A class of stable perturbations for a minimal mass soliton in three dimensional saturated nonlinear Schrödinger equations

by

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Abstract

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In this result, we develop the techniques of (KriSch1) and (BouWa) in order to determine a class of stable perturbations for a minimal mass soliton solution of a saturated, focusing nonlinear Schrödinger equation (NLSE)

$$\begin{cases} iu_t + \Delta u + \beta(|u|^2)u = 0 \\ u(0,x) = u_0(x), \end{cases}$$

in $\mathbb{R}^3$. We first numerically and anaytically study the existence of a minimal mass soliton, R_{min}, as well as the spectrum of the Hamiltonian, $\mathcal{H}$ resulting by linearizing about that soliton. By projecting into a subspace of the continuous spectrum of $\mathcal{H}$ as in (Schlag1), (KriSch1), we are able to use a contraction mapping similar to that from (BouWa) in order to show that there exist solutions of the form

$$e^{i\lambda_{\min} t}(R_{min} + e^{i\mathcal{H}t}\phi + w(x,t)),$$

where $e^{i\mathcal{H}t}\phi + w(x,t)$ disperses as $t \to \infty$. Hence, we have long time persistance of a soliton of minimal mass despite the fact that these solutions are shown to be nonlinearly unstable in (ComPel). Of general interest will be the numerical and analytical spectral results on Hamiltonians linearized about a soliton, numerical descriptions of soliton dynamics, and

improved decay in time using a distorted Fourier basis approach for solutions of the form $e^{i\mathcal{H}t}\phi$, where ϕ is in the continuous spectrum of $\mathcal{H}$ under various regularity and decay assumptions.

Professor Daniel Tataru
Dissertation Committee Chair

I would like to dedicate this work to my wife Amber, as well as all other friends and family members who helped me get to this point.

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Chapter 1

Introduction

1.1 Saturated Nonlinear Schrödinger Equations

In this section, we develop the techniques used to prove stability of solitons for a focusing nonlinear Schrödinger equation (NLS) in $\mathbb{R} \times \mathbb{R}^d$:

$$\begin{aligned} iu_t + \Delta u + \beta(|u|^2)u &= 0 \\ u(0,x) &= u_0(x), \end{aligned}$$

for $\beta : \mathbb{R} \to \mathbb{R}$, $\beta(s) \geq 0$ for all $s \in \mathbb{R}$.

Definition 1.1.1. *Saturated nonlinearities of type* 1 *are of the form*

$$\beta(s) = s^{\frac{q}{2}} \frac{s^{\frac{p-q}{2}}}{1 + s^{\frac{p-q}{2}}}, \tag{1.1}$$

where $p > 2 + \frac{4}{d}$ *and* $\frac{4}{d} > q > 0$ *for* $d \geq 3$ *and* $\infty > p > 2 + \frac{4}{d} > \frac{4}{d} > q > 0$ *for* $d < 3$.

Definition 1.1.2. *Saturated nonlinearities of type* 2 *are of the form*

$$\beta(s) = \frac{s}{(1+s)^{\frac{2-q}{2}}}, \tag{1.2}$$

where $\frac{4}{d} > q > 0$, $d > 2$.

Remark 1.1. *In both cases, for* $|u|$ *large, the behavior is* L^2 *subcritical and for* $|u|$ *small, the behavior is* L^2 *supercritical. For Definition 1.1.1,* p *is chosen much larger than the* L^2 *critical exponent,* $\frac{4}{d}$ *in order to allow sufficient regularity in linearizing the equation.*

In the sequel, we assume that $u_0 \in H^1$ and $|x|u_0 \in L^2$, or in other words, u_0 has finite variance. For this initial data, from the spatial and phase invariance of NLS, we have many interesting conserved quantities discussed below.

1.2 Conserved Quantities

The solution to (NLS) satisfies the following conservation laws:

Conservation of Mass:

$$Q(u) = \frac{1}{2}\int_{\mathbb{R}^n} |u|^2 dx = \frac{1}{2}\int_{\mathbb{R}^n} |u_0|^2 dx,$$

Conservation of Energy:

$$E(u) = \int_{\mathbb{R}^n} |\nabla u|^2 dx - \int_{\mathbb{R}^n} G(|u|^2) dx = \int_{\mathbb{R}^n} |\nabla u_0|^2 dx - \int_{\mathbb{R}^n} G(|u_0|^2) dx,$$

where

$$G(t) = \int_0^t \beta(s) ds.$$

We also have the pseudoconformal conservation law:

$$\|(x + 2it\nabla)u\|_{L^2}^2 - 4t^2 \int_{\mathbb{R}^n} G(|u|^2) dx = \|x\phi\|_{L^2}^2 - \int_0^t \theta(s) ds, \tag{1.3}$$

where

$$\theta(s) = \int_{\mathbb{R}^n} (4(n+2)G(|u|^2) - 4n\beta(|u|^2)|u|^2).$$

Note that $(x + 2it\nabla)$ is the Hamilton flow of the linear Schrödinger equation, so the above identity relates how the solution to the nonlinear equation is affected by the linear flow.

Detailed proofs of these conservation laws can be arrived at easily using energy estimates or Noether's Theorem, which relates conservation laws to symmetries of an equation and is discussed below. Global well-posedness in L^2 of (NLS) with β of type 1 or 2 for finite variance initial data follows from standard theory for L^2 subcritical monomial nonlinearities, a proof of which can be found in Cazenave (Caz) or Sulem-Sulem (SulSul), Chapter 3.

1.2.1 Noether's Theorem and conserved quantities resulting from symmetris of the equation

We present here the version of Noether's Theorem from (FoGe).

Proposition 1.2.1. *If the functional*

$$J[u] = \int F(x, u, \nabla u)dx$$

is invariant under the family of transformations

$$\begin{aligned} x_i^* &= \Phi_i(x, u, \nabla u; \epsilon) \sim x_i + \epsilon\phi_i(x, u, \nabla u), \\ u^* &= \Psi(x, u, \nabla u; \epsilon) \sim u + \epsilon\psi(x, u, \nabla u), \end{aligned}$$

for $i = 1, 2, ..., d$, *then*

$$\sum_{i=1}^{d} \frac{\partial}{\partial_{x_i}}(F_{u_{x_i}}\bar{\psi} + F\phi_i) = 0$$

on each external surface of $J[u]$, *where*

$$\bar{\psi} = \psi - \sum_{i=1}^{d} u_{x_i}\phi_i.$$

Proof. Standard calculus of variations techniques say that the variation of such a functional is given by

$$\delta J = \epsilon \int (F_u - \sum_{i=1}^{d} \frac{\partial}{\partial_{x_i}})\bar{\psi}dx + \epsilon \int \sum_{i=1}^{d} \frac{\partial}{\partial_{x_i}}(F_{u_{x_i}}\bar{\psi} + F\phi_i)dx.$$

□

1.3 Soliton Solutions

A soliton solution is of the form

$$u(t, x) = e^{i\lambda t}R_\lambda(x)$$

where $\lambda > 0$ and $R_\lambda(x)$ is a positive, radially symmetric, exponentially decaying solution of the equation:

$$\Delta R_\lambda - \lambda R_\lambda + \beta(R_\lambda)R_\lambda = 0. \tag{1.4}$$

With this type of nonlinearity, the soliton solutions exist, and ground state solutions are unique. Existence of solitary waves for nonlinearities of the type presented in Definitions 1.1.1 and 1.1.2 is proved by Berestycki and Lions (BerLion) by minimizing the functional

$$T(u) = \int |\nabla u|^2 dx$$

with resptect to the functional

$$V(u) = \int [G(|u|^2) - \frac{\lambda}{2}|u|^2] dx.$$

Then, using a minimizing sequence and Schwarz symmetrization, one sees the existence of the nonnegative, spherically symmetric, decreasing soliton solution. Essentially, they minimize one functional in terms of another and are able to obtain a radially symmetric minimizer. For uniqueness, see (McCleod), where a shooting method is implemented to show that the desired soliton behavior only occurs for one particular initial value. While approached analytically in (McCleod), the behavior will be particularly obvious in later numerical experiments performed. For completeness, we give a brief discussion of existence and uniqueness in Appendix A.

An important fact is that $Q_\lambda = Q(R_\lambda)$ and $E_\lambda = E(R_\lambda)$ are differentiable with respect to λ. This fact can be determined from the early works of Shatah, namely (Shatah1), (Shatah2). For completeness, we also discuss this in Appendix A. By differentiating Equation (1.4), Q and E with respect to λ, we have

$$\partial_\lambda E_\lambda = -\lambda \partial_\lambda Q_\lambda.$$

Numerics show that if we plot Q_λ with respect to λ, we get a curve that goes to ∞ as $\lambda \to 0, \infty$ and has a global minimum at some $\lambda = \lambda_0 > 0$. We will explore this in detail in Chapter 3. Variational techniques developed in (GrilShaStr) and (ShatStr1) tell us that when $\delta(\lambda) = E_\lambda + \lambda Q_\lambda$ is convex or $\delta''(\lambda) > 0$, we are guaranteed stability under small perturbations and for $\delta''(\lambda) < 0$, we are guaranteed that the soliton is unstable under small perturbations. We will explore the nature of this stability in what follows, but for a brief reference on this subject, see (SulSul), Chapter 4. For notational purposes, we refer to a minimal mass soliton as R_{min}.

1.4 Linearization about a Soliton

Let us write down the form of NLS linearized about a soliton solution. First of all, we assume we have a solution $\psi = e^{i\lambda t}(R_\lambda + \phi(x,t))$. For simplicity, set $R = R_\lambda$. Inserting this into the equation we know that since ϕ is a soliton solution we have

$$i(\phi)_t + \Delta(\phi) \quad = \quad -\beta(R^2)\phi - 2\beta'(R^2)R^2\mathrm{Re}(\phi) + O(\phi^2), \tag{1.5}$$

by splitting ϕ up into its real and imaginary parts, then doing a Taylor Expansion. Hence, if $\phi = u + iv$, we get

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{H} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{1.6}$$

where

$$\mathcal{H} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}, \tag{1.7}$$

where

$$L_- = -\Delta + \lambda - \beta(R_\lambda)$$

and

$$L_+ = -\Delta + \lambda - \beta(R_\lambda) - 2\beta'(R_\lambda^2)R_\lambda^2.$$

Definition 1.4.1. *A Hamiltonian, $\mathcal{H}$ is called admissible if the following hold:*

1) There are no embedded eigenvalues in the essential spectrum,

2) The only real eigenvalue in $[-\lambda, \lambda]$ is 0,

3) The values $\pm\lambda$ are not resonances.

Definition 1.4.2. *Let (NLS) be taken with nonlinearity β. We call β admissible if there exists a minimal mass soliton, R_{min}, for (NLS) and the Hamiltonian, $\mathcal{H}$, resulting from linearization about R_{min} is admissible in terms of Definition 1.4.1.*

The spectral properties we need for the linearized Hamiltonian equation in order to prove stability results are precisely those from Definition 1.4.1. Notationally, we refer to P_d

and P_c as the projections onto the discrete spectrum of $\mathcal{H}$ and onto the continuous spectrum of $\mathcal{H}$ respectively.

Analysis of these spectral conditions will be done both numerically and analytically in Chapter 4.

