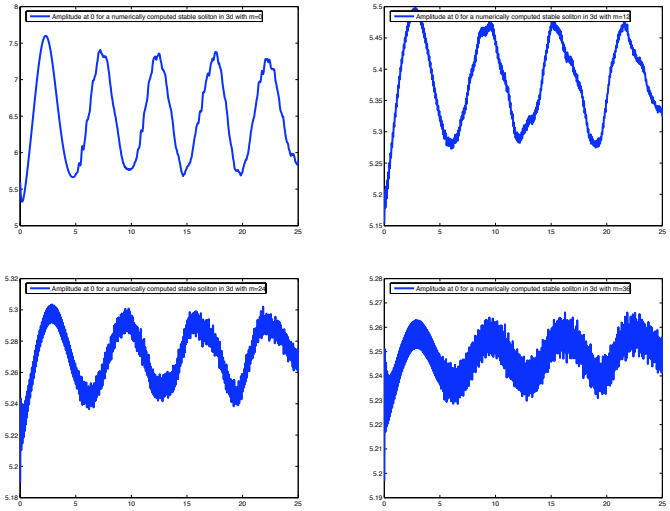

FIGURE 1. Numerical simulation of soliton stability for R_1 computed with improved accuracy for saturated nonlinearities of type 2 in $\mathbb{R}^3$ with $q = 2$. In particular, we plot the change in amplitude at $x = 0$ on a time interval $[0, T]$ for various corrections $m = 0, 12, 24, 36, 48$ with the initial spatial step to be $\Delta x = .1$.

f:solconv

Once we have Proposition 18.6.1, the continuity of the solution map can be established as is Section 5.7 of [**MMT12**]. Namely, we consider a sequence of initial data $\{u_0^n\} \to u_0$ in H^s. Frequency envelope bounds can then be chosen such that there exists a uniform N_ϵ for which

$$\| a_{N_\epsilon}^{(n)} \| \leq \epsilon$$

for all n, which gives a uniform upper bound by Proposition 18.6.1 on the high frequencies of each corresponding solution $u^{(n)}$ to (18.1) with initial data $u_0^{(n)}$ in the $L^\infty H^s$ norm. Separating into low and high frequencies, using the smallness of the high frequencies and the uniform convergence in weaker Sobolev norms provided by (18.29), the result follows. However, we generally gain no more than continuity of the solution map from such arguments. Hence, in order to accurately compare the flow of the full solution map defined by (18.1) and that of the Strang-Splitting method, we rely on differentiating the equation and the balancing of spatial and time regularity, as in the previous section.

18.7. Soliton Stability - Some Numerical Studies

num:stab

Here, we present numerical experiments towards soliton stability/instability. In Figure 1, we see that as we increase the correction parameter m when finding a stable soliton ($\lambda = 1$), the solution converges faster. Similarly, in Figure 2 we that as we increase the correction paramter m when finding an unstable soliton ($\lambda = .025$), it takes the solution longer to diverge. This gives numerical verification of the convergence of our computed solitons, as well as verification that our code for solving NLS adheres to standard theory.

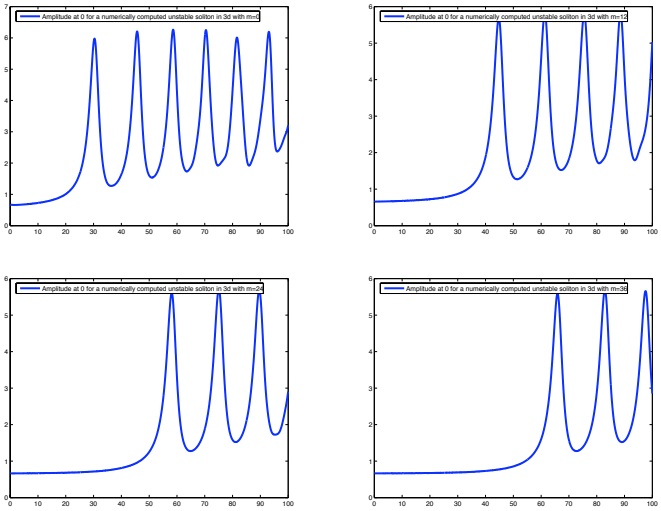

FIGURE 2. Numerical simulation of soliton instability for $R_{.025}$ computed with improved accuracy for saturated nonlinearities of type 2 in $\mathbb{R}^3$ with $q = 2$. In particular, we plot the change in amplitude at 0 on a time interval $[0, T]$ for various corrections $m = 0, 12, 24, 36$ with the initial spatial step to be $\Delta x = .1$.

f:soldiv

In Figure 3, we plot perturbations of solitons computed with high accuracy ($m = 35$) in $\mathbb{R}$ and $\mathbb{R}^3$. Specifically, we plot the evolution of the initial data $1.01 * R$, for R both stable and unstable. The behavior is similar to that predicted by theory. The perturbations of the stable solitons oscillate quickly towards a soliton very close on the soliton curve. For the unstable solitons, the solution radically alters until it once again starts oscillating about a stable soliton solution.

18.8. Soliton Dispersion

num:disp

After accurately calculating an approximation to the minimal mass soliton R_{min}, we plot $\alpha * R_{min}$ for a $1d$ saturated nonlinearity in Figure 4. As seen in the figure, the solution disperses quickly even for $\alpha = .99$. Similarly, in Figure 5, the same phenomenon is observed in $\mathbb{R}^3$ for a saturated nonlinearity of type 2. Future research will include analyzing this behavior analytically to prove that for arbitrary initial data below this minimal mass, scattering occurs.

18.9. Soliton Interaction

num:int

From the work of [**RSS03b**], we know that under various separation conditions, we have asymptotic stability for a collection of N solitons for saturated NLS equations. This phenomenon is observed in Figure 6 where we look at two stationary solitons in $\mathbb{R}$ separated spatially and see that these solitons remain stable. Specifically, in this experiment, one of the solitons is a slightly stable perturbation of the minimal mass soliton.

In Figures 7 and 8, we look at a numerical simulation of the solution with initial data consisting of a slightly stable perturbation of the minimal mass soliton stationary

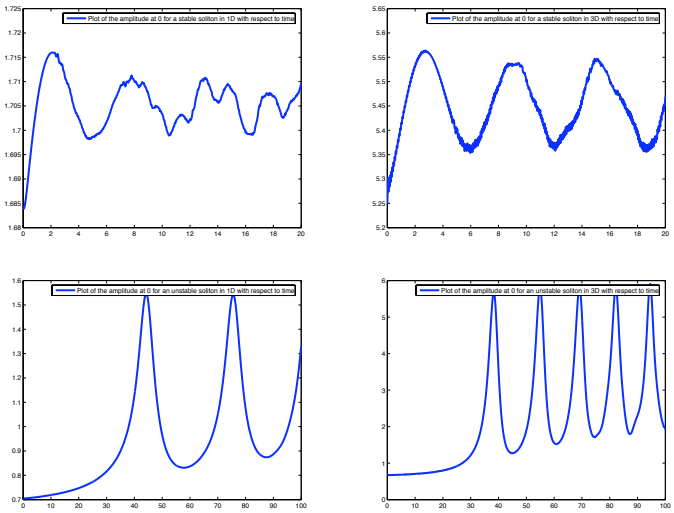

FIGURE 3. Numerical simulation of soliton stability/instability for saturated nonlinearities in $1d$ and type 2 in $\mathbb{R}^3$. In particular, we plot the amplitude at $x = 0$ for a solution with initial data $1.01 * R$ on a time interval $[0, T]$ for T large enough to see the overrall behavior, $R = R_1$ for the stable computation and $R = R_{.025}$ for the unstable computation.

f:sol2stable

at the origin coupled with a more stable soliton given a velocity ($v = 5$). These calculations are in a preliminary phase and will be explored in more detail in further research, but the effect of annihilation of the minimal mass soliton is observed. Phenomenon of this nature are unique to the saturated nonlinearities and provide interesting clues into phyiscal situations where such nonlinearities act as models. The author will continue to explore these phenomena to further numerical accuracy as well as analytically.

