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Solitons
Stability
Linearization
Spectral Theory
Estimates
Main Result
Summary

Stable perturbations of a minimal mass soliton for a saturated NLSE in 3d

Work done for my thesis under the direction of Professor Daniel Tataru

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Department of Mathematics
University of California, Berkeley

University of Toronto - February 12th, 2007

Outline

Preliminaries

Solitons

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Main Result

Saturated Nonlinear Schrödinger Equations

- The Focusing Saturated NLS Problem:

$$\begin{cases} iu_t + \Delta u + \beta(|u|^2)u = 0, & u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \\ u(0,x) = u_0(x), & u_0 : \mathbb{R}^d \to \mathbb{R}. \end{cases}$$

- The structure of β:

$$\beta(s) = s^{\frac{p-1}{2}} \frac{s^{\frac{q-p}{2}}}{1 + s^{\frac{q-p}{2}}},$$

where $q > 2 + \frac{2}{d}$ and $1 + \frac{4}{d} > p > 1$ for $d \geq 3$ and $\infty > q > 1 + \frac{4}{d} > p > 1$ for $d < 3$.

Saturated Nonlinear Schrödinger Equations

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Saturated Nonlinear Schrödinger Equations Cont.

- More interestingly, we also allow for the following perturbations of the cubic nonlinearity

$$\beta(s) = \frac{s}{\langle s \rangle^{1-\frac{p-1}{2}}},$$

where $< x >= (1 + x^2)^{\frac{1}{2}}$ is the standard Japanese bracket and $1 + \frac{4}{d} > p > 1$ as above.
- Description of phenomenon:
For $|u| \gg 1$, the behavior is known as L^2 SUB-CRITICAL.
For $|u| \ll 1$, the behavior is known as L^2 SUPER-CRITICAL.
- Criticality deals with the scale invariance in the typical monomial nonlinearity studied widely in the literature

$$\beta(s^2) = s^{p-1}.$$

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Motivation

- Saturated nonlinearities have applications to describing the dielectric constant of gas vapors where a laser beam propagates, laser beams in plasmas and Bose superfluids at zero temperatures.
- Mathematically, the author's interest in saturated nonlinearities arose out of the work of Rodnianski-Schlag-Soffer, who prove asymptotic stability for a collection of N solitons under various separation conditions for

$$\beta(s) = s^{\frac{p-1}{2}} \frac{s^{\frac{q-p}{2}}}{\epsilon + s^{\frac{q-p}{2}}},$$

where ϵ small. As $\epsilon \to 0$, this system approaches the standard sub-critical monomial nonlinearity.
- Note this is equivalent to the β presented above using a rescaling

$$u(t,x) \to \gamma^{\frac{2}{q}} v(\gamma^2 t, \gamma x).$$

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Conserved Quantities

- Conservation of Mass:

$$Q(u) = \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 dx = \frac{1}{2} \int_{\mathbb{R}^d} |u_0|^2 dx.$$

- Conservation of Energy:

$$\begin{aligned} E(u) &= \int_{\mathbb{R}^d} |\nabla u|^2 dx - \int_{\mathbb{R}^d} G(|u|^2) dx \\ &= \int_{\mathbb{R}^d} |\nabla u_0|^2 dx - \int_{\mathbb{R}^d} G(|u_0|^2) dx, \end{aligned}$$

where

$$G(t) = \int_0^t \beta(s) ds.$$

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Soliton Solutions

- Equation:
  Plugging in the ansatz $u = e^{i\lambda^2 t}\varphi_\lambda(x)$, we have:
  $$\Delta\varphi_\lambda - \lambda^2\varphi_\lambda + \beta(\varphi_\lambda^2)\varphi_\lambda = 0.$$
- Description of the Solution:
  The unique soliton solution φ is positive, radially symmetric, at least C^2 (which gives far better regularity through standard elliptic estimates), and exponentially decaying.
- The soliton curve:
  Set $Q(\lambda) = Q(\varphi_\lambda)$, $E(\lambda) = E(\varphi_\lambda)$. A variational argument shows that $\partial_\lambda Q = -\lambda\partial_\lambda E$. Stability theory for solitons depends heavily upon the sign of $\partial_\lambda Q$. For monomial nonlinearities, this is done by scaling.