1.5 Main Results

To begin, we define the function space

$$\mathcal{P}_1^A = \{\phi \in L^2 | \|\phi\|_{H^A} < \infty,\ \||x|^A \phi\|_{L^2} < \infty,\ \int x^\alpha \phi(x) dx = 0 \text{ for } |\alpha| \leq 2A\},$$

with norm given by

$$\|\phi\|_{\mathcal{P}_1^A} = \left(\|\phi\|_{H^A}^2 + \||x|^A \phi\|_{L^2}^2\right)^{\frac{1}{2}}.$$

We similarly define the function space

$$\mathcal{P}_2^A = \{\phi \in P_c\mathcal{H} | \|\phi\|_{H^A} < \infty,\ \||x|^A \phi\|_{L^2} < \infty,\ \text{condition 6.22 is satisfied for } j \leq A\},$$

with norm given by

$$\|\phi\|_{\mathcal{P}_2^A} = \left(\|\phi\|_{H^A}^2 + \||x|^A \phi\|_{L^2}^2\right)^{\frac{1}{2}}.$$

In this result, we seek to prove that minimal mass solitons for nonlinear Schrödinger equations in three dimensions have stable perturbations for long times. These minimal solitons are unstable as discussed below. The main goal of this thesis is to prove the following three theorems:

Theorem 1. *Take the equation in* $\mathbb{R} \times \mathbb{R}^3$

$$\begin{cases} iu_t + \Delta u + \beta(|u|^2)u = 0 \\ \qquad u(0,x) = u_0(x), \end{cases} \tag{1.8}$$

where β *is an admissible saturated nonlinearity of type* 1. *For any* $\phi \in \mathcal{P}_1^A$, *Equation* (1.8) *has a solution* u *for* $t \in [\frac{1}{\delta}, \infty)$ *of the form*

$$u(x,t) = R_{min} + v(t) = R_{min} + e^{i\Delta t}\phi + w(x,t),$$

where R_{min} is the minimal mass soliton in Definition 1.4.1 and $\|w(\cdot,t)\|_{H^2} \to 0$ as $t \to \infty$. For Equation (1.8) in $\mathbb{R}^3$ of type 1, we have $A > \frac{13}{2}$.

Theorem 2. *Take Equation (1.8), where β is an admissible saturated nonlinearity of type 1. For any $\phi \in \mathcal{P}_1^A$, Equation (1.8) has a solution u for $t \in [\frac{1}{\delta}, \infty)$ of the form*

$$u(x,t) = R_{min} + v(t) = R_{min} + e^{i\mathcal{H}t}\phi + w(x,t),$$

where R_{min} is the minimal mass soliton in Definition 1.4.1 and $\|w(\cdot,t)\|_{L^2} \to 0$ as $t \to \infty$. In this theorem, for Equation (1.8) in $\mathbb{R}^3$ of type 1, we have $A > \frac{5}{2}$.

Theorem 3. *Given Equation (1.8), where β is an admissible saturated nonlinearity of type 2, for any $\phi = P_c\phi \in W^{2,1} \cap H^2$ with $\|\phi\|_{W^{2,1}\cap H^2} < \delta < 1$, Equation (1.8) has a solution for $t \in [0, \left(\frac{1}{2\delta}\right)^{\frac{1}{4}})$ of the form*

$$u(x,t) = R_{min} + v(t) = R_{min} + e^{i\mathcal{H}t}\phi + w(x,t),$$

where R_{min} is the minimal mass soliton in Definition 1.4.1 and

$$u(x,0) = R_{min} + \phi.$$

Remark 1.2. *In Theorems 1 and 2, the stable perturbations can be shown to live on a finite codimension manifold for p large enough compared to d. This will be explored further in Chapter 7.*

The class of functions $\mathcal{P}_i^A$ for $i = 1, 2$ will be developed throughout the course of this work. They will result from projecting onto a distorted Fourier basis for the linearized problem. For a further discussion these topics and the notion of distorted Fourier basis, see Chapters 6 and 7.

Note, in the Theorems 1 and 2, we are able to prove many of the spectral properties of $\mathcal{H}$ analytically, though some of them must be verified numerically in specific cases. However, one particular analytical result is expressed in the following:

Theorem 4. *Given an admissible Hamiltonian $\mathcal{H}$, there exists some $M > 0$ such that for $|\mu| > M$, there are no solutions u_μ such that*

$$\mathcal{H}u_\mu = \mu u_\mu.$$

Remark 1.3. *The result in Theorem 4 states that there are no large eigenvalues embedded in the continuous spectrum of $\mathcal{H}$. There is a similar result for ruling out large spherical harmonic expansions of eigenvalues embedded in the continous spectrum. These will be discussed more in Chapter 4.*

We derive the existence of and important properties for distorted Fourier bases, and hence a distorted Fourier transform, for a general class of linearized Hamiltonians. Let $\mathcal{S}$ be the Schwartz class of functions. Then, we have the following:

Theorem 5. *Given an admissible Hamiltonian $\mathcal{H}$, P_c the projection on the continuous spectrum of $\mathcal{H}$, for initial data $\phi \in \mathcal{S}$,*

$$\|e^{it\mathcal{H}} P_c \phi\|_{L^\infty} \leq t^{-\frac{d}{2}}.$$

Although Theorem 4 and Theorem 5 are used to prove Theorems 1 and 2, we mention them here as they are of independent interest to the study of nonlinear dispersive PDE's and particularly to soliton theory. The notations used in these results will be clarified in the sequel.

1.6 Motivation and possibilities for future research

According to the book by Catherine and Pierre Sulem (SulSul), saturated nonlinearities have applications to describing the dielectric constant of gas vapors where a laser beam propagates, laser beams in plasmas and Bose superfluids at zero temperatures. Mathematically, the author's interest in saturated nonlinearities arose out of the work of Rodnianski-Schlag-Soffer (RodSchSof), who prove asymptotic stability for a collection of N solitons under various separation conditions for λ large enough and

$$\beta(s) = s^{\frac{q}{2}} \frac{s^{\frac{p-q}{2}}}{\epsilon + s^{\frac{p-q}{2}}},$$

for ϵ small. This is equivalent to the β presented in Equation (1.1) using a rescaling

$$u(t,x) \to \gamma^{\frac{2}{q}} v(\gamma^2 t, \gamma x),$$

where $\gamma^{-\frac{2(p-q)}{q}} = \epsilon$. Hence, for large enough values of λ, we expect stability. This is done using closeness of β to the monomial problem as $\epsilon \to 0$. We will explore this proof further in the stability section below. As a result of the analysis in (RodSchSof), we are able to describe analytically the slope of the soliton curve for large values. However, for mid-range and small values this will continue to be explored by the author and Justin Holmer. In addition, the requirements on the nonlinearity β have to do with requiring enough smoothness in order to decompose the operator into a linear flow with quadratic error in the perturbation. In the case of nonlinearities without smoothness, very oscillatory behavior has been observed numerically. An attempt to explain a lack of asymptotic stability based upon the lack of smoothness of the nonlinearity is ongoing work also with Justin Holmer.

As described below numerically, for small values of λ, we have orbital instability. It is possible that perturbations of unstable solitons will eventually resolve into stable solitons, however this issue falls under the very broad and poorly understood phenomenon of soliton resolution. The work presented below focuses on λ in an intermediate range, specifically at the point where the behavior switches from instability to stability. Despite the fact that these solitons are in fact unstable, we prove stability for a large class of perturbations. This is in the same direction as the works of Schlag (Schlag1) and Krieger-Schlag (KriSch1), who prove there exist manifolds of stable perturbations in $\mathbb{R}$ and $\mathbb{R}^3$ respectively for supercritical monomial nonlinearities. As seen in Chapter 7, for p large enough in β, we are able to prove that our perturbations lie on a manifold as well. Improving the codimension of that manifold by taking advantage of symmetries of the equation will be studied in future work. In addition, the author will study the interaction of minimal mass solitons with other solitons during collisions, as well as with more general perturbations. For minimal mass solitons, very small perturbations may annihilate the soliton. Preliminary numerical results in this direction are stated in Chapter 3. The complete description of the soliton curve, dispersion below the minimal mass, and a better understanding analytically of the spectrum of the linearized operator will also be topics of future research.

Chapter 2

Stability Theory and Variational Methods

2.1 Stability Theory

We begin with a few definitions.

Definition 2.1.1. *A soliton R is orbitally stable if for any $\epsilon > 0$, there exists $\delta > 0$ such that if the initial condition u_0 is such that*

$$\inf_{\theta,y} \|u_0(x) - e^{i\theta}R(x+y)\|_{H^1(x)} < \delta,$$

then the solution $u(x,t)$ of NLS satisfies

$$\inf_{\theta,y} \|u(x,t) - e^{i\theta}R(x+y)\|_{H^1(x)} < \epsilon.$$

Definition 2.1.2. *A soliton R is said to be linearly stable if we can control the growth in some sense of the solution to the problem linearized around the soliton. This linearization will be explored later, but for now, let $\mathcal{H}$ be the Hamiltonian resulting from linearizing the equation about a soliton. Then, the new solution space is $L^2 \times L^2 = \mathcal{N} + \mathcal{N}^{*\perp}$ for*

$$\mathcal{N} = \bigcup_{l \geq 1} ker\mathcal{H}^l,$$

and $\mathcal{N}^*$ defined similarly for $\mathcal{H}^*$. If P is the projection onto ${\mathcal{N}^*}^\perp$, then we have linear stability if

$$\sup_t \|e^{i\mathcal{H}t} P\psi\|_{L^2} < \infty,$$

where $\psi = e^{i\theta}\left(R_\lambda(x-vt) + \phi(t, x-vt)\right)$.

Note that the notion of linear stability is closely related to that of orbital stability.

Definition 2.1.3. *Let $u_0(x) = e^{i\theta_0 + v_0 x} R_{\lambda_0}(x+y_0) + \phi(x)$. Then, the corresponding soliton to (NLS) is said to be asymptotically stable if there exist $w(x)$ and $\gamma_\infty = (\lambda_\infty, \theta_\infty, v_\infty, y_\infty)$ such that*

$$\|u(x,t) - e^{i\theta_\infty + v_\infty x} R_{\lambda_\infty}(x + 2v_\infty t + y_\infty) - e^{i\frac{t}{2}\Delta} w\|_{L^2(x)} \to 0. \tag{2.1}$$

In general, asymptotic stability says a small perturbation of a soliton eventually converges to a soliton solution (of possibly different soliton parameter) plus dispersion. This is closely related to the notion of scattering for dispersive equations and soliton resolution in general. The particular stability discussed in (GrilShaStr) and affiliated with the variational function δ is orbital stability.

We present here the result of (Wein1) proving orbital stability of solitons for the nonlinear Schrödinger equation. To begin, we restrict ourselves to pure power NLS equations

$$iu_t + \Delta u + |u|^{p-1}u,$$

for $1 < p < \frac{4}{n}$.