18.10. Double Well Dynamics

sec:randw

18.10.1. Polar Coordinates. As it will simplify the process of building initial conditions for numerically solving (12.13) that display the behaviors we study above, let us discuss here an alternative set of coordinates for (12.48). Namely, we set $\rho_0 = r_0 e^{i\theta_0}$ and $\rho_1 = r_1 e^{i\theta_1}$. This leads to the following system of ODE's:

eqn:sys-fd-pol

$$\begin{cases} \dot{r_0} = r_1^2 r_0 \sin(2\Delta\theta), \\ \dot{r_1} = -r_0^2 r_1 \sin(2\Delta\theta), \\ (\dot{\Delta\theta}) = \Omega_1 - \Omega_0 + (r_1^2 - r_0^2)(1 + \cos(2\Delta\theta)), \end{cases} \tag{18.33}$$

where $\Delta\theta = \theta_1 - \theta_0$. Given the system above, we can say that the bifurcation of stability occurs at

$$N_{cr}^{FD} = \frac{\Omega_1 - \Omega_0}{2}. \tag{18.34}$$

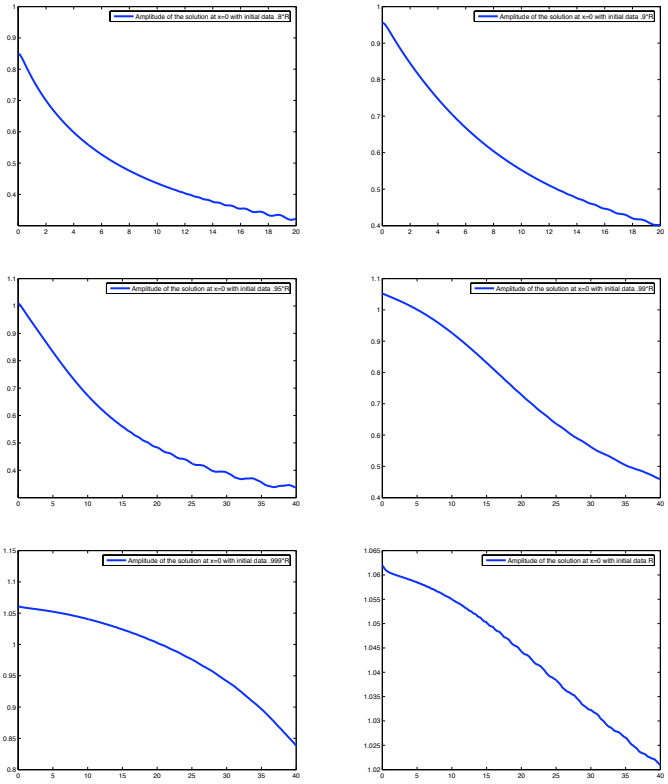

FIGURE 4. Numerical simulation of dispersion for perturbations below the minimal mass soliton for saturated nonlinearities. In particular, given the minimal mass soliton R, we solve the equation with initial data αR for $\alpha = .8, .9, .95, .99, .999, 1$. Presented here are plots of the amplitude at $x = 0$ with respect to time t for each α with saturated nonlinearity of type 1 in $\mathbb{R}$ with $p = 7$, $q = 3$.

`f:soldisp`

As we are interested in the behavior quite near the bifurcation point, we define new parameters ϵ_0, ϵ_1 and n such that

$$r_0 = \sqrt{N_{cr}^{FD}} + \epsilon_0, \tag{18.35}$$

$$r_1 = \epsilon_1, \tag{18.36}$$

$$N = N_{cr}^{FD} + n, \tag{18.37}$$

where

$$n = \epsilon_0^2 + \epsilon_1^2 + 2\sqrt{N_{cr}^{FD}}\epsilon_0. \tag{18.38}$$

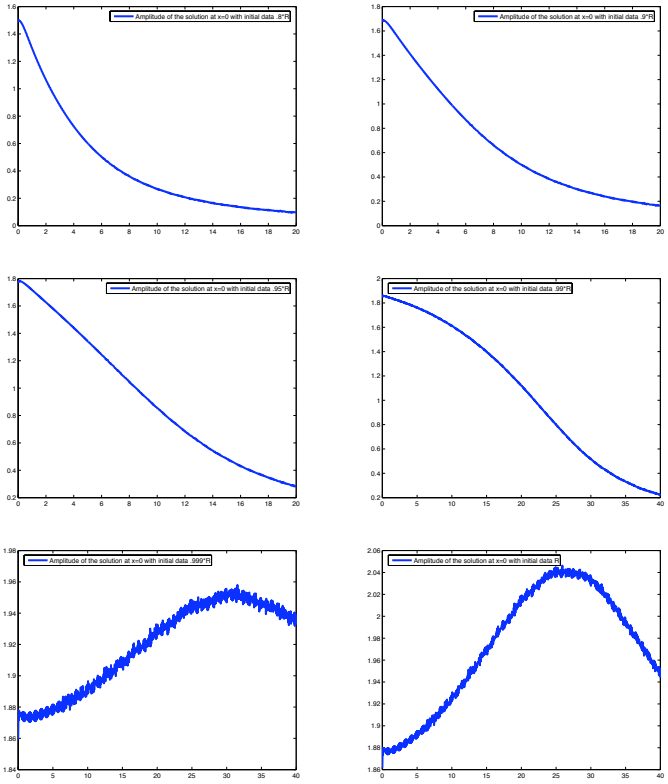

FIGURE 5. Numerical simulation of dispersion for perturbations below the minimal mass soliton for saturated nonlinearities. In particular, given the minimal mass soliton R, we solve the equation with initial data αR for $\alpha = .8, .9, .95, .99, .999, 1$. Presented here are plots of the amplitude at $x = 0$ with respect to time t for each α with saturated nonlinearity of type 2 in $\mathbb{R}^3$ with $q = 2$.

f:sol2disp

Then, we have

$$\dot{\epsilon_0} = \epsilon_1^2\left(\sqrt{N_{cr}^{FD}}+\epsilon_0\right)\sin(2\Delta\theta), \tag{18.39}$$

$$\dot{\epsilon_1} = -\epsilon_1\left(\sqrt{N_{cr}^{FD}}+\epsilon_0\right)^2\sin(2\Delta\theta), \tag{18.40}$$

$$(\dot{\Delta\theta}) = \Omega_1-\Omega_0+\left(\epsilon_1^2-\left(\sqrt{N_{cr}^{FD}}+\epsilon_0\right)^2\right)(1+\cos(2\Delta\theta)). \tag{18.41}$$

It should be noted, using the conservation laws we can write the system for ϵ_1 and $\Delta\theta$ independently

eqn:radodesys

$$\begin{cases} \dot{\epsilon_1} = -\epsilon_1\left(N_{cr}^{FD}+n-\epsilon_1^2\right)\sin(2\Delta\theta), \\ (\dot{\Delta\theta}) = \Omega_1-\Omega_0-\left(\left(N_{cr}^{FD}+n-2\epsilon_1^2\right)\right)(1+\cos(2\Delta\theta)) \end{cases} \tag{18.42}$$

and hence analyze phase plane diagrams, see Figures 9 and 10. In particular, the behavior of $\Delta\theta$ in these regions leads to interesting oscillatory behavior as shown in the phase