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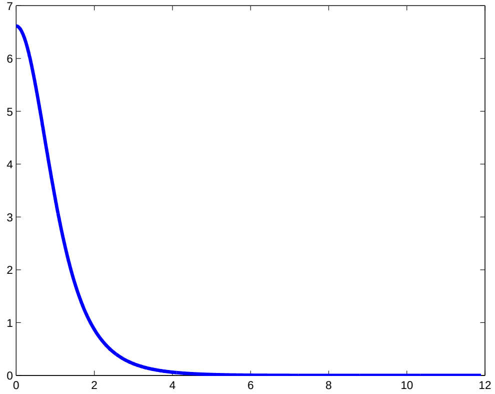

- Figure: A plot of a soliton.

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Soliton Solutions Cont.

- $Q(\lambda)$ and $E(\lambda)$ are C^1.
- A numerical plot shows a curve for a saturated nonlinearity:

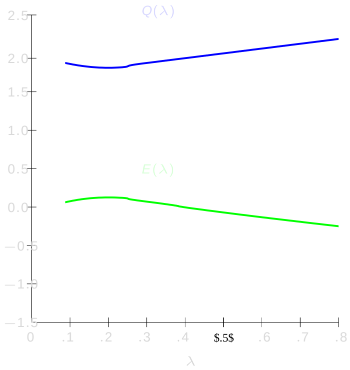

Figure: A sample soliton curve.

- Say the minimal mass soliton occurs at λ_0.

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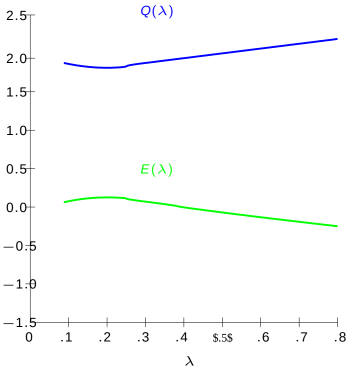

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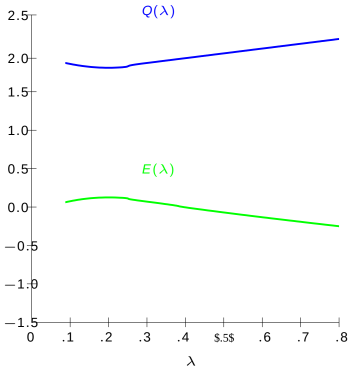

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Soliton Stability.

- From the works of Weinstein, Shatah, and Grillakis-Shatah-Strauss, we know that if

$$\partial_\lambda Q(\lambda) > 0, \ (< 0)$$

  the solitons are orbitally stable (unstable) under small perturbations.
- For the minimal mass soliton, we are at the intersection of these regimes. From the work of Comech-Pelinovsky, we know that the minimal mass soliton is unstable under general perturbations.

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Orbital Stability

- A soliton φ is orbitally stable if for any $\epsilon > 0$, there exists $\delta > 0$ such that if the initial condition u_0 is such that

$$\inf_{\theta, y} \| u_0(x) - e^{i\theta} \varphi(x+y) \|_{H^1(x)} < \delta,$$

then the solution $u(x,t)$ of NLS satisfies

$$\inf_{\theta, y} \| u(x,t) - e^{i\theta} \varphi(x+y) \|_{H^1(x)} < \epsilon.$$

Linearized Hamiltonian

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- Solution:
  Set $\lambda = 1$. We take a solution of the form $u(x,t) = e^{it}(\varphi(x) + R(x,t))$.
- The equation for $R = v_1 + iv_2$:

$$
\begin{aligned}
iR_t + \Delta R &= -\beta(|\varphi|^2)R \\
&- 2\beta'(|\varphi|^2)\varphi \mathrm{Re}(R) + O(R^2),
\end{aligned}
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  by splitting R up into its real and imaginary parts, then doing a Taylor Expansion.