Using the ansatz $u(x,t) = e^{i\lambda t} R(x)$, we get a semilinear elliptic equation for R:

$$-\Delta R + \lambda^2 R - |R|^{p-1} R.$$

From the standard variational techniques discussed in Appendix A, we can prove there exists a minimizer to the resulting Lagrangian, say R_λ, the soliton solution, which is positive, radially symmetric, and exponentially decaying. Unlike the case of saturated nonlinearities, with a monomial nonlinearity we can actually write $R_\lambda = \lambda^{\frac{2}{p-1}} R_1(\lambda x)$ by scaling. Set

$R = R_1$ for simplicity. We call R the "ground state" solution. As proved in (Wein2), R minimizes the functional

$$J[u] = \frac{\|\nabla u\|_{L^2}^{\frac{p-1}{2}n}\|u\|_{L^2}^{2+(p-1)(1-\frac{n}{2})}}{\|u\|_{L^{p+1}}^{p+1}}.$$

The result we desire to prove is that these soliton solutions are orbitally stable. Let us make this more precise by introducing the following norm:

$$[\rho_\lambda(\phi(t), \mathcal{G}_{R_\lambda})]^2 =$$

$$\inf_{\gamma, x_0} \left\{ \|\nabla\phi(\cdot + x_0, t)e^{i\gamma} - \nabla R_\lambda\|_{L^2}^2 + \lambda\|\phi(\cdot + x_0, t)e^{i\gamma} - R_\lambda\|_{L^2}^2 \right\},$$

where $\mathcal{G}_R$ is the set of functions generated by the symmetries of NLS applied to R. Hence, for all times t, we measure how the profile of ϕ compares to that of R.

As in Section 1.2, we have conservation of mass

$$Q(u) = \int |u_0|^2 dx,$$

and conservation of energy

$$E(u) = \int |\nabla u_0|^2 - G(|u_0|^2) dx.$$

From these, we define the Lyapunov functional

$$\mathcal{E}(u) = E(u) - \lambda Q(u).$$

Note that if u were moving purely along the soliton curve, $\mathcal{E}(u_\lambda) = \delta(\lambda)$. We are now ready to state the main result of this section.

Theorem 6. *Let $R(x)$ be the unique ground state soliton. For any $\epsilon > 0$, there exists $\delta > 0$ such that if $\rho(\phi_0, \mathcal{G}_R) < \delta$, then for all $t > 0$, we have $\rho(\phi(t), \mathcal{G}_R) < \epsilon$.*

Proof. Set $\phi(x + x_0, t)e^{i\gamma} = R(x) + w$, $w = u + iv$. Then, we have

$$\begin{aligned}
\Delta\mathcal{E} &= \mathcal{E}(\phi_0(\cdot)) - \mathcal{E}(R(\cdot)) \\
&= \mathcal{E}(\phi(t)) - \mathcal{E}(R(\cdot)) \\
&= \mathcal{E}(\phi(\cdot + x_0, t)e^{i\gamma}) - \mathcal{E}(R) \\
&= \mathcal{E}(R + w) - \mathcal{E}(R) \\
&\geq (L_+u, u) + (L_-v, v) - C_1\|w\|_{H^1}^{2+\theta} - C_2\|w\|_{H^1}^6,
\end{aligned}$$

for $\theta > 0$, $L_- = -\Delta + 1 - R^{2\sigma}$, $L_+ = -\Delta + 1 - (2\sigma + 1)R^{2\sigma}$.

To see this, set

$$\tilde{G}(t) = G(R + tw).$$

Then,

$$\tilde{G}(t) = \tilde{G}(0) + \int_0^t \tilde{G}'(s)ds,$$

which is the farthest we care to expand due to regularity requirements on the nonlinearity. However, the above quadratic terms come from the expansion

$$\tilde{G}(0) + \tilde{G}'(0) + \frac{1}{2}\tilde{G}''(0).$$

Due to the nature of the soliton solution R and the Gagliardo-Nirenberg inequality, it a simple exercise to bound the difference in the above expansions by the given error terms.

The result then depends upon the following lemma.

Lemma 2.1.4. *If x_0, γ are chosen to minimize*

$$\left\{ \|\nabla\phi(\cdot + x_0, t)e^{i\gamma} - \nabla R_\lambda\|_{L^2}^2 + \lambda\|\phi(\cdot + x_0, t)e^{i\gamma} - R_\lambda\|_{L^2}^2 \right\},$$

then we have the following estimate

$$(L_+u, u) + (L_-v, v) \geq C_3\|w\|_{H^1}^2 - C_4\|w\|_{H^1}^3 - C_5\|w\|^4.$$

According to our ansatz, $\rho(\phi, \mathcal{G}_R) \sim \|w\|_{H^1}$, so we have

$$\Delta\mathcal{E} \geq g(\rho)$$

for $g(x) = cx^2(1 - ax^\theta - bx^4)$. Since $\mathcal{E}$ is continuous for functions in H^1, choose δ such that $\rho_0 < \delta$ implies $\Delta\mathcal{E}(0) < g(\epsilon)$. Since $\Delta\mathcal{E}$ is constant,

$$g(\rho(t)) < g(\epsilon)$$

for all times t. Hence, since g is increasing on a small interval around 0, we get that $\rho < \epsilon$ for all times t. This gives us the result. □

All that remains is to prove the lemma.

Proof. From the minimization of ρ over x_0, γ implies the following integral identities:

$$\int R^{p-1} \frac{\partial R}{\partial x_j} u(x,t) = 0, \tag{2.2}$$

and

$$\int R^p v(x,t) dx = 0. \tag{2.3}$$

We arrive at these equations by differentiating ρ with respect to x_0 and γ respectively, then using that w is the minimum solution by assumption.

We have that $L_- R = 0$ and R is a positive ground state solution, hence by oscillation theory, we must have $(L_- v, v) \geq (v, v)$ since $v \neq \alpha R$ for $\alpha \neq 0$ by Equation (2.3) above. For a reference on oscillation theory, see Reed-Simon (RSv4). Hence, using standard elliptic estimates, we see

$$(L_- v, v) \geq C \|v\|^2_{H^1}.$$

Let us now make a simplifying assumption that we will remove later. Assume that

$$\int |\phi|^2 dx = \int R^2 dx.$$

Hence, we have that $(u, R) = -\frac{1}{2}[(u,u) + (v,v)]$.

Now, we need some results about the operator L_+.

Proposition 2.1.5. *The operator L_+ has exactly one negative eigenvalue.*

Proof. For $n = 1$, this follows from oscillation theory. R' has a single zero at $x = 0$, so 0 is the second eigenvalue of L_+. Since first eigenvalues are simple in this setting, we have the result.

For $n \geq 2$, let us first look at L_+ resulting from a soliton for NLS with a standard monomial nonlinearity. Calculate the second variation of the functional J above. As R is a minimizer, this quantity is positive, but it is also equivalent to $L_+ + r_1$, where r_1 is a rank one operator. Therefore, L_+ can have at most one negative eigenvalue. Since ∇R is not positive, this unique negative eigenvalue exists. □

Proposition 2.1.6. *Given L_+ as defined above, $\alpha = \inf_{(f,R)=0}(L_+f, f) = 0$.*

Proof. Since $L_+\nabla R = 0$ and $(R, \nabla R) = 0$, we have $\alpha \leq 0$.

Let $\{f_\eta\}$ be a minimizing sequence, or $\|f_\eta\|_{L^2} = 1$ and $\lim_{\eta\to\infty}(L_+f_\eta, f_\eta) \to \alpha$ and f_η satisfies the orthogonality condition. For any $\nu > 0$, we have an $f = f_\eta$ such that

$$\alpha < \int |\nabla f|^2 dx + \lambda^2 \int |f|^2 dx < \int [\beta(R^2) + 2\beta'(R^2)] f^2 dx + \alpha + \nu.$$

Hence $\|f_\eta\|_{H^1}$ is uniformly bounded and there exists a subsequence which converges weakly in H^1 to some function, say f^*. By weak convergence, the orthogonality condition still holds for f^*.

Also, we have

$$\int [\beta(R^2) + 2\beta'(R^2)R^2](f_\eta)^2 dx \to \int [\beta(R^2) + 2\beta'(R^2)R^2](f^*)^2 dx,$$

by the weak convergence, Hölder's inequality and the decay of R. Hence,

$$f^* \neq 0$$

since ν is arbitrary.

By Fatou's lemma, we have that $\|f^*\|_{L^2} \leq 1$. Suppose that the minimun is obtained at f^* and that $\|f^*\|_{L^2} < 1$. Let $\psi \in L^2$, $\|\psi\|_{L^2} = 1$, then by weak convergence in H^1, we have

$$\begin{aligned} (\psi, \nabla f^*) &= \liminf_{\eta\to\infty}(\psi, \nabla f_\eta) \\ &\leq \liminf_{\eta\to\infty} \|\nabla f_\eta\|_{L^2}. \end{aligned}$$

Maximizing over all such ψ, we have

$$\|\nabla f^*\|_{L^2} \leq \liminf_{\eta\to\infty} \|\nabla f_\eta\|_{L^2}.$$

Then, using the convergence principles above, we have

$$(L_+f^*, f^*) < \liminf_{\eta\to\infty}(L_+f_\eta, f_\eta) = \alpha,$$

a contradiction to α being a minimizer.

Since the minimum is then obtained at some function $f^* \neq 0$ satisfying the orthogonality conditions, there exists (f^*, α, β) among the critical points to the Lagrange multiplier problem

$$(L_+ - \alpha)f = \beta R, \tag{2.4}$$

$$\|f\|_{L^2} = 1, \tag{2.5}$$

$$(R, f) = 0. \tag{2.6}$$

We must show that $\alpha \geq 0$. Otherwise, if $\beta = 0$, then f is a ground state and hence does not satisfy the orthogonality condition. If $\beta \neq 0$, then α cannot be the the lowest eigenvalue of L_+, otherwise taking the inner product of Equation (2.4) with the ground state of L_+ would contradict the orthogonality condition.

Consider the function

$$g(\lambda) = ((L_+ - \lambda)^{-1}R, R), \tag{2.7}$$

which is well-defined and smooth for $\lambda \in (-\lambda_0, 0)$ because R is orthogonal to the null space of L_+, where $-\lambda_0$ is the negative eigenvalue for L_+. For (2.3) to hold, we need $g(\alpha) = 0$. Differentiation of (2.4) gives

$$g'(\lambda) = \|(L_+ - \lambda)^{-1}R\|^2.$$

Thus, g is increasing on $(-\lambda_0, 0)$. However, $g(0) < 0$ since

$$L_+ \partial_\lambda R_\lambda = 2\lambda R,$$

and by scaling $\partial_\lambda(R_\lambda, R_\lambda) < 0$. Note that this will be a necessary condition for us to address in the case of general nonlinearities. □

Proposition 2.1.7. *Given L_+ as defined above,*

$$Ker(L_+) = span\{\frac{\partial R}{\partial x}\}.$$

Proof. It is clear simply from differentiation that $\nabla R \in \mathrm{Ker}(L_+)$. Hence, we must prove the reverse containment.

In one dimension, this involves using an argument by contradiction. Assume that there is something else, say v, in the kernel. A Wronskian relation with $w = \frac{R'}{R''(0)}$ gives us an expression for v in terms of R that can be manipulated to show $v \leq -\infty$ as $x \to \infty$.

For higher dimensions, note that L_+ is only dependent upon $r = |x|$. Hence, we can expand any L^2 solution into eigenvalues of the form $f(r)Y(\theta)$, for $f \in L^2(0, \infty; r^{n-1}dr)$ and $Y \in L^2(S^{n-1})$. Then, these functions must satisfy

$$\begin{aligned} A_k f &= \left(-\frac{d^2}{dr^2} - \frac{n-1}{r}\frac{d}{dr} + 1 - pR^{p-1} + \frac{\lambda_k}{r^2} \right) f = 0, \\ -\Delta_{S^{n-1}} Y &= \lambda_k Y, \end{aligned}$$

for $\lambda_k = k(n-2+k)$. The functions $\partial_j R$ represent the solutions to these equations for $k = 1$. Hence, no solutions exist for $k = 0, 2, 3, 4, ...$

For $k \geq 2$, we have $A_k = A_1 + \frac{\delta(n+\delta)}{r^2}$. Also, R' is an eigenfunction for A_1 with no interior zeros. Hence, by oscillation theory once again, A_1 is a nonnegative operator. Thus, A_k is a positive operator and for $f \in L^2$, $A_k f = 0$, we have $f = 0$.