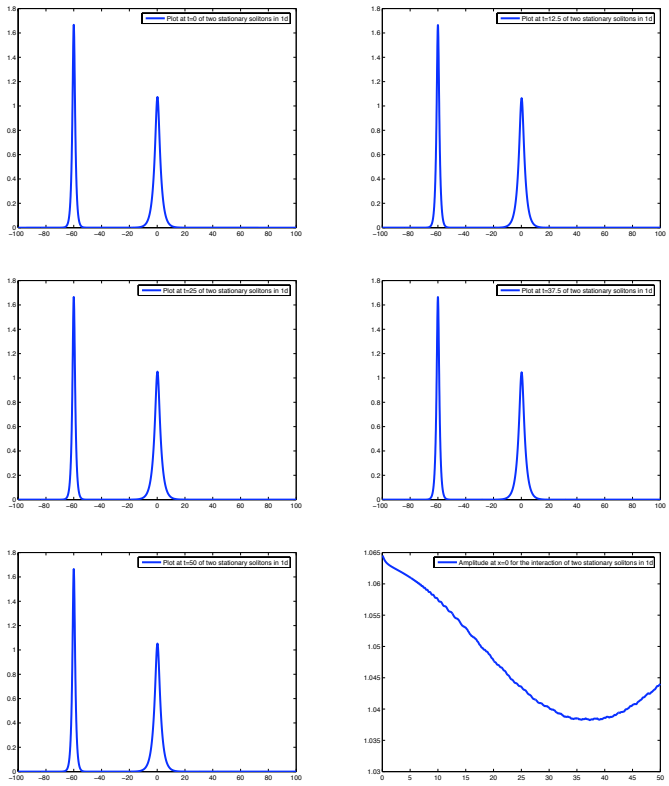

FIGURE 6. Numerical simulation of a stationary stable soliton interacting with another stationary soliton which is stable, yet close to the minimal mass soliton for saturated nonlinearities of type 1 in $\mathbb{R}$ with $p = 7$, $q = 3$. Presented here are plots of the absolute value of the solution with respect to the spatial grid for times $t = 0, 12.5, 25, 37.5, 50$ as well as a plot of the amplitude at $x = 0$ with respect to time.

f:solint

diagrams featured in Figures 9 and 10. For $n > 0$, a simple calculation shows that the equilibrium solutions occur for $\epsilon_1 = \sqrt{\frac{n}{2}}$, $\Delta\theta = k\pi$ for $k \in \mathbb{Z}$.

For $n > 0$, we have trapped orbits near the values $\Delta\theta = k\pi$ for $k \in \mathbb{Z}$. This is a manifestation of the orbital stability of these mixed states. However, oscillations between wells can be generated by a large enough phase shift to leave the region where such trapped orbits occur. These oscillations are then large, in particular ϵ_1 must reach some ϵ_{max} before decreasing.

For $n < 0$, we see the oscillations still attain a maximum at $\Delta\theta = k\pi$, however their amplitude approaches 0 with the initial $\epsilon_1(0)$. This is a manifestation of the stability of the symmetric state in this regime. One may ask if such finite dimensional Hamiltonian dynamics would appear in the infinite dimension dynamics of the PDE. For this, see Figures 11, 12 and 13 for numerical evidence of their existence for long times.

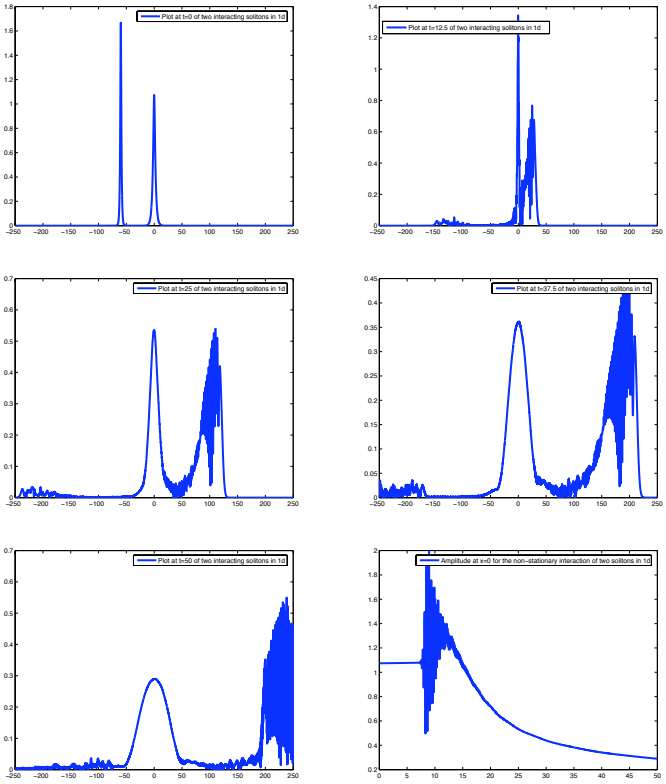

FIGURE 7. Numerical simulation of a stable soliton moving with velocity $v = 5$ interacting with a stationary soliton which is stable, yet close to the minimal mass soliton for saturated nonlinearities of type 1 in $\mathbb{R}$ with $p = 7$, $q = 3$. Presented here are plots of the absolute value of the solution with respect to the spatial grid for times $t = 0, 12.5, 25, 37.5, 50$ as well as a plot of the amplitude at 0 with respect to time.

f:sol2int

To begin, we locate the symmetry breaking point for a particular system. To do this, we use spectral renormalization with an asymmetric initial function to find the symmetry breaking point, see Fig. 3.

In the remainder of this section, we run simulations with wells of the form

eqn:ddw

$$V_L = q(\delta(x - \frac{L}{2}) + \delta(x + \frac{L}{2})). \tag{18.43}$$

Similar results hold and the same numerical analysis tools are applicable for potentials with more regularity. Knowing the bifurcation point as discussed in [**JW04**], we can numerically integrate using a finite element method similar to that in [**HMZ07b**], where scattering of soliton solutions across single delta function potentials was analyzed (see [**?**] for analysis of finite element methods for nonlinear Schrödinger equations without potential). It is quite simple to adapt the method presented there to allow for a double-well potential (for delta functions or smoother potentials). The initial data

