Stable perturbations of a minimal mass soliton for a saturated NLSE in 3d
Work done for my thesis under the direction of Professor Daniel Tataru

Jeremy Marzuola

Department of Mathematics
University of California, Berkeley

University of Toronto - February 12th, 2007
The Focusing Saturated NLS Problem:

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\begin{cases}
  i u_t + \Delta u + \beta(|u|^2)u = 0, & u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \\
  u(0, x) = u_0(x), & u_0 : \mathbb{R}^d \to \mathbb{R}.
\end{cases}
\]

The structure of \( \beta \):

\[
\beta(s) = s^{\frac{p-1}{2}} \frac{s^{\frac{q-p}{2}}}{1 + s^{\frac{q-p}{2}}},
\]

where \( q > 2 + \frac{2}{d} \) and \( 1 + \frac{4}{d} > p > 1 \) for \( d \geq 3 \) and \( \infty > q > 1 + \frac{4}{d} > p > 1 \) for \( d < 3 \).
Saturated Nonlinear Schrödinger Equations

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More interestingly, we also allow for the following perturbations of the cubic nonlinearity

$$\beta(s) = \frac{s}{\langle s \rangle^{1 - \frac{p-1}{2}}}$$

where $\langle x \rangle = (1 + x^2)^{1/2}$ is the standard Japanese bracket and $1 + \frac{4}{d} > p > 1$ as above.

Description of phenomenon:
For $|u| \gg 1$, the behavior is known as $L^2$ SUB-CRITICAL.
For $|u| \ll 1$, the behavior is known as $L^2$ SUPER-CRITICAL.

Criticality deals with the scale invariance in the typical monomial nonlinearity studied widely in the literature

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Motivation

- Saturated nonlinearities have applications to describing the dielectric constant of gas vapors where a laser beam propagates, laser beams in plasmas and Bose superfluids at zero temperatures.

- Mathematically, the author’s interest in saturated nonlinearities arose out of the work of Rodnianski-Schlag-Soffer, who prove asymptotic stability for a collection of $N$ solitons under various separation conditions for

$$\beta(s) = s^{\frac{p-1}{2}} \frac{s^{\frac{q-p}{2}}}{\epsilon + s^{\frac{q-p}{2}}}$$

where $\epsilon$ small. As $\epsilon \to 0$, this system approaches the standard sub-critical monomial nonlinearity.

- Note this is equivalent to the $\beta$ presented above using a rescaling

$$u(t, x) \to \gamma^{\frac{q}{2}} v(\gamma^2 t, \gamma x).$$
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Conserved Quantities

- Conservation of Mass:

\[ Q(u) = \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 dx = \frac{1}{2} \int_{\mathbb{R}^d} |u_0|^2 dx. \]

- Conservation of Energy:

\[
E(u) = \int_{\mathbb{R}^d} |\nabla u|^2 dx - \int_{\mathbb{R}^d} G(|u|^2) dx \\
= \int_{\mathbb{R}^d} |\nabla u_0|^2 dx - \int_{\mathbb{R}^d} G(|u_0|^2) dx,
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\[ G(t) = \int_0^t \beta(s) ds. \]
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Soliton Solutions

- Equation:
  Plugging in the ansatz \( u = e^{i\lambda^2 t} \varphi_\lambda(x) \), we have:

\[
\Delta \varphi_\lambda - \lambda^2 \varphi_\lambda + \beta(\varphi_\lambda^2) \varphi_\lambda = 0.
\]

- Description of the Solution:
  The unique soliton solution \( \varphi \) is positive, radially symmetric, at least \( C^2 \) (which gives far better regularity through standard elliptic estimates), and exponentially decaying.

- The soliton curve:
  Set \( Q(\lambda) = Q(\varphi_\lambda) \), \( E(\lambda) = E(\varphi_\lambda) \). A variational argument shows that \( \partial_\lambda Q = -\lambda \partial_\lambda E \). Stability theory for solitons depends heavily upon the sign of \( \partial_\lambda Q \). For monomial nonlinearities, this is done by scaling.
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Soliton Solution.

Figure: A plot of a soliton.
Soliton Solutions Cont.

- $Q(\lambda)$ and $E(\lambda)$ are $C^1$.
- A numerical plot shows a curve for a saturated nonlinearity:

![A sample soliton curve.](image)

**Figure**: A sample soliton curve.

- Say the minimal mass soliton occurs at $\lambda_0$. 
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![Graph showing $Q(\lambda)$ and $E(\lambda)$ as functions of $\lambda$]

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Soliton Stability.