The $k = 0$ case is simply resolved by the uniqueness of the ground state as determined in the results of Coffman, Kwong, and McLeod. See (McCleod) for a survey of all this work. □

Proposition 2.1.8. *We have*

$$(L_+ u, u) \geq D\|u\|_{H^1}^2 - D'\|\nabla w\|_{L^2}^2 \|w\|_{L^2}^2 - D''\|w\|_{H^1}^4. \tag{2.8}$$

Proof. To begin, we write $u = u_{perp} + u_{par}$ where

$$u_{par} = (u, R)R = -\frac{1}{2}[(u, u) + (v, v)]R,$$

and

$$u_{perp} = u - (u, R)R = u + \frac{1}{2}[(u, u) + (v, v)]R.$$

Without loss of generality, we assume $(R, R) = 1$. If not, there is simply a factor of $\frac{1}{(R,R)}$ floating around in the constants.

Then, $(L_+u, u) = (L_+u_{par}, u_{par}) + (L_+u_{perp}, u_{perp}) + 2(L_+u_{par}, u_{perp})$. We wish to analyze each term separately.

By assumption, $(u_{perp}, R) = 0$, hence $(L_+u_{perp}, u_{perp}) \geq c(u_{perp}, u_{perp})$. Otherwise, $u_{perp} = c \cdot \nabla R$ which would violate the integral identity above. Hence,

$$(L_+u_{perp}, u_{perp}) \geq c(u_{perp}, u_{perp}) = d[(u, u) - \frac{1}{4}[(u, u) + (v, v)]^2].$$

Now,

$$(L_+u_{par}, u_{par}) = \frac{1}{4}(L_+R, R)[(u, u) + (v, v)]^2,$$

where (L_+R, R) is some negative constant.

Finally,

$$\begin{aligned} (L_+u_{perp}, u_{par}) &= (u, R)(L_+u_{perp}, R) \\ &\geq -d'\|w\|^2\|\nabla w\| - d''\text{h.o.t.}(\|w\|), \end{aligned}$$

by expanding the above expression and using the fact the regularity of R.

Putting all of these estimates gives us the desired inequality (2.8). □

So, if in fact $\|\phi\|_{L^2}^2 = \|R\|_{L^2}^2$, then we are done. If not, find λ such that $\|\phi\|_{L^2}^2 = \|R_\lambda\|_{L^2}^2$ and use the above result along with the fact that $\|R - R_\lambda\|_{H^1}$ is small. Then, we have the result in general.

□

The proofs above clearly rely upon the structure of the monomial nonlinearity. However, under certain spectral assumptions, the results above hold for general nonlinearities with mild changes to the argument above.

Lemma 2.1.9. *The conditions for orbital stability of the above linearized Hamiltonian equation are:*

1) Given $\lambda > 0$, there exists a unique ground state R_λ,

2) L_+ has a null space that is spanned by the set $\{\frac{\partial R}{\partial x_j}$ for $j = 1, 2, ..., n\}$,

3) $\frac{d}{d\lambda}Q(R_\lambda) > 0$.

Under these assumptions, the only place we must change the proof as given is:

Proposition 2.1.10. *L_+ has exactly one negative eigenvalue.*

Proof. We present here the general proof given in Section 13 of (RodSchSof). An alternative proof using a form of the equation created as a linear combination of a power nonlinearity and a generic nonlinearity satisfying the conditions above can be found in (Wein1).

As seen in (BerLion) and proved in Appendix A, for a large class of nonlinearities, including those addressed in this note, a soliton is constructed by the constrained minimization problem for the functional

$$J[u] = \{T(u)|V(u) = 1\},$$

for functionals T and V defined above.

A minimizer w will then satisfy

$$-\Delta w + \mu(\lambda^2 w - \beta(w^2)w) = 0,$$

where μ is the Lagrange multiplier determined by the constraint. The soliton is then given by the rescaling

$$R(x) = w(\mu^{-\frac{1}{2}}),$$

for a particular choice of μ. Now, take a smooth family of functions w_z such that $V(w_z) = 1$ for all z and $w_0 = w$, the minimizer. We then have

$$\begin{aligned} \frac{d}{dz} J[w_z]|_{z=0} &= 0, \\ \frac{d^2}{dz^2} J[w_z]|_{z=0} &\geq 0. \end{aligned}$$

Let $\dot{w} = \partial_z w$. Hence,

$$2\int \nabla w_0 \nabla \dot{w}_0 = 0$$

and

$$-2\int \Delta \dot{w}_0 \dot{w}_0 + \Delta w_0 \ddot{w}_0 \geq 0.$$

Since $V(w_z)$, is constant with respect to z, we get

$$\begin{aligned}\int (\beta(w_0^2)w_0 - \lambda^2 w_0)\dot{w}_0 &= 0,\\ \int (\beta(w_0^2) + 2\beta'(w_0^2)w_0^2 - \lambda^2 w_0)|\dot{w}_0|^2 + (\beta(w_0^2)w_0 - \lambda^2 w_0)\ddot{w}_0 &= 0.\end{aligned}$$

Taking the inner product of the equation for w_0 with $\ddot{w}_0$, we have

$$-\int \Delta w_0 \ddot{w}_0 = \mu \int (\beta(w_0^2)w_0 - \lambda^2 w_0)\ddot{w}.$$

Combining the above results, we have

$$-2\int (\Delta \dot{w}_0 + \mu(\beta(w_0^2) + 2\beta'(w_0^2)w_0^2 - \lambda^2)\dot{w}_0)\dot{w}_0 \geq 0.$$

By rescaling, we then get

$$\mu^{1+\frac{n}{2}} \langle L_+ \dot{w}_0(\mu^{-\frac{1}{2}}\cdot), \dot{w}_0(\mu^{-\frac{1}{2}}\cdot)\rangle \geq 0.$$

From above, we see that $\dot{w}_0(\mu^{-\frac{1}{2}}\cdot)$ is orthogonal to

$$\Xi = (\beta(R^2)R - \lambda^2 R).$$

Let Π be the orthogonal projection onto the subspace of functions orthogonal to Ξ. Then, from above, $\Pi L_+ \Pi \geq 0$. Let ν and v be a negative eigenvalue and the corresponding eigenfunction of L_+ respectively. Let

$$v = v_1 + \Xi,$$

where $\Pi v_1 = v_1$. We have

$$L_+ v = L_+ v_1 + L_+ \Xi = \nu(v_1 + \Xi),$$

which implies

$$(\Pi L_+ \Pi - \nu)v_1 = -\Pi L_+ \Xi.$$

Since $\Pi L_+ \Pi \geq 0$ and $\nu < 0$, we have that $(\Pi L_+ \Pi - \nu)$ is invertible on the space of functions orthogonal to Ξ and

$$v_1 = -(\Pi L_+ \Pi - \nu)^{-1} \Pi L_+ \Xi.$$

Therefore, v is simple as it is uniquely defined.

Using the above representation for v and an inner product with Ξ, we obtain

$$-\langle L_+(\Pi L_+\Pi - \nu)^{-1}\Pi L_+\Xi, \Xi\rangle + \langle L_+\Xi, \Xi\rangle = \nu\langle \Xi, \Xi\rangle.$$

Assume there exists another negative eigenvalue $\tilde{\nu}$. The same holds and we have

$$-\langle L_+(\Pi L_+\Pi - \nu)^{-1}\Pi L_+\Xi, \Xi\rangle + \langle L_+(\Pi L_+\Pi - \tilde{\nu})^{-1}\Pi L_+\Xi, \Xi\rangle = (\nu - \tilde{\nu})\langle \Xi, \Xi\rangle.$$

Since

$$\nu - \tilde{\nu} = (\Pi L_+\Pi - \tilde{\nu}) - (\Pi L_+\Pi - \nu),$$

we have

$$(\Pi L_+\Pi - \nu)^{-1} - (\Pi L_+\Pi - \tilde{\nu})^{-1} = (\nu - \tilde{\nu})(\Pi L_+\Pi - \nu)^{-1}(\Pi L_+\Pi - \tilde{\nu})^{-1}.$$

Using this resolvent identity, plus the fact that L_+ is self-adjoint and $(\Pi L_+\Pi - \nu)^{-1}$ maps the space of functions orthogonal to Ξ to itself, we can rewrite the above as

$$\langle (\Pi L_+\Pi - \nu)^{-1}(\Pi L_+\Pi - \tilde{\nu})^{-1}\Pi L_+\Xi, \Pi L_+\Xi\rangle = -\langle \Xi, \Xi\rangle.$$

This is a contradiction since the above is a positive operator as $\Pi L_+\Pi$ is positive and $\nu, \tilde{\nu} < 0$.

□

Note that in the sequel, we will address all of the above assumptions both analytically and numerically in order to allow one to test a given problem for the likelihood that these conditions are satisfied.

2.2 Variational Techniques and Stability Results

Once we have the existence of soliton solutions, we would like to know if those solutions are stable. There are several forms of stability that are defined above. All of these are tied very closely to the spectral properties of the linearized equation. From analysis due

to (Wein1) and (GrilShaStr), these spectral properties depend heavily on the function $\delta(\lambda)$ described above. As we saw earlier, from a Lagrange multiplier approach, we see that

$$\frac{d}{d\lambda}E(R_\lambda) = -\lambda\frac{d}{d\lambda}Q(R_\lambda). \tag{2.9}$$

Hence, the condition for orbital stability and instability becomes

$$\frac{d}{d\lambda}Q(R_\lambda) > 0 \tag{2.10}$$

and

$$\frac{d}{d\lambda}Q(R_\lambda) < 0, \tag{2.11}$$

respectively. For power nonlinearities, a simple scaling argument shows that L^2 subcritical powers ($p < \frac{4}{d}$) give stable solitons and L^2 supercritical powers ($p > \frac{4}{d}$) give unstable solitons. In particular,

$$\delta(\omega) = E(\phi_\omega) - \omega Q(\phi_\omega).$$

Then, we have the following results.

Theorem 7. *The statement that $\delta(\omega_0)$ is convex at ω_0 is equivalent to the statement that the Hamiltonian $E(u)$ restricted to the manifold $M_0 = \{u \in H_r^1 | Q(u) = Q(\phi_{\omega_0})\}$ has a local minimum at $u = \phi_{\omega_0}$.*

Theorem 8. *Fix ω_0 and consider the curve*

$$\lambda \to q(\omega, x) = \phi_\omega(\frac{x}{\eta\omega}),$$

where $\eta(\omega)^d = \frac{Q(\omega_0)}{Q(\omega)}$. Hence, $Q(q(\omega)) = Q(\phi_{\omega_0})$. Then, one has

$$(i)\ \partial_\omega^2 E(q(\omega_0)) \leq \delta''(\omega_0),$$

and (ii)

If $\delta(\omega)$ is strictly concave at ω_0, then $E(q(\omega)) < E(\phi_{\omega_0})$ for $\omega \neq \omega_0$ near ω_0.

Theorem 9. *If $\delta(\omega)$ is convex at ω_0, then the associated standing wave $\phi_{\omega_0}e^{i\omega_0 t}$ is orbitally stable. Conversely, if $\delta(\omega)$ is concave at ω_0, the orbit of ϕ_{ω_0} is unstable.*

The proofs of the above statements about orbital stability/instability are discussed in (SulSul). Note that the stability result is a restatement of the result above in Section 2.1, however presented in a more abstract setting.