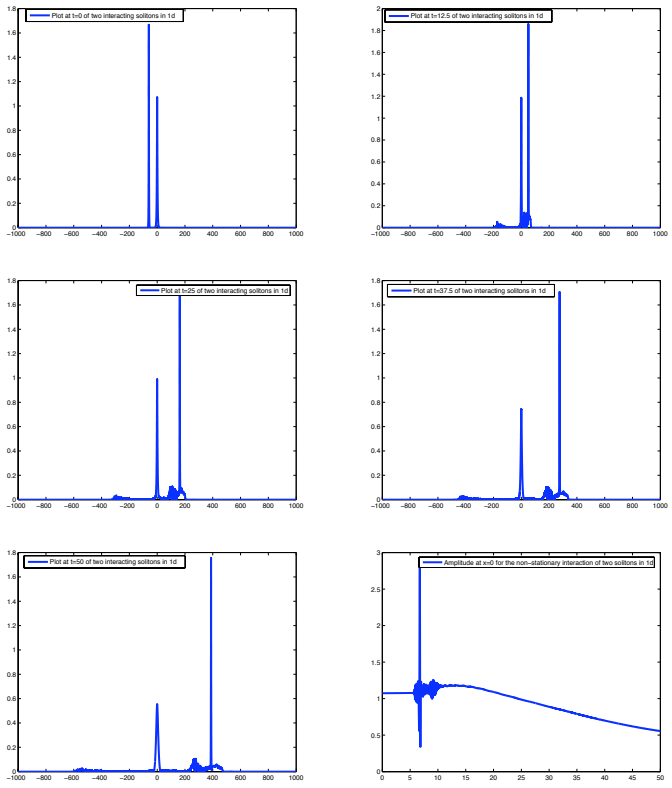

FIGURE 8. A more accurate numerical simulation of a stable soliton moving with velocity $v = 5$ interacting with a stationary soliton which is stable, yet close to the minimal mass soliton for saturated nonlinearities of type 1 in $\mathbb{R}$ with $p = 7$, $q = 3$. Presented here are plots of the absolute value of the solution with respect to the spatial grid for times $t = 0, 12.5, 25, 37.5, 50$ as well as a plot of the amplitude at 0 with respect to time.

f:sol3int

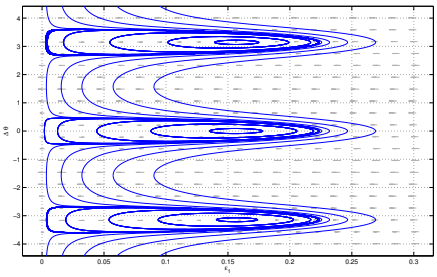

FIGURE 9. A numerical plot of a phase plane with $\Delta\theta$ plotted versus ϵ_1 for $n => 0$. Here, we have $N_{cr}^{FD} = .2$, $n = N - N_{cr}^{FD} = .05$.

fig1

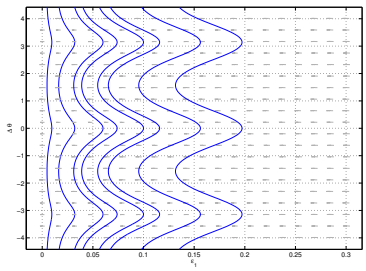

FIGURE 10. A numerical plot of a phase plane with $\Delta\theta$ plotted versus ϵ_1 for $n<0$. Here, we have $N_{cr}^{FD}=.2$, $n=N-N_{cr}^{FD}=-.05$. fig2

is generated by finding the lowest entries of the spectrum of the discretized representation of $H=-\Delta+V$ in the Galerkin approximation, which is an operation embedded in many numerical software programs. For simplicity, we use the eig function from *Matlab*. Then, we may numerically solve the PDE system (12.13) with initial data corresponding to that necessary for the three types of oscillation described in Section 12.9. Note, one could also use the solitons from the spectral renormalization code (see [**?**]) as initial data quite easily, however these represent true nonlinear structures and we wish to observe structures derived from the finite dimensional dynamics, which we only expect to persist on finite time scales due to the nonlinear structure. The orbital stability of the nonlinear objects is an interesting question in its own right and was explored in [**?**]. The equilibrium point of our dynamical system is in fact the finite dimensional part of a soliton solution, so there is no question the orbital stability of the soliton and the long time existence of oscillations near an equilibrium point are related. The phase plane diagrams for the finite dimensional dynamics are plotted using the MATLAB software program *pplane7* [**?**].

Finally, by taking a system such that $|\Omega_0-\Omega_1|$ is comparably large (or well-separation distance, L, comparably close but sufficiently large to guarantee the hypothesized discrete spectrum), we can observe for large perturbations coupling to the continuous spectrum and hence decay of a fully oscillatory solution to a ground state in a finite time. In general, the mass dispersion is rather rapid, hence after a prescribed number of time steps determined by the computational domain, we cut off the solution near the origin and continue solving with the cut-off initial data. See Figures 14, 15, 16 and 17 for various computed time evolutions of the (radiation) damped oscillations observed in such a system.

18.11. Exercises

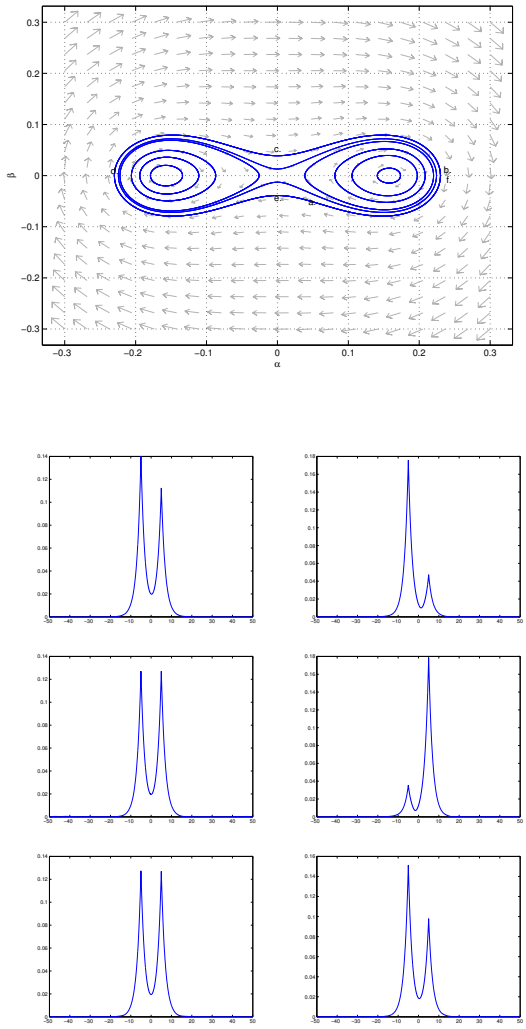

FIGURE 11. At top, a numerical plot of the phase plane diagram for $n = N - N_{cr}^{FD} > 0$ with specific points chosen along a closed orbit which shows full oscillation of mass from one well to another. Below numerical plots of the absolute value of the solution to Equation (12.13) at various times with initial data such that $n = N - N_{cr}^{FD} > 0$ and $\Delta\theta(0) = 1$. The plots correspond to points a, b, c, d, e, and f respectively from the specified orbit.

fig3

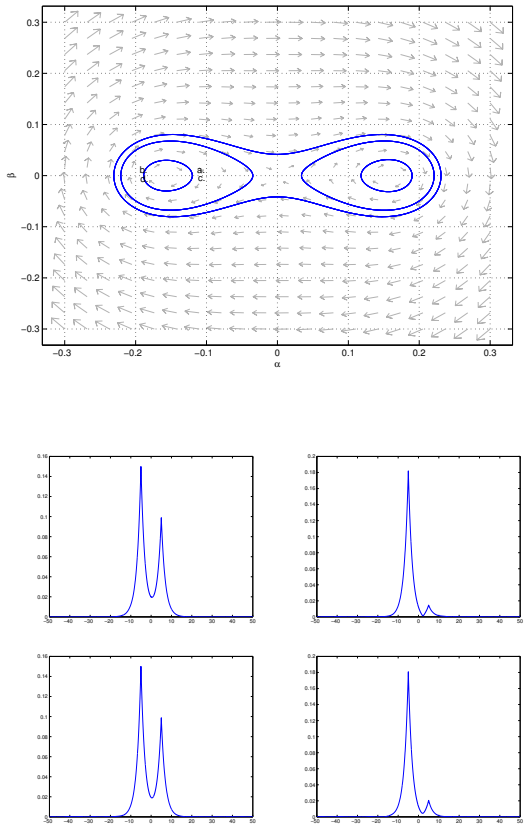

FIGURE 12. At top, a numerical plot of the phase plane diagram for $n = N - N_{cr}^{FD} > 0$ with specific points chosen along a closed orbit which shows localization of the mass on one side of the well. Below, numerical plots of the absolute value of the solution to Equation (12.13) at various times with initial data such that $n = N - N_{cr}^{FD} > 0$ and $\Delta\theta(0) = 0$. The plots correspond to points a, b, c, d respectively from the specified orbit.