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Linearized Hamiltonian Cont.

- Hence, we have

$$\partial_t \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathcal{H} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

- Above,

$$\mathcal{H} = \begin{pmatrix} 0 & iL_- \\ -iL_+ & 0 \end{pmatrix},$$

where

$$L_- = -\Delta + \lambda - \beta(\varphi_\lambda^2)$$

and

$$L_+ = -\Delta + \lambda - \beta(\varphi_\lambda^2) - 2\beta'(\varphi_\lambda^2)\varphi_\lambda^2.$$

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Spectral Properties of L_+, $L-$.

- L_+, L_- are self-adjoint operators.
- $L_- \geq 0$.
- There exists one simple, negative eigenvalue for L_+.
- The continuous spectrum of L_-, L_+ is (λ^2, ∞).
- The null space of L_- is spanned by φ_λ.
- The null space of L_+ is spanned by $\partial_j \varphi_\lambda$ for $j = 1, ..., d$.

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Spectral Properties of $\mathcal{H}$.

- The generalized null space is at least of order 2. To see this, look at $\partial_\lambda \varphi_\lambda$. We have $L_+ \partial_\lambda \varphi_\lambda = 2\varphi_\lambda$. Hence, $L_- L_+ \partial_\lambda \varphi_\lambda = 0$.
- At a minimal mass soliton, the generalized null space is of order 4.
- An embedded resonance is in fact an embedded eigenvalue.
- The continuous spectrum of $\mathcal{H}$ is $(-\infty, -\lambda^2) \cup (\lambda^2, \infty)$.

- **Theorem**

  There are no large embedded eigenvalues for $\mathcal{H}$.

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There are a finite number of allowed spherical harmonics in an expansion for an embedded eigenvalue.

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Spectral Assumptions on $\mathcal{H}$.

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- For $\mathcal{H}_{\lambda_0}$, the only real eigenvalue in $[-\lambda^2, \lambda^2]$ is 0.
- There are no resonances at $\pm\lambda^2$.
- Note that we have designed numerical experiments that will allow us to test these assumptions in the limited range they apply.
- We say that an $\mathcal{H}$ satisfying all of the above conditions is *admissible*.

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Dispersive Estimates for $\mathcal{H}$.

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- Theorem (Schlag-Erdogan,Bourgain)

$$\begin{aligned}
\|e^{it\mathcal{H}} P_c \varphi\|_{H^1} &\leq C\|\varphi\|_{H^1} \\
\|e^{it\mathcal{H}} (P_c \varphi)\|_{H^s} &\leq C\|\varphi\|_{H^s} \\
\|e^{it\mathcal{H}} (P_d \varphi)\|_{H^s} &\leq C(1+|t|^3) \int e^{-c|x|} |\varphi(x)| dx \\
\|e^{it\mathcal{H}} (P_c \varphi)\|_{L^2} &\leq C(\||x|^\alpha \varphi\|_{L^2} + (1+|t|^\alpha)\|\varphi\|_{H^\alpha} \\
\|e^{it\mathcal{H}} (P_d \varphi)\|_{L^2} &\leq C(1+|t|^3) \int |\varphi| e^{-c|x|} dx,
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\|e^{it\mathcal{H}}(P_c\varphi)\|_{L^2} &\leq C(\||x|^\alpha\varphi\|_{L^2}+(1+|t|^\alpha)\|\varphi\|_{H^\alpha} \\
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$$\begin{cases} iu_t + \Delta u + \beta(|u|^2)u = 0, & u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C} \\ u(0,x) = u_0(x), & u_0 : \mathbb{R}^3 \to \mathbb{C} \end{cases} \tag{1}$$

  where β is an admissible saturated nonlinearity, there exists a codimension 10(*) *(at most) manifold $\mathcal{M}$ embedded in function space Σ, such that given $\psi \in \mathcal{M}$, Eqn.* (1) *has a solution of the form $u(x,t) = \varphi_{\lambda_0} + e^{i\mathcal{H}t}\psi + w(x,t)$, where $w(x,t) \to 0$ as $t \to \infty$ uniformly in x.*

  - * This is work in progress, but the general idea is that most of the functions in the generalized null space will still produce stable perturbations.