Now, for the types of nonlinearities described above, no scaling occurs. Hence, we must assume the differentiability of $Q(R_\lambda)$ and analyze at this curve numerically. As discussed in Chapter 1, in (RodSchSof) it is proved that condition (2.10) holds for a slightly different class of nonlinearities that depends on a small parameter θ and are small perturbations of stable power nonlinearities for any λ provided θ small enough. Numerics show that for the nonlinearities of type 1 and 2 above, the graph of $Q(R_\lambda)$ with respect to λ for $0 < \lambda < \infty$ is a map which is ∞ at $\lambda = 0, \infty$ and has a global minimum at some $\lambda_0 > 0$ depending upon p, q. Hence, we have orbital instability for $0 < \lambda < \lambda_0$ and orbital stability for $\lambda_0 < \lambda$. At λ_0, we should expect to see instability as some perturbations can lead to orbitally unstable solitons and some perturbations can lead to energies for which soliton solutions are not allowed (below $Q(R_{\lambda_0})$).

Recently, in (ComPel) it is proved using spectral analysis that in fact the minimal energy ground state is unstable. The proof is rather similar to the idea of modulation stability presented in (Wein1). In general, they use as an ansatz a minimal mass soliton coupled with the various constants from Galilean invariance. Then, from that ansatz they linearize about the soliton, develop a system of ODE's for the parameters of the equation and show there exists initial data such that the parameters of the equation change dramatically, while the initial deviation is still quite small. In general, they actually build a solution that moves the solution along the soliton curve. Hence, the orbital stability condition is violated. However, it is important to note that this is strict nonlinear instability. The precise statement they prove is

Theorem 10. *Let $e^{i\omega t}R_\omega$ be the standing wave solutions to Equation* (1.4)*. Assume that $\omega_0 \in (\omega_1, \omega_2)$ is the inflection point of $d(\omega)$ such that $d''(\omega) = 0$ and $d''(\omega) \leq 0$ in a one-sided open neighborhood of ω_0. Then, there exists $\epsilon > 0$ such that for any $\delta > 0$, there exist $t_1 < \infty$ and a pair of functions*

$$(\omega, \rho) \in C^1([0, t_1], (\omega_1, \omega_2)) \times C^1([0, t_1], X)$$

such that

$$u(t) = e^{\int_0^t \omega(t')dt'}(R_{\omega_t} + \rho(t))$$

is a solution to Equation (1.8) *and*

$$|\omega(0) - \omega_0| < \delta,\ \|\rho(t)\|_X \leq \beta(\omega(t) - \omega_0),\ \textit{and}\ |\omega(t_1) - \omega_0| > \epsilon,$$

where $\beta \in C(\mathbb{R})$ *is such that* $\beta(s) \geq 0$, $\beta(s) = o(s)$.

In general, the heuristic is that the linearized operator has polynomial growth of order t^3 in time due to an increased null space so, though linear stability/instability theory breaks down, there is enough growth to be unstable.

Also, the recent results in (Schlag1) and (KriSch1) prove that though solitons for L^2 supercritical NLS are unstable, there is in fact a finite codimension submanifold of perturbations along which there is in fact asymptotic stability in three and one dimensions respectively. To do this, they project onto on of the eigenvalues of the linearized Hamiltonian that does not cause instability. As shown in (ComPel) and discussed later, we linearize around a minimal energy ground state, all the eigenvalues collapse to degree 4 generalized null space at 0. By projecting onto some subset of the kernel for this Hamiltonian, we see a similar result at λ_0 provided p is large enough. These results will be discussed in more detail in Chapter 7.

Let us take R to be the soliton solution for the equation

$$-\Delta R + \gamma R - \frac{|R|^{p-1}}{1 + |R|^{p-q}} R = 0.$$

Then, setting $\tilde{R} = \lambda^{\alpha} R(\lambda^{\frac{1}{2}} x)$, we see that $\tilde{R}$ satisfies

$$-\Delta \tilde{R} + \gamma\lambda \tilde{R} - \frac{|\tilde{R}|^{p-1}}{\lambda^{\beta} + |\tilde{R}|^{p-q}} \tilde{R} = 0,$$

where $\alpha = \frac{2}{q-1}$ and $\beta = (p-q)\alpha$. Hence, for $\gamma = \frac{1}{\lambda}$, we see the correspondence between large soliton parameters γ and the solutions to the subcritical monomial problem and for small soliton parameters γ and solutions for the supercritical monomical problem. A variational argument in (RodSchSof) shows that given NLS with nonlinearity

$$\beta_\theta(s^2) = s^{q-1} \frac{f(s^2)}{\theta + s^2},$$

where

$$C_1 s^{p+1} \leq |f(s^2)| \leq C_2 s^{p+1},\ s^2|f'(s^2)| \leq C_3|f(s^2)|,\ q \in (1, \frac{4}{d}),\ q > -1,$$

for the ground state R_1, there exists θ_0, such that for $\theta < \theta_0$, $\langle L_+^{-1} R, R\rangle > 0$. In other words, the soliton curve is increasing, so the ground state is stable. From this analysis, one has that for λ large enough, the soliton curve is increasing much like the monomial subcritical soliton curve.

Chapter 3

Behavioral Numerical Methods and Results

3.1 Finding a Soliton

In (McCleod), uniqueness of the soliton is proved for a large class of nonlinearities. To prove this, they work in radial coordinates and hence prove that for solutions of the resulting Ordinary Differential Equation (ODE), there is a unique initial value at the origin which leads to a decreasing, positive solution by shooting. If this value is shifted downward, there are an infinite number of extrema and the solution is always positive. If the value is shifted upward, the solution becomes negative and is in fact, monotone decreasing. Hence, we use this analysis to numerically find the unique soliton.

Specifically, we use the method of spectral deferred correction coupled with a Linearly Implicit Euler (LIE) scheme on the radial ODE. It is possible for solitons of NLS equations with monomial nonlinearities to be found using a spectral scheme described in (DS), however, the convergence of the method depends heavily upon a scaling factor. Developing such a sophisticated method for saturated nonlinearities will be a topic of further research. We specifically choose linearly implicit Euler because it allows us to avoid the singularity

at the origin in radial coordinates. We use deferred correction to get an incredibly accurate method without having a large number of grid points.

To begin, we compute a first approximation u^0 to

$$\partial_t u = f(t, u),$$

at m stretched Legendre points

$$s_1 = T + H(1 + t_1)/(1 + t_m), \ldots, s_m = T + H(1 + t_m)(1 + t_m) = T + H,$$

where t_i is the i-th root of the m-th Legendre polynomial. Then, we have

$$(I - h_j Df(s_{j+1}, u_j^0))(u_{j+1}^0 - u_j^0) = h_j f(s_{j+1}, u_j^0),$$

for $j = 0, 1, \ldots, m-1$, $h_j = s_{j+1} - s_j$. Then, we need to estimate the error caused by the numerical approximation by a repeated correction step taking $u_k = (u_1^k, ..., u_m^{k+1})$ to $u^{k+1} = (u_1^{k+1}, ..., u_m^{k+1})$.

We have the itegral formulation given by

$$u(t) = u_0 + \int_0^t f(s, u(s))ds,$$

and define the residual to be

$$R(t) = u_0 + \int_0^t f(s, u(s))ds - u(t).$$

An order m accurate appoximate solution for R at the Legendre points is given by

$$R_n^k = u_0 + \sum_{j=1}^{m} S_{nj} f(s_j, u_j^k) - u_n^k,$$

where S_{nj} are elements of the spectral integration matrix derived from Gaussian quadrature. If y is the exact solution, the error E can be defined as

$$\begin{aligned} E(t) &= y(t) - u(t) \\ &= u_0 + \int_0^t f(s, y(s))ds - (u_0 + \int_0^t f(s, u(s))ds - R(t)) \\ &= R(t) + \int_0^t (f(s, u(s) + E(s)) - f(s, u(s)))ds \\ &= R(t) + \int_0^t G(s, E(s); u(s))ds, \end{aligned}$$

where $G(s, E; u) = f(s, u + E) - f(s)$. Then using linearly implicit Euler, we have

$$(I - hDf(s_{n+1}, u^k_{n+1}))E^k_{n+1} = E^k_n + R^k_{n+1} - R^k_n.$$

Hence, we set

$$u^{k+1}_n = u^k_n + E^k_n,$$

for $n = 1, 2, ..., m$. Each time we correct, we gain one order of accuracy, up to order m. Hence, we have a highly accurate method with a small number of relative grid points.

Now, that we have established the method, we begin with a range of values and based on the analysis of (McCleod), we bisect the interval until we have a very accurate approximation to the soliton. A plot of a known soliton for (NLS) on $\mathbb{R}$ compared to the computed soliton is shown in Figure 3.2. While it is not a particularly useful exercise, we show a plot of a soliton for a saturated nonlinearity in Figure 3.1. Using such a method, we can also approximately track the quantities $Q(\lambda)$ and $E(\lambda)$ from Section 1.2. We call the plots of Q and E with respect to λ the *soliton curve* for a particular nonlinearity. A sketch of the soliton curves for various nonlinearities is given in Figure 3.3. Notice that for the saturated nonlinearities, Q has a unique minimal mass attained at a unique λ.

3.2 Solving NLS numerically

We outline numerical methods for one dimension used to produce the results described at the end of this section. Note that these methods generalize to radial problems in higher dimensions quite easily.

We discretize our equation,

$$iu_t + u_{xx} + \beta(|u|^2)u = 0,$$

$$u(0, x) = u_0,$$

using a finite element scheme in space and the standard Crank-Nicholson scheme in time (at first order, this is the midpoint rule). Just as the equation itself this method is L^2 conservative. A similar scheme was implemented in (ADKM3), where the blow-up for NLS

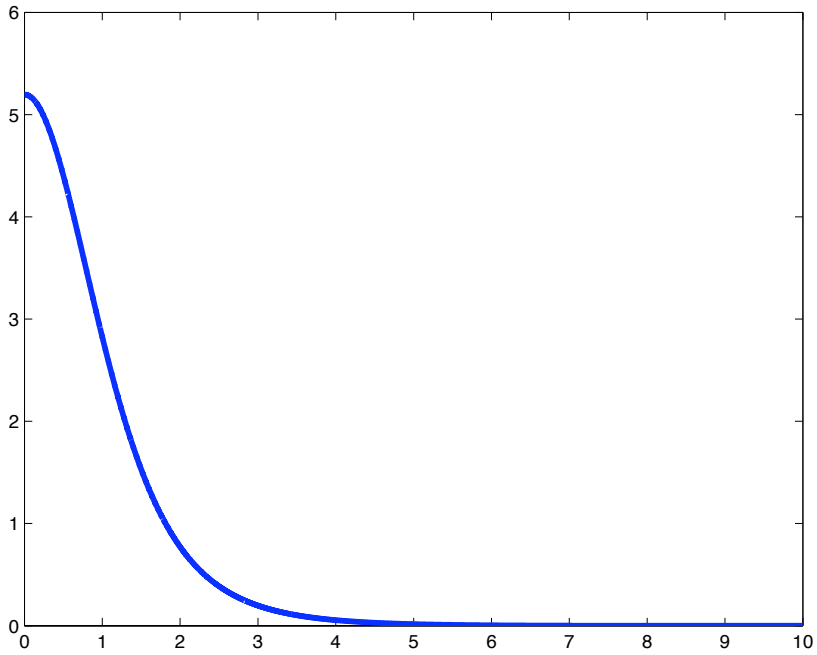

Figure 3.1. The plot of a numerically found soliton for a saturated NLS of type 2 in $\mathbb{R}^3$ with $q = 1$.

in several dimensions was analyzed. This method was applied by the author with Justin Holmer and Maciej Zworski in (2) and (3) to study soliton interactions with delta potentials.