fig4

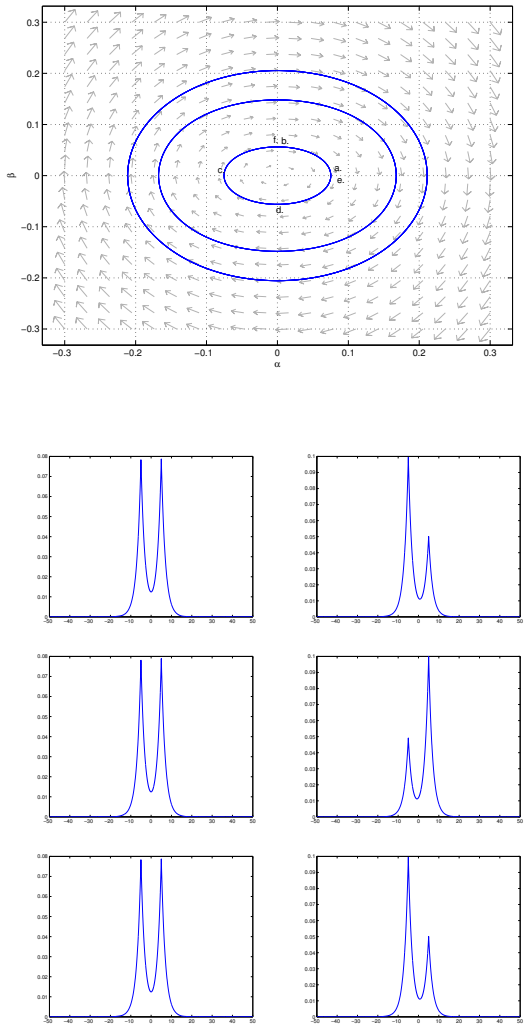

FIGURE 13. At top, a numerical plot of the phase plane diagram for $n = N - N_{cr}^{FD} < 0$ with specific points chosen along a closed orbit. Below, numerical plots of the absolute value of the solution to Equation (12.13) at various times with initial data such that $n = N - N_{cr}^{FD} < 0$ and $\Delta\theta(0) = 0$. The plots correspond to points a, b, c, d, e, and f respectively from the specified orbit.

fig5

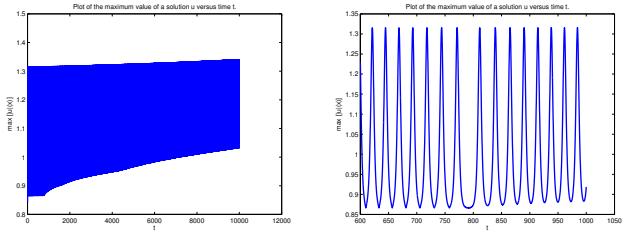

FIGURE 14. A numerical plot of the maximum amplitude of an oscillatory solution decaying to a ground state and in particular the transition region blown up to capture the fast oscillations.

fig9

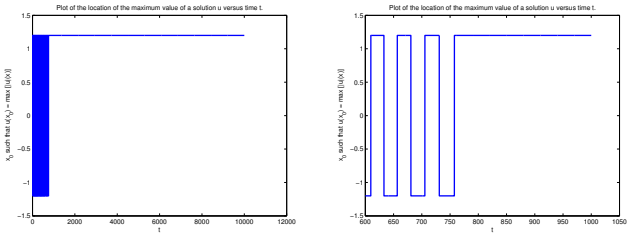

FIGURE 15. A numerical plot of the location of maximum amplitude of an oscillatory solution decaying to a ground state and in particular the transition region blown up to capture the fast oscillations.

fig10

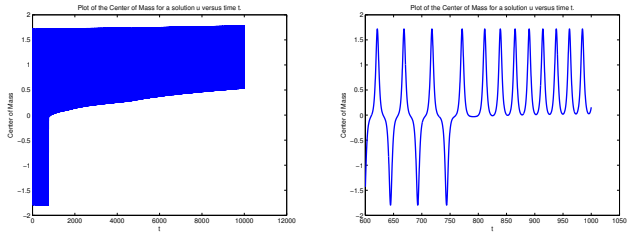

FIGURE 16. A numerical plot of the center of mass of an oscillatory solution decaying to a ground state and in particular the transition region blown up to capture the fast oscillations.

fig11

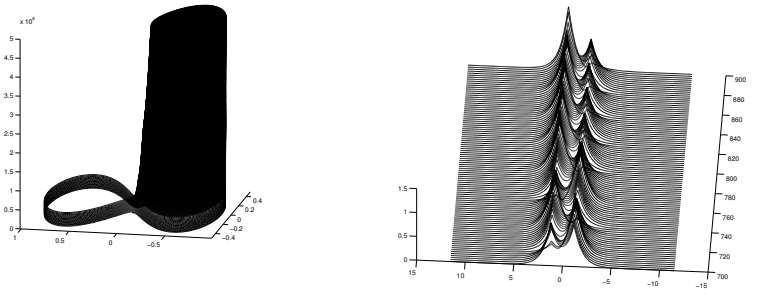

FIGURE 17. A numerical plot of transition across the separatrix for the full infinite dimensional problem in both the projection onto the phase diagram and a plot of the amplitude with respect to time.

fig12

Bibliography

[ABB13] Xavier Antoine, Weizhu Bao, and Christophe Besse. Computational methods for the dynamics of the nonlinear schrodinger/gross-pitaevskii equations. *arXiv preprint arXiv:1305.1093*, 2013.

[ADKM91] G. Akrivis, V. Dougalis, O. Karakashian, and W. McKinney. On fully discrete Galerkin methods of second-order temporal accuracy for the nonlinear Schrödinger equation. *Numerische Mathematik*, 59(194):31–53, 1991.

[ADKM97] G. Akrivis, V. Dougalis, O. Karakashian, and W. McKinney. Solving the systems of equations arising in the discretization of some nonlinear p.d.e's by implicit Runge-Kutta methods. *RAIRO ModåŻŻ. Math. Anal. Numåśĭ.*, 31(3):251–288, 1997.

[ADKM03] G. Akrivis, V. Dougalis, O. Karakashian, and W. McKinney. Numerical approximation of blow-up of radially symmetric solutions of the nonlinear Schrödinger equation. *SIAM Journal of Scientific Computing*, 25(1):186–212, 2003.

[AG] H. Appel and E.K.U. Gross. *Static and time-dependent many-body effects via density-functional theory*. NIC Series Vol. 10. John von Neumann Institute for Computing, Jülich.

[AG90] E.L. Allgower and K. Georg. *Numerical Continuation Methods: An Introduction*. Springer, 1990.

[Agm75a] S. Agmon. Spectral properties for Schrödinger operators and scattering theory. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 2:151–218, 1975.

[Agm75b] Shmuel Agmon. Spectral properties of schrödinger operators and scattering theory. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 2(2):151–218, 1975.