- From the works of Weinstein, Shatah, and Grillakis-Shatah-Strauss, we know that if

\[ \partial_\lambda Q(\lambda) > 0, \quad (< 0) \]

the solitons are orbitally stable (unstable) under small perturbations.

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Orbital Stability

- A soliton $\varphi$ is orbitally stable if for any $\epsilon > 0$, there exists $\delta > 0$ such that if the initial condition $u_0$ is such that

$$\inf_{\theta, y} \| u_0(x) - e^{i\theta} \varphi(x + y) \|_{H^1(x)} < \delta,$$

then the solution $u(x, t)$ of NLS satisfies

$$\inf_{\theta, y} \| u(x, t) - e^{i\theta} \varphi(x + y) \|_{H^1(x)} < \epsilon.$$
Solution: 
Set $\lambda = 1$. We take a solution of the form 
$$ u(x, t) = e^{it}(\varphi(x) + R(x, t)). $$

The equation for $R = v_1 + iv_2$:

$$ iR_t + \Delta R = -\beta(|\varphi|^2)R - 2\beta'(|\varphi|^2)\varphi \text{Re}(R) + O(R^2), $$

by splitting $R$ up into its real and imaginary parts, then doing a Taylor Expansion.
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Hence, we have

$$\partial_t \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathcal{H} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$ 

Above,

$$\mathcal{H} = \begin{pmatrix} 0 & iL_- \\ -iL_+ & 0 \end{pmatrix},$$

where

$$L_- = -\Delta + \lambda - \beta(\varphi_\lambda^2)$$

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$$L_+ = -\Delta + \lambda - \beta(\varphi_\lambda^2) - 2\beta'(\varphi_\lambda^2)\varphi_\lambda^2.$$
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Spectral Properties of $L_+, L_-$.  

- $L_+, L_-$ are self-adjoint operators.
- $L_- \geq 0$.
- There exists one simple, negative eigenvalue for $L_+$. 
- The continuous spectrum of $L_-, L_+$ is $(\lambda^2, \infty)$.
- The null space of $L_-$ is spanned by $\varphi_{\lambda}$.
- The null space of $L_+$ is spanned by $\partial_j \varphi_{\lambda}$ for $j = 1, ..., d$. 

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Spectral Properties of $\mathcal{H}$.

- The generalized null space is at least of order 2. To see this, look at $\partial_\lambda \varphi_\lambda$. We have $L_+ \partial_\lambda \varphi_\lambda = 2 \varphi_\lambda$. Hence, $L_- L_+ \partial_\lambda \varphi_\lambda = 0$.

- At a minimal mass soliton, the generalized null space is of order 4.

- An embedded resonance is in fact an embedded eigenvalue.

- The continuous spectrum of $\mathcal{H}$ is $(-\infty, -\lambda^2) \cup (\lambda^2, \infty)$.

**Theorem**

*There are no large embedded eigenvalues for $\mathcal{H}$.*

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- Note that we have designed numerical experiments that will allow us to test these assumptions in the limited range they apply.
- We say that an $\mathcal{H}$ satisfying all of the above conditions is *admissible*. 
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Dispersive Estimates for $\mathcal{H}$.

Theorem (Schlag-Erdogan, Bourgain)

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\| e^{it\mathcal{H}} P_c \varphi \|_{H^1} \leq C \| \varphi \|_{H^1} \\
\| e^{it\mathcal{H}} (P_c \varphi) \|_{H^s} \leq C \| \varphi \|_{H^s} \\
\| e^{it\mathcal{H}} (P_d \varphi) \|_{H^s} \leq C (1 + |t|^3) \int e^{-c|x|} |\varphi(x)| dx \\
\| e^{it\mathcal{H}} (P_c \varphi) \|_{L^2} \leq C (\| |x|^\alpha \varphi \|_{L^2} + (1 + |t|^\alpha) \| \varphi \|_{H^\alpha}) \\
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Main Theorem

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Given the equation in

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\begin{cases}
iu_t + \Delta u + \beta(|u|^2)u = 0, & u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C} \\
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where $\beta$ is an admissible saturated nonlinearity, there exists a codimension 10 (*) (at most) manifold $M$ embedded in function space $\Sigma$, such that given $\psi \in M$, Eqn. (1) has a solution of the form

\[u(x, t) = \varphi_{\lambda_0} + e^{iHt} \psi + w(x, t),\]

where $w(x, t) \to 0$ as $t \to \infty$ uniformly in $x$.