Note that we require the spatial grid to be large enough to insure negligible interaction with the boundary. The convergence of such methods without potentials was studied in (ADKM1), (ADKM2).

We select a symmetric region about the origin, $[-R, R]$, upon which we place a mesh of N elements. The standard hat function basis is used in the Galerkin approximation. We allow for a finer grid in a neighbourhood of length 1 centered at the origin to better study the effects of the interaction with the delta potential. In terms of the hat basis the problem becomes:

$$\langle u_t, v\rangle + i\langle u_x, v_x\rangle/2 - i\langle\beta(|u|^2)u, v\rangle = 0,$$
$$u(0,x) = u_0\,, \quad u(t,x) = \textstyle\sum_v c_v(t)v\,,$$

where $\langle\cdot,\cdot\rangle$ is the standard L^2 inner product, v is a basis function and u, u_0 are linear

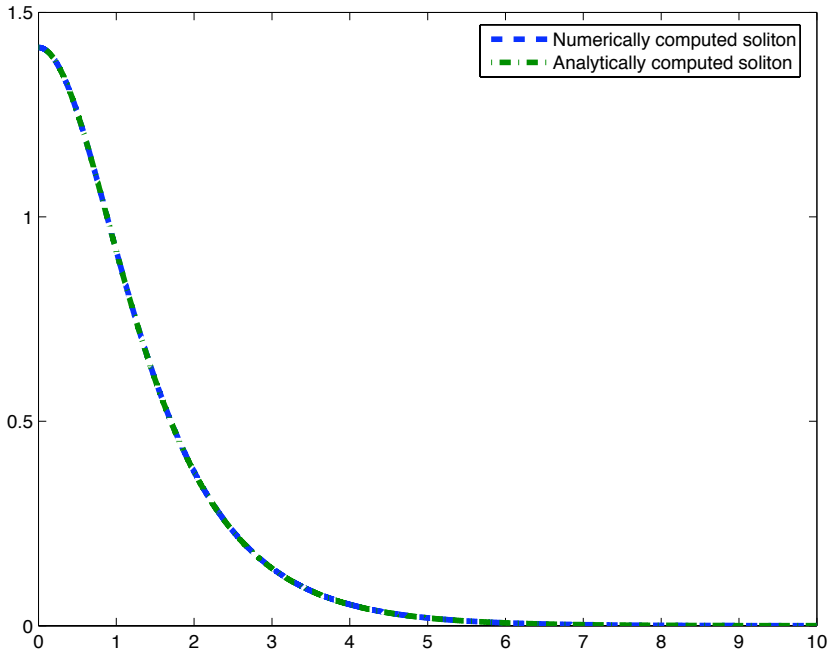

Figure 3.2. The plot of a numerically found soliton for cubic NLS in $\mathbb{R}$ vs. the known analytic soliton $R_1(x) = \sqrt{2}\text{sech}(x)$.

combinations of the v's. We remark that since v's are continuous the pairing of uv with the delta function is justified.

Since v's are hat functions, we have created a tridiagonal linear system with one contribution to the central element resulting from the delta function. Let $h_t > 0$ be a uniform time step,

$$u_n = \sum_v c_v(nh_t)v\,,$$

be the approximate solution at the nth time step. Implementing the midpoint rule in time, the system becomes:

$$\begin{aligned}&\langle u_{n+1} - u_n, v\rangle + ih_t\,\langle((u_{n+1}+u_n)/2)_x\,, v_x\rangle \\ &\quad = ih_t\,\langle\beta((u_{n+1}+u_n)/2)(u_{n+1}+u_n)/2, v\rangle\,, \qquad u_0 = \sum_v \alpha_v v,\end{aligned}$$

By defining

$$y_n = (u_{n+1} + u_n)/2\,,$$

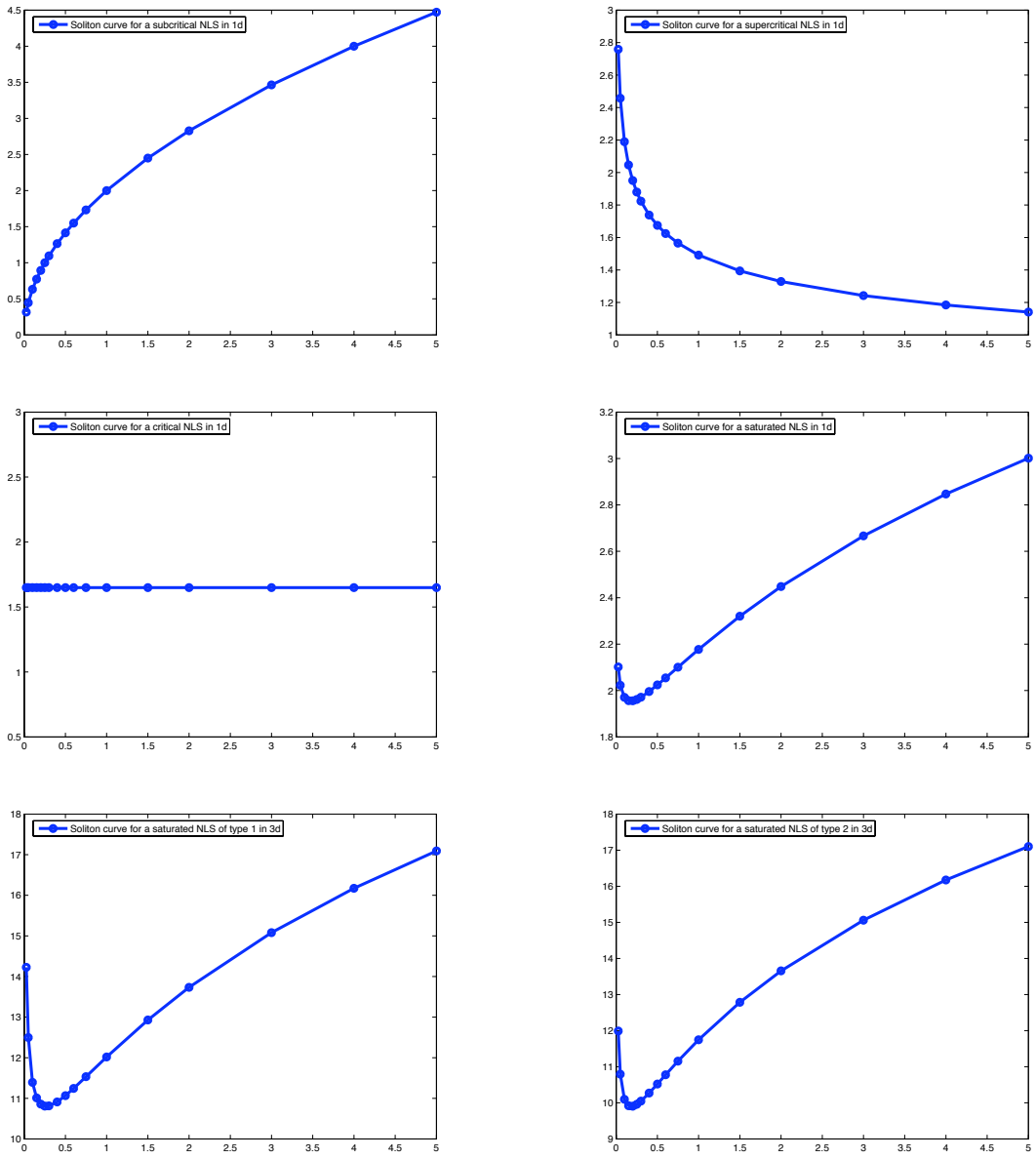

Figure 3.3. Plots of the soliton curves ($Q(\lambda)$ with respect to λ) for a subcritical nonlinearity ($d = 1$, $p = 3$), supercritical nonlinearity ($d = 3$, $p = 3$), critical nonlinearity ($d = 1$, $p = 5$), saturated nonlinearity of type 1 ($p = 7$, $q = 3$) in $\mathbb{R}$, saturated nonlinearity of type 1 in $3d$ ($p = 4$, $q = 2$), saturated nonlinearity of type 2 in $\mathbb{R}^3$ ($q = 2$). The curves for the monomial nonlinearities are found analytically, while the curves for the saturated nonlinearities are found numerically.

we have simplified our system to:

$$\langle y_n, v\rangle + i\frac{h_t}{4}\langle (y_n)_x, v_x\rangle = i\frac{h_t}{2}\langle \beta(|y_n|^2)y_n, v\rangle + \langle u_n, v\rangle.$$

An iteration method from (ADKM3) is now used to solve this nonlinear system of equations. To wit,

$$\langle y_n^{k+1}, v\rangle + i\frac{h_t}{4}\langle (y_n^{k+1})_x, v_x\rangle = i\frac{h_t}{2}\langle \beta(|y_n^k|)y_n^k, v\rangle + \langle u_n, v\rangle.$$

We take $y_n^0 = u_n$ and perform three iterations in order to obtain an approximate solution.

In order to take this radial problems in higher dimensions, we refine our inner product to be

$$\langle u, v\rangle = \int_0^\infty u(r)v(r)r^{d-1}dr.$$

The remainder of the analysis follows trivially. Note that a finite difference argument can also be applied here by linearizing the problem and discretizing derivatives directly.

In the remainder of this chapter, we use the finite element method to display soliton stability/instability for the stable/unstable parts of the soliton curve with saturated nonlinearities, dispersion for initial data with L^2 mass below the minimal mass for a saturated nonlinearity and interactions between stable solitons and close to minimal mass solitons for saturated nonlinearities in $\mathbb{R}$.

3.3 Soliton Stability

Here, we present numerical experiments towards soliton stability/instability. In Figure 3.4, we see that as we increase the correction parameter m from Section 3.1 when finding a stable soliton ($\lambda = 1$), the solution converges faster. Similarly, in Figure 3.5 we that as we increase the correction paramter m when finding an unstable soliton ($\lambda = .025$), it takes the solution longer to diverge. This gives numerical verification of the convergence of our computed solitons, as well as verification that our code for solving NLS adheres to standard theory.