[AM05] M.J. Ablowitz and Z.H. Musslimani. Spectral renormalization method for computing self-localized solutions to nonlinear systems. *Optics letters*, 30(16):2140–2142, 2005.

[ARW95] Uri M Ascher, Steven J Ruuth, and Brian TR Wetton. Implicit-explicit methods for time-dependent partial differential equations. *SIAM Journal on Numerical Analysis*, 32(3):797–823, 1995.

[BC12] Weizhu Bao and Yongyong Cai. Mathematical theory and numerical methods for bose-einstein condensation. *arXiv preprint arXiv:1212.5341*, 2012.

[BCOR09] Sergio Blanes, F Casas, JA Oteo, and José Ros. The magnus expansion and some of its applications. *Physics Reports*, 470(5):151–238, 2009.

[BD04] W. Bao and Q. Du. Computing the ground state solution of Bose-Einstein condensates by a normalized gradient flow. *SIAM Journal of Scientific Computing*, 25:1674–1697, 2004.

[Bea77] R. Beals. Characterization of pseudodifferential operators and applications. *Duke Mathematical Journal*, 44(1):45–57, 1977.

[BJM03a] Weizhu Bao, Dieter Jaksch, and Peter A Markowich. Numerical solution of the gross–pitaevskii equation for bose–einstein condensation. *Journal of Computational Physics*, 187(1):318–342, 2003.

[BJM03b] Weizhu Bao, Shi Jin, and Peter A Markowich. Numerical study of time-splitting spectral discretizations of nonlinear schrödinger equations in the semiclassical regimes. *SIAM Journal on Scientific Computing*, 25(1):27–64, 2003.

[BL76] Jöran Bergh and Jörgen Löfström. *Interpolation spaces. An introduction*. Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[BL83] H. Berestycki and P.-L. Lions. Nonlinear scalar field equations, I, existence of a ground state. *Arch. Rat. Mech. Anal.*, 82:313–345, 1983.

[BMS03] Weizhu Bao, NJ Mauser, and HP Stimming. Effective one particle quantum dynamics of electrons: A numerical study of the schrodinger-poisson-xalpha model. *Communications in Mathematical Sciences*, 1(4):809–828, 2003.

[Bon75] Jerry Bona. On the stability of solitary waves. *Proc. Roy. Soc. Lond. A*, 328:153–183, 1975.

[Boy01] J.P. Boyd. *Chebyshev and Fourier Spectral Methods*. Courier Dover Publications, 2001.

[BP95] V. S. Buslaev and G. S. Perel′man. On the stability of solitary waves for nonlinear Schrödinger equations. In *Nonlinear evolution equations*, volume 164 of *Amer. Math. Soc. Transl. Ser. 2*, pages 75–98. Amer. Math. Soc., Providence, RI, 1995.

[BS03] Vladimir S. Buslaev and Catherine Sulem. On asymptotic stability of solitary waves for nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 20(3):419–475, 2003.

[BS05] Weizhu Bao and Jie Shen. A fourth-order time-splitting laguerre–hermite pseudospectral method for bose–einstein condensates. *SIAM Journal on Scientific Computing*, 26(6):2010–2028, 2005.

[BW98] J. Bourgain and W. Wang. Construction of blowup solutions for the nonlinear Schrödinger equation with critical nonlinearity. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze Sér. 4*, 25(1–2):197–215, 1998.

[CGNT08] Shu-Ming Chang, Stephen Gustafson, Kenji Nakanishi, and Tai-Peng Tsai. Spectra of linearized operators for NLS solitary waves. *SIAM J. Math. Anal.*, 39(4):1070–1111, 2007/08.

[Chi07] Siu A Chin. Higher-order splitting algorithms for solving the nonlinear schrödinger equation and their instabilities. *Physical Review E*, 76(5):056708, 2007.

[CK03] M. Christ and A. Kiselev. Maximal functions associated with filtrations. *Comm. Pure Appl. Math.*, 56(11):1565–1607, 2003.

[CM02] SM Cox and PC Matthews. Exponential time differencing for stiff systems. *Journal of Computational Physics*, 176(2):430–455, 2002.

[CMMT12] H. Christianson, J. Marzuola, J. Metcalfe, and M. Taylor. Stability of energy-minimizing standing waves, submitted. 2012.

[CMR04] Alberto Castro, Miguel AL Marques, and Angel Rubio. Propagators for the time-dependent kohn–sham equations. *The Journal of chemical physics*, 121:3425, 2004.

[CP05] Scipio Cuccagna and Dmitry Pelinovsky. Bifurcations from the endpoints of the essential spectrum in the linearized nonlinear Schrödinger problem. *J. Math. Phys.*, 46(5):053520, 15, 2005.

[CPV05] Scipio Cuccagna, Dmitry Pelinovsky, and Vitali Vougalter. Spectra of positive and negative energies in the linearized NLS problem. *Comm. Pure Appl. Math.*, 58(1):1–29, 2005.

[Cuc03] Cuccagna. On asymptotic stability of ground states of NLS. *Rev. Math. Phys.*, 15(8):877–903, 2003.

[dBHNS99] Anne de Bouard, Nakao Hayashi, Pavel I. Naumkin, and Jean-Claude Saut. Scattering problem and asymptotics for a relativistic nonlinear Schrödinger equation. *Nonlinearity*, 12(5):1415–1425, 1999.

[dBHS97] Anne de Bouard, Nakao Hayashi, and Jean-Claude Saut. Global existence of small solutions to a relativistic nonlinear Schrödinger equation. *Comm. Math. Phys.*, 189(1):73–105, 1997.

[DF06] Piero D'Ancona and Luca Fanelli. L^p-boundedness of the wave operator for the one dimensional Schrödinger operator. *Comm. Math. Phys.*, 268(2):415–438, 2006.

[DS05] L. Demanet and W. Schlag. Numerical cubic nls. *Nonlinearity*, 2005.

[DT79] P. Deift and E. Trubowitz. Inverse scattering on the line. *Comm. Pure Appl. Math.*, 32(2):121–251, 1979.

[DT12] Stéphane Descombes and Mechthild Thalhammer. The lie–trotter splitting for nonlinear evolutionary problems with critical parameters: a compact local error representation and application to nonlinear schrödinger equations in the semiclassical regime. *IMA Journal of Numerical Analysis*, 2012.

[DW11] Vincent Duchêne and Michael I. Weinstein. Scattering, homogenization, and interface effects for oscillatory potentials with strong singularities. *Multiscale Model. Simul.*, 9(3):1017–1063, 2011.

[ES06] B. Erdogan and W. Schlag. Dispersive estimates for Schrödinger operators in the presence of a resonance and/or eigenvalue at zero energy in dimension three. II. *J. Anal. Math*, 99:199–248, 2006.

[Eva98] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[EZ06] L.C. Evans and M. Zworski. Lectures on semiclassical analysis. 2006.

[FGL09] Erwan Faou, Vasile Gradinaru, and Christian Lubich. Computing semiclassical quantum dynamics with Hagedorn wavepackets. *SIAM J. Sci. Comput.*, 31(4):3027–3041, 2009.

[FJ08] Reika Fukuizumi and Louis Jeanjean. Stability of standing waves for a nonlinear Schrödinger equation with a repulsive Dirac delta potential. *Discrete Contin. Dyn. Syst.*, 21(1):121–136, 2008.

[Fol95] Gerald B. Folland. *Introduction to partial differential equations*. Princeton University Press, Princeton, NJ, second edition, 1995.