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Construction of $\mathcal{M}$ and Σ.

- To begin, for simplicity assume $\lambda_0 = 1$. Using an argument similar to that of Bourgain-Wang, we look for solutions of the form

$$u(x,t) = e^{ixt}\varphi + z_\psi + w(x,t),$$

where $z_\psi = e^{i\mathcal{H}t}\psi$.

- Note that Bourgain-Wang used z_ψ a solution to the critical (monomial) nonlinear problem in dimensions $d = 1, 2$ to look at stable perturbations of a blow-up solution via the pseudoconformal transform.
- It is also possible, with restrictions similar to those of B-W to build a solution for $z_\psi = e^{i\Delta t}\psi$.

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Construction of $\mathcal{M}$ and Σ cont.

- To build solutions of this type, we build a contraction map backwards from ∞.
- Note that if we can show w is small, then we can say that the map $\Psi : \psi \to w(t_0)$ is Lipschitz and hence, we have a smooth, finite codimension manifold of perturbations. The codimension will be at most $2d + 4$ since $H^1 \times H^1 = N_g(\mathcal{H}) \oplus \{N_g(\mathcal{H}^*)\}^\perp$ and $N_g(\mathcal{H}) = 2d + 4$. However, in the spirit of Schlag, Krieger-Schlag, the null space functions up to order 2 should not effect the convergence. Hence, the manifold should be of codimension 2.

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Linearization scheme.

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- Let $v_0 = z_\varphi e^{-it}$. Then, we set

$$
\begin{aligned}
f_0 &= \beta(|\varphi + v_0|^2)(\varphi + v_0) - \beta(\varphi^2)\varphi \\
&- (\beta(\varphi^2) + \beta'(\varphi^2)\varphi^2)v_0 - \beta(\varphi^2)\varphi^2\bar{v}_0, \\
a &= [\beta(|\varphi + v_0|^2) + \beta'(|\varphi + v_0|^2)|\varphi + v_0|^2] \\
&- [\beta(\varphi^2) + \beta'(\varphi^2)\varphi^2], \\
b &= \beta'(|\varphi + v_0|^2)(\varphi + v_0)^2 \\
&- \beta'(\varphi^2)\varphi^2, \\
G(w) &= \beta(|\varphi + v_0 + w|^2)(\varphi + v_0 + w) \\
&- \beta(|\varphi + v_0|^2)(\varphi + v_0) \\
&- [\beta(|\varphi + v_0|^2) + \beta'(|\varphi + v_0|^2)|\varphi + v_0|^2]w \\
&- \beta'(|\varphi + v_0|^2)(\varphi + v_0)^2\bar{w}.
\end{aligned}
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Linearization scheme cont.

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- Under the smoothness assumptions on β, we have

$$\begin{aligned} f_0 &= O(|v_0|^2), \\ a &= O(|v_0|), \\ b &= O(|v_0|), \\ G(w) &= O(|w|^2). \end{aligned}$$

- Then, we have

$$\begin{aligned} -iw_t &= \Delta w - w + (\beta(\varphi^2) + \beta'(\varphi^2)\varphi^2)w \\ &+ \beta'(\varphi^2)\varphi^2\bar{w} + f_0 + aw + b\bar{w} + G(w). \end{aligned}$$

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Integral equation for w.

- In other words, we have

$$iw_t + \mathcal{H}w + aw + b\bar{w} + f_0 + G(w) = 0.$$

  We have G to be at least quadratic in w.
- Using the integral formulation of the equation for w

$$\begin{aligned} w(t) &= -i\int_t^\infty e^{i(\tau-t)\mathcal{H}} P_c[f_0 + aw + b\bar{w} + G(w)]d\tau \\ &+ -i\int_t^\infty e^{i(\tau-t)\mathcal{H}} P_d[f_0 + aw + b\bar{w} + G(w)]d\tau, \end{aligned}$$

  where P_c projects onto the continuous part of the spectrum and P_d projects onto the discrete part of the spectrum.