In Figure 3.6, we plot perturbations of solitons computed with high accuracy ($m = 35$) in $\mathbb{R}$ and $\mathbb{R}^3$. Specifically, we plot the evolution of the initial data $1.01 * R$, for R both stable

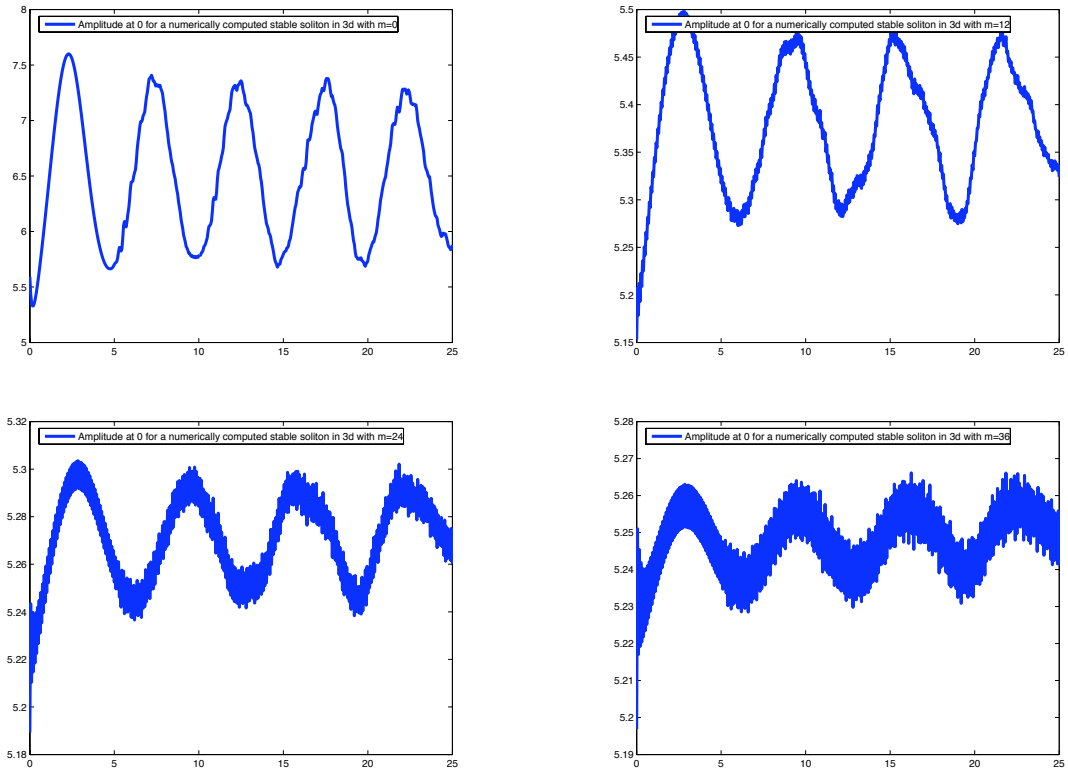

Figure 3.4. Numerical simulation of soliton stability for R_1 computed with improved accuracy for saturated nonlinearities of type 2 in $\mathbb{R}^3$ with $q = 2$. In particular, we plot the change in amplitude at $x = 0$ on a time interval $[0, T]$ for various corrections $m = 0, 12, 24, 36, 48$ with the initial spatial step to be $\Delta x = .1$.

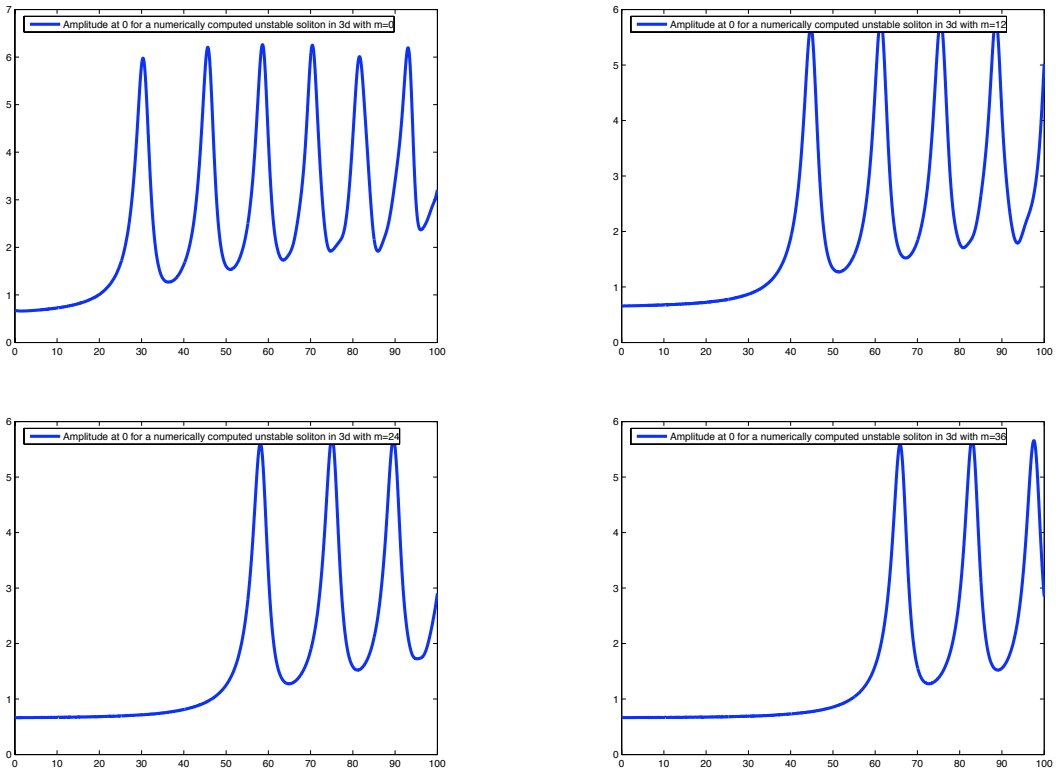

Figure 3.5. Numerical simulation of soliton instability for $R_{.025}$ computed with improved accuracy for saturated nonlinearities of type 2 in $\mathbb{R}^3$ with $q = 2$. In particular, we plot the change in amplitude at 0 on a time interval $[0, T]$ for various corrections $m = 0, 12, 24, 36$ with the initial spatial step to be $\Delta x = .1$.

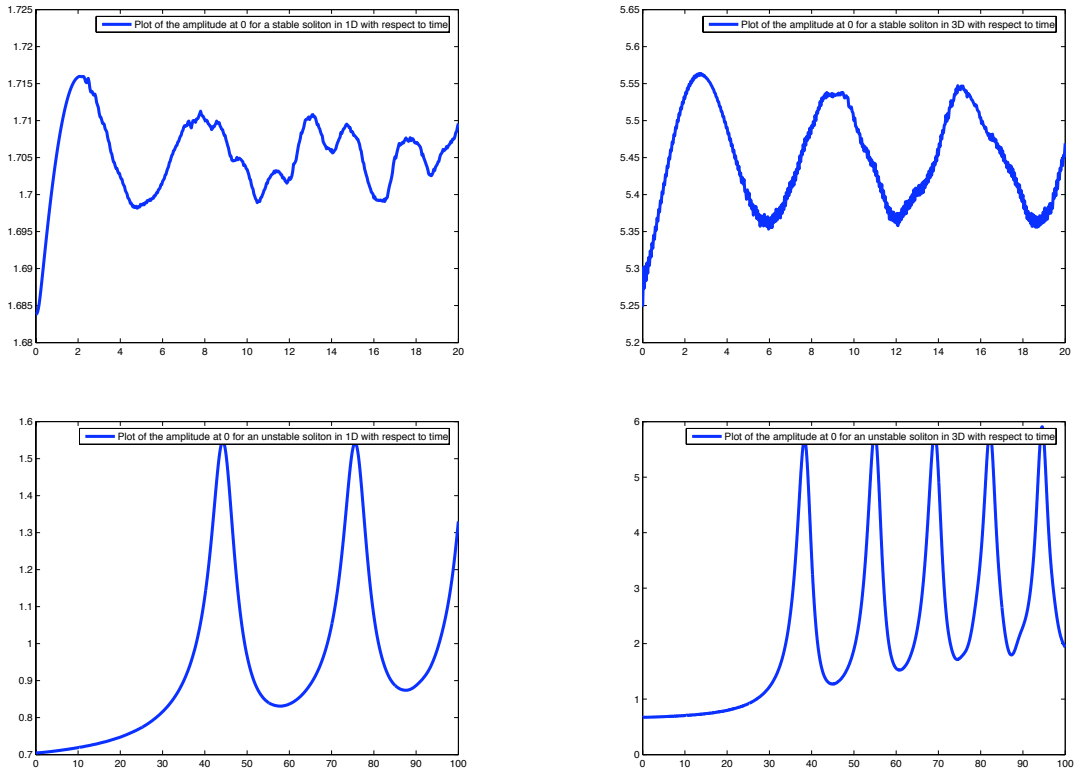

Figure 3.6. Numerical simulation of soliton stability/instability for saturated nonlinearities in $1d$ and type 2 in $\mathbb{R}^3$. In particular, we plot the amplitude at $x = 0$ for a solution with initial data $1.01 * R$ on a time interval $[0, T]$ for T large enough to see the overrall behavior, $R = R_1$ for the stable computation and $R = R_{.025}$ for the unstable computation.

and unstable. The behavior is similar to that predicted by theory. The perturbations of the stable solitons oscillate quickly towards a soliton very close on the soliton curve. For the unstable solitons, the solution radically alters until it once again starts oscillating about a stable soliton solution.

3.4 Soliton Dispersion

After accurately calculating an approximation to the minimal mass soliton R_{min}, we plot $\alpha * R_{min}$ for a $1d$ saturated nonlinearity in Figure 3.7. As seen in the figure, the solution disperses quickly even for $\alpha = .99$. Similarly, in Figure 3.8, the same phenomenon is observed in $\mathbb{R}^3$ for a saturated nonlinearity of type 2. Future research will include analyzing

this behavior analytically to prove that for arbitrary initial data below this minimal mass, scattering occurs.

3.5 Soliton Interaction

From the work of (RodSchSof), we know that under various separation conditions, we have asymptotic stability for a collection of N solitons for saturated NLS equations. This phenomenon is observed in Figure 3.9 where we look at two stationary solitons in $\mathbb{R}$ separated spatially and see that these solitons remain stable. Specifically, in this experiment, one of the solitons is a slightly stable perturbation of the minimal mass soliton.

In Figures 3.10 and 3.11, we look at a numerical simulation of the solution with initial data consisting of a slightly stable perturbation of the minimal mass soliton stationary at the origin coupled with a more stable soliton given a velocity ($v = 5$). These calculations are in a preliminary phase and will be explored in more detail in further research, but the effect of annihilation of the minimal mass soliton is observed. Phenomenon of this nature are unique to the saturated nonlinearities and provide interesting clues into phyiscal situations where such nonlinearities act as models. The author will continue to explore these phenomena to further numerical accuracy as well as analytically.