* This is work in progress, but the general idea is that most of the functions in the generalized null space will still produce stable perturbations.
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Construction of $\mathcal{M}$ and $\Sigma$.

To begin, for simplicity assume $\lambda_0 = 1$. Using an argument similar to that of Bourgain-Wang, we look for solutions of the form

$$u(x, t) = e^{ixt} \phi + z\psi + w(x, t),$$

where $z\psi = e^{i\mathcal{H}t}\psi$.

Note that Bourgain-Wang used $z\psi$, a solution to the critical (monomial) nonlinear problem in dimensions $d = 1, 2$ to look at stable perturbations of a blow-up solution via the pseudoconformal transform.

It is also possible, with restrictions similar to those of B-W to build a solution for $z\psi = e^{i\Delta t}\psi$. 
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Construction of $\mathcal{M}$ and $\Sigma$.

- To begin, for simplicity assume $\lambda_0 = 1$. Using an argument similar to that of Bourgain-Wang, we look for solutions of the form

$$u(x, t) = e^{ixt} \varphi + z_\psi + w(x, t),$$

where $z_\psi = e^{i\mathcal{H}t} \psi$.

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To build solutions of this type, we build a contraction map backwards from $\infty$.

Note that if we can show $w$ is small, then we can say that the map $\Psi : \psi \rightarrow w(t_0)$ is Lipschitz and hence, we have a smooth, finite codimension manifold of perturbations. The codimension will be at most $2d + 4$ since $H^1 \times H^1 = N_g(\mathcal{H}) \oplus \{N_g(\mathcal{H}^*)\}^\perp$ and $N_g(\mathcal{H}) = 2d + 4$. However, in the spirit of Schlag, Krieger-Schlag, the null space functions up to order 2 should not effect the convergence. Hence, the manifold should be of codimension 2.
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Linearization scheme.

- Let \( v_0 = z\varphi e^{-it} \). Then, we set

\[
\begin{align*}
    f_0 &= \beta(|\varphi + v_0|^2)(\varphi + v_0) - \beta(\varphi^2)\varphi \\
         &\quad - (\beta(\varphi^2) + \beta'(\varphi^2)\varphi^2)v_0 - \beta(\varphi^2)\varphi^2\bar{v}_0, \\
    a &= [\beta(|\varphi + v_0|^2) + \beta'(|\varphi + v_0|^2)|\varphi + v_0|^2] \\
        &\quad - [\beta(\varphi^2) + \beta'(\varphi^2)\varphi^2], \\
    b &= \beta'(|\varphi + v_0|^2)(\varphi + v_0)^2 \\
        &\quad - \beta'(\varphi^2)\varphi^2, \\
    G(w) &= \beta(|\varphi + v_0 + w|^2)(\varphi + v_0 + w) \\
          &\quad - \beta(|\varphi + v_0|^2)(\varphi + v_0) \\
          &\quad - [\beta(|\varphi + v_0|^2) + \beta'(|\varphi + v_0|^2)|\varphi + v_0|^2]w \\
          &\quad - \beta'(|\varphi + v_0|^2)(\varphi + v_0)^2\bar{w}.
\end{align*}
\]
Linearization scheme cont.

- Under the smoothness assumptions on $\beta$, we have

\[
\begin{align*}
f_0 &= O(|v_0|^2), \\
a &= O(|v_0|), \\
b &= O(|v_0|), \\
G(w) &= O(|w|^2).
\end{align*}
\]

- Then, we have

\[
-iw_t = \Delta w - w + (\beta(\varphi^2) + \beta'(\varphi^2)\varphi^2)w \\
+ \beta'(\varphi^2)\varphi^2 \bar{w} + f_0 + aw + b\bar{w} + G(w).
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Then, we have

\[
-iw_t = \Delta w - w + (\beta(\varphi^2) + \beta'(\varphi^2)\varphi^2)w + \beta'(\varphi^2)\varphi^2 \tilde{w} + f_0 + aw + b\tilde{w} + G(w).
\]
In other words, we have

\[ iw_t + \mathcal{H}w + aw + b\bar{w} + f_0 + G(w) = 0. \]

We have \( G \) to be at least quadratic in \( w \).