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Existence proof for w.

- Assume that $\|w\|_{L^2} \leq t^{-N}$ for N large.
- Make the assumption $\psi \in \mathcal{M}$, which gives sufficient decay in t for z. Most of this argument will involve standard scattering theory techniques, but applied to a nonselfadjoint operator.
- Prove the existence of w through a bootstrapping argument.

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Distorted Fourier Basis.

- There exists a distorted Fourier basis, u_{ξ_0} for $\mathcal{H}$ which gives a unique representation for the continuous spectrum.
- u_{ξ_0} is a solution to the equation

$$[L_+L_-]u_{\xi_0} = (\lambda^2 + \xi_0^2)^2 u_{\xi_0}.$$

- For each ξ_0, u_{ξ_0} is of the form

$$u_{\xi_0} = e^{ix\cdot\xi_0} + f_{\xi_0},$$

where $f_{\xi_0} \in C^\infty \cap L^p$, $p \geq 4$ and

$$f_{\xi_0} \sim \frac{\cos(|x||\xi_0|)}{|x|}$$

as $x \to \infty$.
- f_{ξ_0} is uniformly bounded and differentiable in ξ_0.

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Time Decay.

- For the linear Schrödinger operator, we can build solutions with better time decay simply by making the assumption on the initial data u_0 that

$$\int_{\mathbb{R}^d} x^\alpha u_0 = 0,$$

for some multi-index α. We then have

$$\|e^{i\Delta t} u_0\|_{L^\infty} \leq t^{-\frac{d+2|\alpha|}{2}}.$$

Time Decay cont.

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- For the linearized matrix Schrödinger operator, from Erdogan-Schlag, we have the standard spectral decomposition from Stone's formula and nice dispersive estimates. We can build solutions with better time decay by using the fact that along the continuous spectrum, there is a representation

$$P_c \mathcal{H} f = \tilde{\mathcal{F}}^{-1} |\xi|^2 \tilde{\mathcal{F}} f,$$

where $\tilde{\mathcal{F}}$ is the Fourier transform with respect to the distorted basis u_{ξ_0}.

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$$\tilde{\mathcal{F}}^{-1} = C \cdot \tilde{\mathcal{F}}^*$$

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Time Decay cont.

- We then have

$$\langle e^{i\mathcal{H}t} P_c f, g \rangle = e^{it} \int e^{it\xi^2} \langle f, u_\xi \rangle \overline{\langle g, \tilde{u}_\xi \rangle} d\xi,$$

  on which we can carefully use stationary phase, nonstationary phase and duality arguments to get time decay. It is here that we must put enough regularity and decay at ∞ on Σ in order to get time decay. This is very similar to the development of Krieger-Schlag in $\mathbb{R} \times \mathbb{R}$ using Wronskian methods to construct the distorted Fourier basis and prove the spectral decomposition.
- Note that a similar and far more well-known approach could be taken for the linear Schrödinger, but the exact formula provides us with a more definitive solution.

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Conclusion

- Despite the fact that the linearized operator is nonselfadjoint, at the minimal mass soliton standard scattering theory techiniques apply and hence stable perturbations can be constructed.
- Along the way to constructing this class, we have extensively studied the spectrum of the linearized operator and constructed bounds for embedded eigenvalues.
- This result distinctly depends upon the smoothness of the nonlinearity.

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Future Work

- The proof that the perturbations exist is fairly well established, but proving that those perturbations live on a manifold with reduced codimension is work in progress.
- Analysis which describes the soliton curve for small values of λ and looks at dispersion below the minimal mass soliton is future work underway with Justin Holmer.
- Studying the necessity of the smoothness condition on the nonlinearity both analytically and numerically is work in progress also with Justin Holmer.
- Studying the dynamics of interactions with minimal mass solitons could lead to interesting annihilation effects.

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