[FOO08] Reika Fukuizumi, Masahito Ohta, and Tohru Ozawa. Nonlinear Schrödinger equation with a point defect. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(5):837–845, 2008.

[Gau11] Ludwig Gauckler. Convergence of a split-step Hermite method for the Gross-Pitaevskii equation. *IMA J. Numer. Anal.*, 31(2):396–415, 2011.

[GHW04] Roy H. Goodman, Philip J. Holmes, and Michael I. Weinstein. Strong NLS soliton-defect interactions. *Phys. D*, 192(3-4):215–248, 2004.

[GS01] David J Griffiths and Carl A Steinke. Waves in locally periodic media. *American Journal of Physics*, 69:137, 2001.

[GSS90] M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry. ii. *J. Funct. Anal.*, 94(2):308–348, 1990.

[GT92] David J Griffiths and Nicholas F Taussig. Scattering from a locally periodic potential. *American journal of physics*, 60:883, 1992.

[GT01] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[HÖ3a] L. Hörmander. *The Analysis of Linear Partial Differential Operators III*. Classics in Mathematics. Springer-Verlag, 2003.

[HÖ3b] L. Hörmander. *The Analysis of Linear Partial Differential Operators IV*. Classics in Mathematics. Springer-Verlag, 2003.

[HMZ07a] J. Holmer, J. Marzuola, and M. Zworski. Soliton splitting by external delta potentials. *J. Nonlinear Sci.*, 17(4):349–367, 2007.

[HMZ07b] Justin Holmer, Jeremy Marzuola, and Maciej Zworski. Fast soliton scattering by delta impurities. *Comm. Math. Phys.*, 274(1):187–216, 2007.

[Hol11] Justin Holmer. Dynamics of KdV solitons in the presence of a slowly varying potential. *Int. Math. Res. Not. IMRN*, (23):5367–5397, 2011.

[Hör05] Lars Hörmander. *The analysis of linear partial differential operators. II*. Classics in Mathematics. Springer-Verlag, Berlin, 2005. Differential operators with constant coefficients, Reprint of the 1983 original.

[HS96] P. Hislop and I.M. Sigal. *Introduction to Spectral Theory With Applications to Schrödinger Operators*, volume 113 of *Applied Mathematical Sciences*. Springer-Verlag, 1996.

[HT73] RH Hardin and FD Tappert. Applications of the split-step fourier method to the numerical solution of nonlinear and variable coefficient wave equations. *SIAM rev*, 15(2):423, 1973.

[HZ07] Justin Holmer and Maciej Zworski. Slow soliton interaction with delta impurities. *J. Mod. Dyn.*, 1(4):689–718, 2007.

[JL00] Tobias Jahnke and Christian Lubich. Error bounds for exponential operator splittings. *BIT*, 40(4):735–744, 2000.

[JOP$^+$] Eric Jones, Travis Oliphant, Pearu Peterson, et al. SciPy: Open source scientific tools for Python, 2001–.

[JW04] R. K. Jackson and M. I. Weinstein. Geometric analysis of bifurcation and symmetry breaking in a Gross-Pitaevskii equation. *J. Statist. Phys.*, 116(1-4):881–905, 2004.

[KPRV05] C. E. Kenig, G. Ponce, C. Rolvung, and L. Vega. Variable coefficient Schrödinger flows for ultrahyperbolic operators. *Adv. Math.*, 196(2):373–486, 2005.

[KPRV06] Carlos E. Kenig, Gustavo Ponce, Christian Rolvung, and Luis Vega. The general quasilinear ultrahyperbolic Schrödinger equation. *Adv. Math.*, 206(2):402–433, 2006.

[KPV93] Carlos E. Kenig, Gustavo Ponce, and Luis Vega. Small solutions to nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(3):255–288, 1993.

[KPV98] Carlos E. Kenig, Gustavo Ponce, and Luis Vega. Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations. *Invent. Math.*, 134(3):489–545, 1998.

[KPV04] Carlos E. Kenig, Gustavo Ponce, and Luis Vega. The Cauchy problem for quasi-linear Schrödinger equations. *Invent. Math.*, 158(2):343–388, 2004.

[KS06a] J. Krieger and W. Schlag. Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension. *J. Amer. Math. Soc.*, 19(4):815–920, 2006.

[KS06b] J. Krieger and W. Schlag. Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension. *J. Amer. Math. Soc.*, 19(4):815–920, 2006.

[KT05a] Aly-Khan Kassam and Lloyd N. Trefethen. Fourth-order time-stepping for stiff PDEs. *SIAM J. Sci. Comput.*, 26(4):1214–1233 (electronic), 2005.

[KT05b] Aly-Khan Kassam and Lloyd N. Trefethen. Fourth-order time-stepping for stiff PDEs. *SIAM J. Sci. Comput.*, 26(4):1214–1233 (electronic), 2005.

[Kur81] Susumu Kurihara. Large-amplitude quasi-solitons in superfluid films. *Journal of the Physical Society of Japan*, 50:3262, 1981.

[LB92] J. Lund and K.L. Bowers. *Sinc Methods for Quadrature and Differential Equations*. Society for Industrial Mathematics, 1992.

[LCFF+08] Stefan Le Coz, Reika Fukuizumi, Gadi Fibich, Baruch Ksherim, and Yonatan Sivan. Instability of bound states of a nonlinear Schrödinger equation with a Dirac potential. *Phys. D*, 237(8):1103–1128, 2008.

[Lio84a] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(2):109–145, 1984.

[Lio84b] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. II. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(4):223–283, 1984.

[LL01] Elliott H. Lieb and Michael Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.

[Lub08] Christian Lubich. On splitting methods for Schrödinger-Poisson and cubic nonlinear Schrödinger equations. *Math. Comp.*, 77(264):2141–2153, 2008.

[LY07] T.I. Lakoba and J. Yang. A generalized Petviashvili iteration method for scalar and vector Hamiltonian equations with arbitrary form of nonlinearity. *Journal of Computational Physics*, 226(2):1668–1692, 2007.

[Mag54] Wilhelm Magnus. On the exponential solution of differential equations for a linear operator. *Comm. Pure Appl. Math.*, 7:649–673, 1954.

[Mar10] J. Marzuola. A class of stable perturbations for a minimal mass soliton in three dimensional saturated nonlinear Schrödinger equation. 2010.

[McC93] K. McCleod. Uniqueness of Positive Radial Solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$, II. *Transactions of the American Mathematical Society*, 339(2):495–505, 1993.

[MMT08] Jeremy Marzuola, Jason Metcalfe, and Daniel Tataru. Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations. *J. Funct. Anal.*, 255(6):1497–1553, 2008.

[MMT12] Jeremy L. Marzuola, Jason Metcalfe, and Daniel Tataru. Quasilinear Schrödinger equations I: Small data and quadratic interactions. *Adv. Math.*, 231(2):1151–1172, 2012.

[MMT13] Jeremy L. Marzuola, Jason Metcalfe, and Daniel Tataru. Quasilinear Schrödinger equations II: Small data and cubic interactions. *Preprint*, 2013.

[MQ02] Robert I. McLachlan and G. Reinout W. Quispel. Splitting methods. *Acta Numer.*, 11:341–434, 2002.

[MRS10] J.L. Marzuola, S. Raynor, and G. Simpson. Dynamics in saturated nls. *Journal of Nonlinear Science*, 2010.

[MS11a] Jeremy L. Marzuola and Gideon Simpson. Spectral analysis for matrix Hamiltonian operators. *Nonlinearity*, 24(2):389–429, 2011.