Using the integral formulation of the equation for \( w \)

\[
\begin{align*}
w(t) &= -i \int_t^\infty e^{i(\tau-t)\mathcal{H}} P_c [f_0 + aw + b\bar{w} + G(w)] d\tau \\
&\quad + -i \int_t^\infty e^{i(\tau-t)\mathcal{H}} P_d [f_0 + aw + b\bar{w} + G(w)] d\tau,
\end{align*}
\]

where \( P_c \) projects onto the continuous part of the spectrum and \( P_d \) projects onto the discrete part of the spectrum.
Integral equation for $w$.

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Existence proof for $w$.

- Assume that $\|w\|_{L^2} \leq t^{-N}$ for $N$ large.
- Make the assumption $\psi \in \mathcal{M}$, which gives sufficient decay in $t$ for $z$. Most of this argument will involve standard scattering theory techniques, but applied to a nonselfadjoint operator.
- Prove the existence of $w$ through a bootstrapping argument.
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Distorted Fourier Basis.

- There exists a distorted Fourier basis, $u_{\xi_0}$ for $\mathcal{H}$ which gives a unique representation for the continuous spectrum.

  - $u_{\xi_0}$ is a solution to the equation
    \[
    [L_+ L_-] u_{\xi_0} = (\lambda^2 + \xi_0^2)^2 u_{\xi_0}.
    \]

  - For each $\xi_0$, $u_{\xi_0}$ is of the form
    \[
    u_{\xi_0} = e^{ix \cdot \xi_0} + f_{\xi_0},
    \]
    where $f_{\xi_0} \in C^\infty \cap L^p$, $p \geq 4$ and
    \[
    f_{\xi_0} \sim \frac{\cos(|x||\xi_0|)}{|x|}
    \]
    as $x \to \infty$.

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Time Decay.

For the linear Schrödinger operator, we can build solutions with better time decay simply by making the assumption on the initial data \( u_0 \) that

\[
\int_{\mathbb{R}^d} x^\alpha u_0 = 0,
\]

for some multi-index \( \alpha \). We then have

\[
\| e^{i\Delta t} u_0 \|_{L^\infty} \leq t^{-\frac{d+2|\alpha|}{2}}.
\]
For the linearized matrix Schrödinger operator, from Erdogan-Schlag, we have the standard spectral decomposition from Stone’s formula and nice dispersive estimates. We can build solutions with better time decay by using the fact that along the continuous spectrum, there is a representation

$$P_c \mathcal{H} f = \tilde{F}^{-1} |\xi|^2 \tilde{F} f,$$

where $\tilde{F}$ is the Fourier transform with respect to the distorted basis $u_{\xi_0}$.

$$\tilde{F}^{-1} = C \cdot \tilde{F}^*$$

$\tilde{F}$, $\tilde{F}^{-1}$ are $L^2$ bounded operators.
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We then have
\[
\langle e^{iHt} P_c f, g \rangle = e^{it} \int e^{it\xi^2} \langle f, u_\xi \rangle \overline{\langle g, \tilde{u}_\xi \rangle} \, d\xi,
\]
on which we can carefully use stationary phase, nonstationary phase and duality arguments to get time decay. It is here that we must put enough regularity and decay at \( \infty \) on \( \Sigma \) in order to get time decay. This is very similar to the development of Krieger-Schlag in \( \mathbb{R} \times \mathbb{R} \) using Wronskian methods to construct the distorted Fourier basis and prove the spectral decomposition.

Note that a similar and far more well-known approach could be taken for the linear Schrödinger, but the exact formula provides us with a more definitive solution.
Time Decay cont.

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Conclusion

- Despite the fact that the linearized operator is nonselfadjoint, at the minimal mass soliton standard scattering theory techniques apply and hence stable perturbations can be constructed.

- Along the way to constructing this class, we have extensively studied the spectrum of the linearized operator and constructed bounds for embedded eigenvalues.

- This result distinctly depends upon the smoothness of the nonlinearity.
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Future Work

- The proof that the perturbations exist is fairly well established, but proving that those perturbations live on a manifold with reduced codimension is work in progress.
- Analysis which describes the soliton curve for small values of $\lambda$ and looks at dispersion below the minimal mass soliton is future work underway with Justin Holmer.
- Studying the necessity of the smoothness condition on the nonlinearity both analytically and numerically is work in progress also with Justin Holmer.
- Studying the dynamics of interactions with minimal mass solitons could lead to interesting annihilation effects.
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