[MS11b] J.L. Marzuola and G. Simpson. Absence of embedded eigenvalues. *Nonlinearity*, 2011.

[MW10] Jeremy L. Marzuola and Michael I. Weinstein. Long time dynamics near the symmetry breaking bifurcation for nonlinear Schrödinger/Gross-Pitaevskii equations. *Discrete Contin. Dyn. Syst.*, 28(4):1505–1554, 2010.

[Per01] Galina Perelman. On the formation of singularities in solutions of the critical nonlinear Schrödinger equation. *Ann. Henri Poincaré*, 2(4):605–673, 2001.

[Pet76] V.I. Petviashvili. Equation of an extraordinary soliton. *Soviet Journal of Plasma Physics*, 2:469–472, 1976.

[PGL03] VıÌĄctor M Pérez-GarcıÌĄa and Xiao-yan Liu. Numerical methods for the simulation of trapped nonlinear schrödinger systems. *Applied Mathematics and Computation*, 144(2):215–235, 2003.

[PM90] D Pathria and J Ll Morris. Pseudo-spectral solution of nonlinear schrödinger equations. *Journal of Computational Physics*, 87(1):108–125, 1990.

[Poh65] S. Pohozaev. Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. *Soviet Math.*, 5:1408–1411, 1965.

[Pop01] Markus Poppenberg. On the local well posedness of quasilinear Schrödinger equations in arbitrary space dimension. *J. Differential Equations*, 172(1):83–115, 2001.

[Pot09] Trevor Potter. Effective dynamics for n-solitons of the gross-pitaevskii equation. *arXiv:1009.4910*, 2009.

[PS04] Dmitry E. Pelinovsky and Yury A. Stepanyants. Convergence of Petviashvili's iteration method for numerical approximation of stationary solutions of nonlinear wave equations. *SIAM J. Numer. Anal.*, 42(3):1110–1127 (electronic), 2004.

[RS78a] M. Reed and B. Simon. *Methods of mathematical physics, vol. 4*. Academic Press Inc., New York, 1978.

[RS78b] Michael Reed and Barry Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.

[RS79] Michael Reed and Barry Simon. *Methods of modern mathematical physics. III*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1979. Scattering theory.

[RSS03a] I. Rodnianski, W. Schlag, and A. Soffer. Asymptotic stability of N-soliton states of NLS. Preprint, 2003.

[RSS03b] I. Rodnianski, W. Schlag, and A. Soffer. Asymptotic stability of N-soliton states of NLS. Preprint, 2003.

[Sch02] Martin Schechter. *Operator methods in quantum mechanics*. Dover Publications Inc., Mineola, NY, 2002. Reprint of the 1981 original.

[Sch09] W. Schlag. Stable manifolds for an orbitally unstable NLS. *Annals of Math*, 169:139–227, 2009.

[Sha83] J. Shatah. Stable Standing Waves of Nonlinear Klein-Gordon Equations. *Communications in Mathematical Physics*, 91:313–327, 1983.

[Sha85] J. Shatah. Unstable Ground State of Nonlinear Klein-Gordon Equations. *Transactions of the American Mathematical Society*, 290(2):701–710, 1985.

[SS84] JM Sanz-Serna. Methods for the numerical solution of the nonlinear schrödinger equation. *Mathematics of Computation*, 43(167):21–27, 1984.

[SS85] J. Shatah and W. Strauss. Instability of Nonlinear Bound States. *Communications in Mathematical Physics*, 100:173–190, 1985.

[SS99] Catherine Sulem and Pierre-Louis Sulem. *The nonlinear Schrödinger equation*, volume 139 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1999. Self-focusing and wave collapse.

[Ste81] F. Stenger. Numerical methods based on the whittaker cardinal, or sinc functions. *SIAM Review*, 23:165–224, 1981.

[Ste93a] E. Stein. *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, 1993.

[Ste93b] F. Stenger. *Numerical Methods Based on Sinc and Analytic Functions*. Springer, 1993.

[Ste00] F. Stenger. Summary of sinc numerical methods. *Journal of Computational and Applied Mathematics*, 121(1-2):379–420, 2000.

[Str68] Gilbert Strang. On the construction and comparison of difference schemes. *SIAM Journal on Numerical Analysis*, 5(3):506–517, 1968.

[Str03] Robert S. Strichartz. *A guide to distribution theory and Fourier transforms*. World Scientific Publishing Co. Inc., River Edge, NJ, 2003. Reprint of the 1994 original [CRC, Boca Raton; MR1276724 (95f:42001)].

[Str05] J. Strain. Numerical Analysis. Unpublished Lecture Notes, 2005.

[Suz93] Masuo Suzuki. Improved trotter-like formula. *Physics Letters A*, 180(3):232–234, 1993.

[SW13] Jie Shen and Zhong-Qing Wang. Error analysis of the strang time-splitting laguerre–hermite/hermite collocation methods for the gross–pitaevskii equation. *Foundations of Computational Mathematics*, 13(1):99–137, 2013.

[TA84] Thiab R Taha and Mark I Ablowitz. Analytical and numerical aspects of certain nonlinear evolution equations. ii. numerical, nonlinear schrödinger equation. *Journal of Computational Physics*, 55(2):203–230, 1984.

[Tat00] Daniel Tataru. Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation. *Amer. J. Math.*, 122(2):349–376, 2000.

[Tay97a] M. Taylor. *Partial Differential Equations I*, volume 115 of *Classics in Mathematics*. Springer-Verlag, 1997.

[Tay97b] M. Taylor. *Partial Differential Equations II*, volume 116 of *Classics in Mathematics*. Springer-Verlag, 1997.

[Tha08] Mechthild Thalhammer. High-order exponential operator splitting methods for time-dependent schrödinger equations. *SIAM Journal on Numerical Analysis*, 46(4):2022–2038, 2008.

[TZ] S.H. Tang and M. Zworski. *Potential Scattering on the Real Line*. Unpublished Lecture Notes.

[Wed99] Ricardo Weder. The $W_{k,p}$-continuity of the Schrödinger wave operators on the line. *Comm. Math. Phys.*, 208(2):507–520, 1999.

[Wei83] M. Weinstein. Nonlinear Schrödinger equations and sharp interpolation estimates. *Commun. Math. Phys.*, 87:567–576, 1983.

[Wei85a] M. Weinstein. Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.*, 16:472–491, 1985.

[Wei85b] M. I. Weinstein. Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.*, 16:472–491, 1985.

[Wei86] M. I. Weinstein. Lyapunov stability of ground states of nonlinear dispersive evolution equations. *Comm. Pure Appl. Math.*, 39:472–491, 1986.

[WH86] JAC Weideman and BM Herbst. Split-step methods for the solution of the nonlinear schrödinger equation. *SIAM Journal on Numerical Analysis*, 23(3):485–507, 1986.

[Yaj95] Kenji Yajima. The $W^{k,p}$-continuity of wave operators for Schrödinger operators. *J. Math. Soc. Japan*, 47(3):551–581, 1995.

[YL07] J. Yang and T.I. Lakoba. Universally-convergent squared-operator iteration methods for solitary waves in general nonlinear wave equations. *Studies in Applied Mathematics*, 118(2):153, 2007.

[YL08] J. Yang and T.I. Lakoba. Accelerated imaginary-time evolution methods for the computation of solitary waves. *Studies in Applied Mathematics*, 120(3):265–292, 2008.

[Yos90] Haruo Yoshida. Construction of higher order symplectic integrators. *Physics Letters A*, 150(5):262–268, 1990.