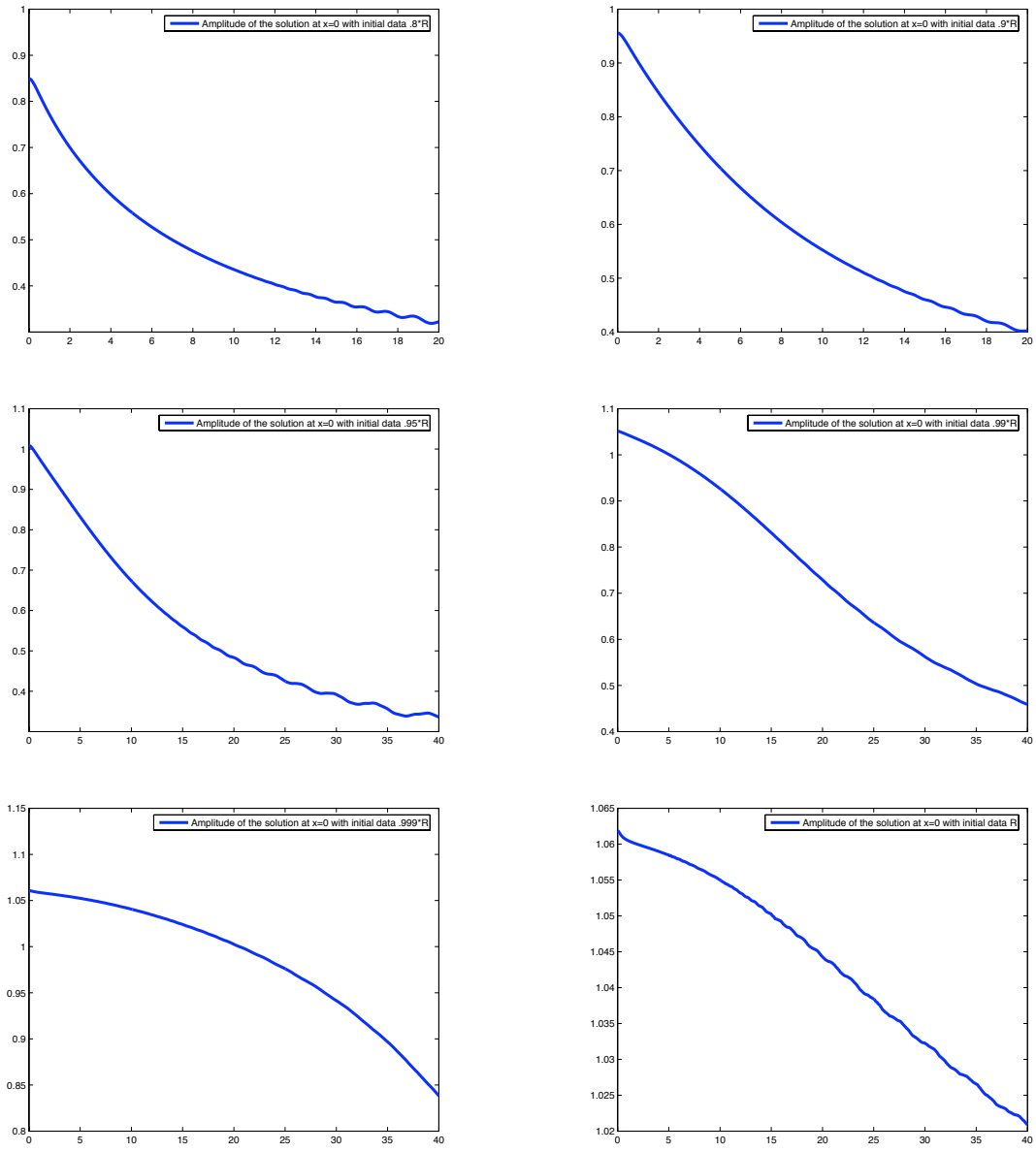

Figure 3.7. Numerical simulation of dispersion for perturbations below the minimal mass soliton for saturated nonlinearities. In particular, given the minimal mass soliton R, we solve the equation with initial data αR for $\alpha = .8, .9, .95, .99, .999, 1$. Presented here are plots of the amplitude at $x = 0$ with respect to time t for each α with saturated nonlinearity of type 1 in $\mathbb{R}$ with $p = 7$, $q = 3$.

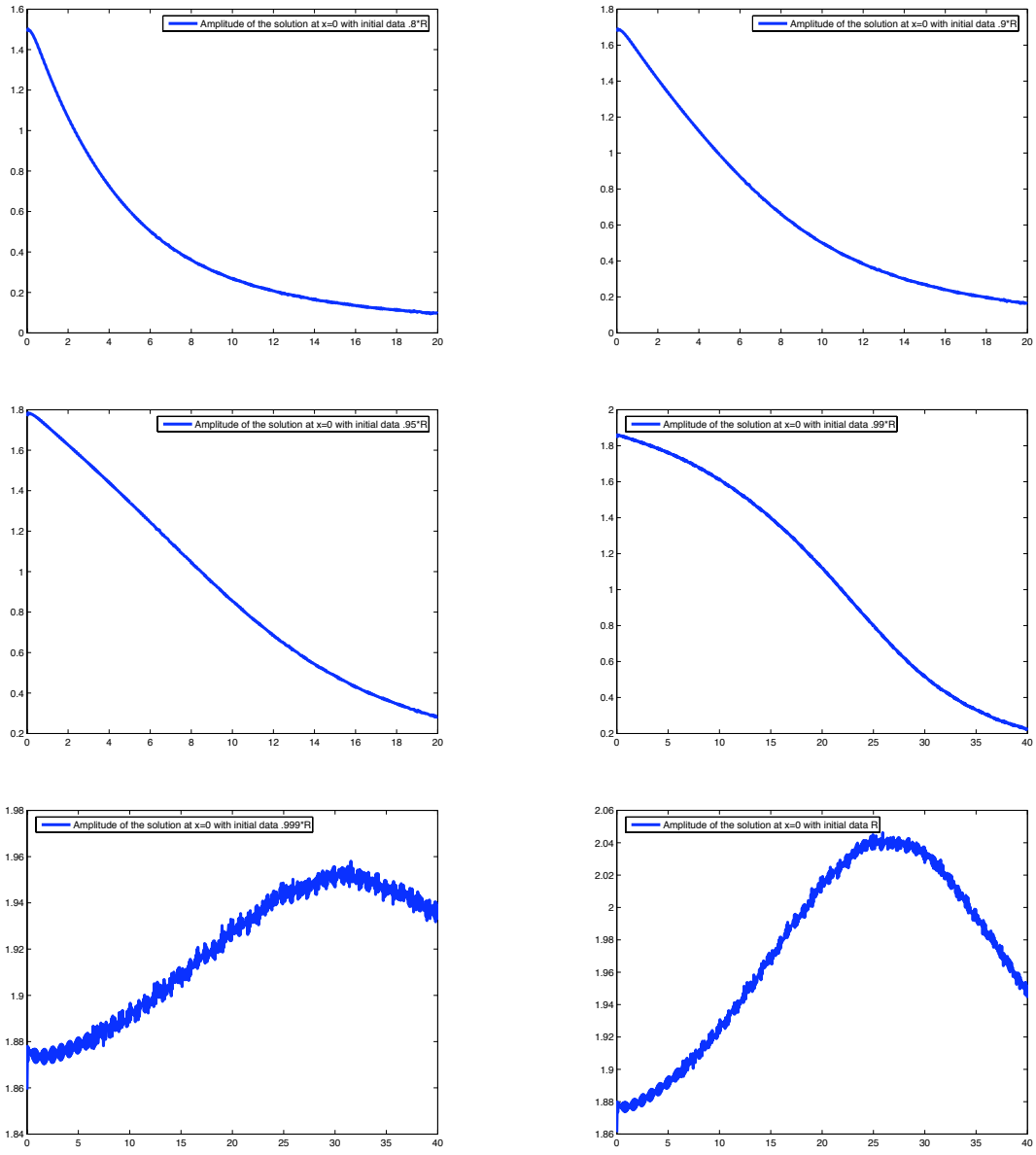

Figure 3.8. Numerical simulation of dispersion for perturbations below the minimal mass soliton for saturated nonlinearities. In particular, given the minimal mass soliton R, we solve the equation with initial data αR for $\alpha = .8, .9, .95, .99, .999, 1$. Presented here are plots of the amplitude at $x = 0$ with respect to time t for each α with saturated nonlinearity of type 2 in $\mathbb{R}^3$ with $q = 2$.

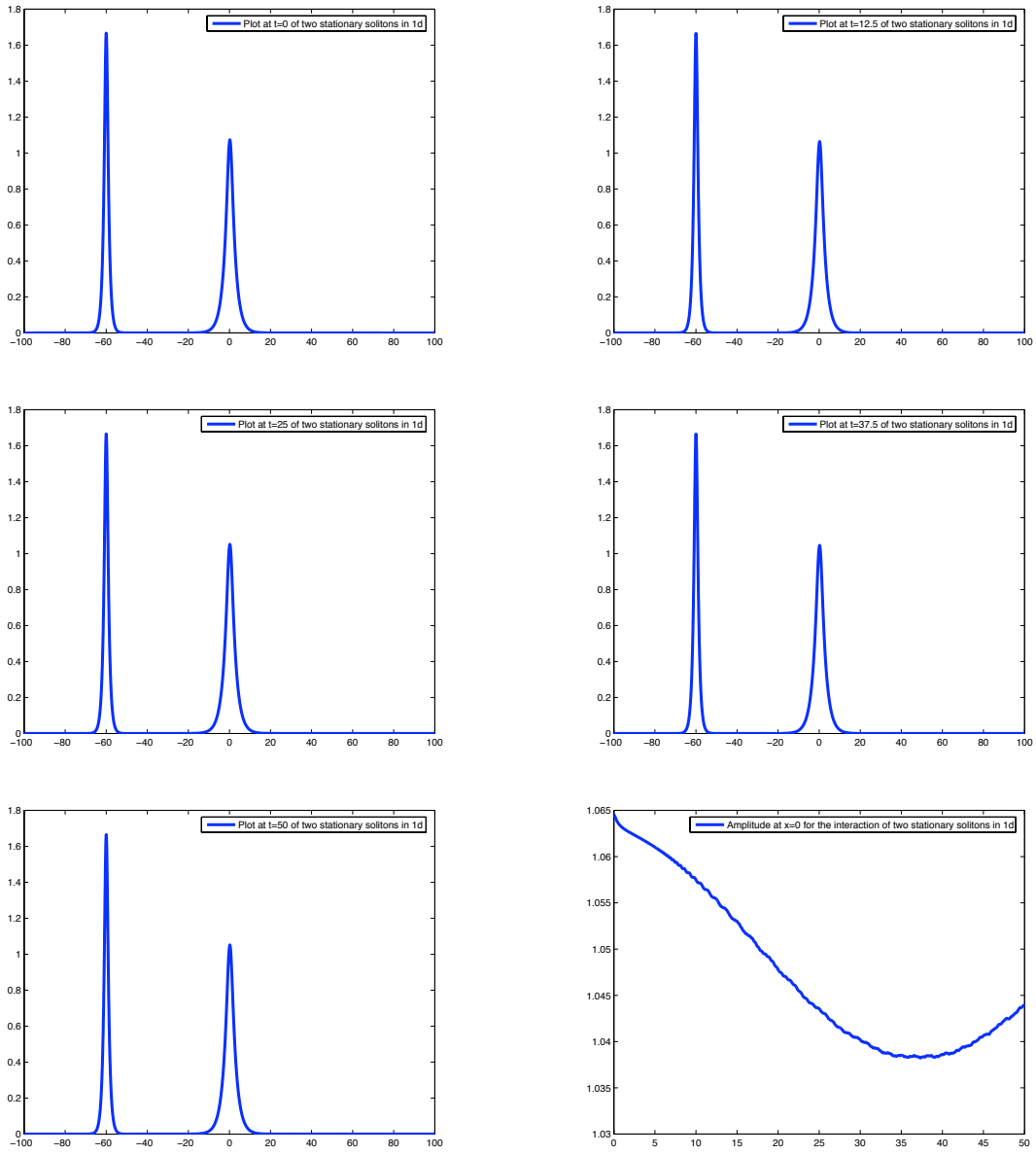

Figure 3.9. Numerical simulation of a stationary stable soliton interacting with another stationary soliton which is stable, yet close to the minimal mass soliton for saturated nonlinearities of type 1 in $\mathbb{R}$ with $p = 7$, $q = 3$. Presented here are plots of the absolute value of the solution with respect to the spatial grid for times $t = 0, 12.5, 25, 37.5, 50$ as well as a plot of the amplitude at $x = 0$ with respect to time.

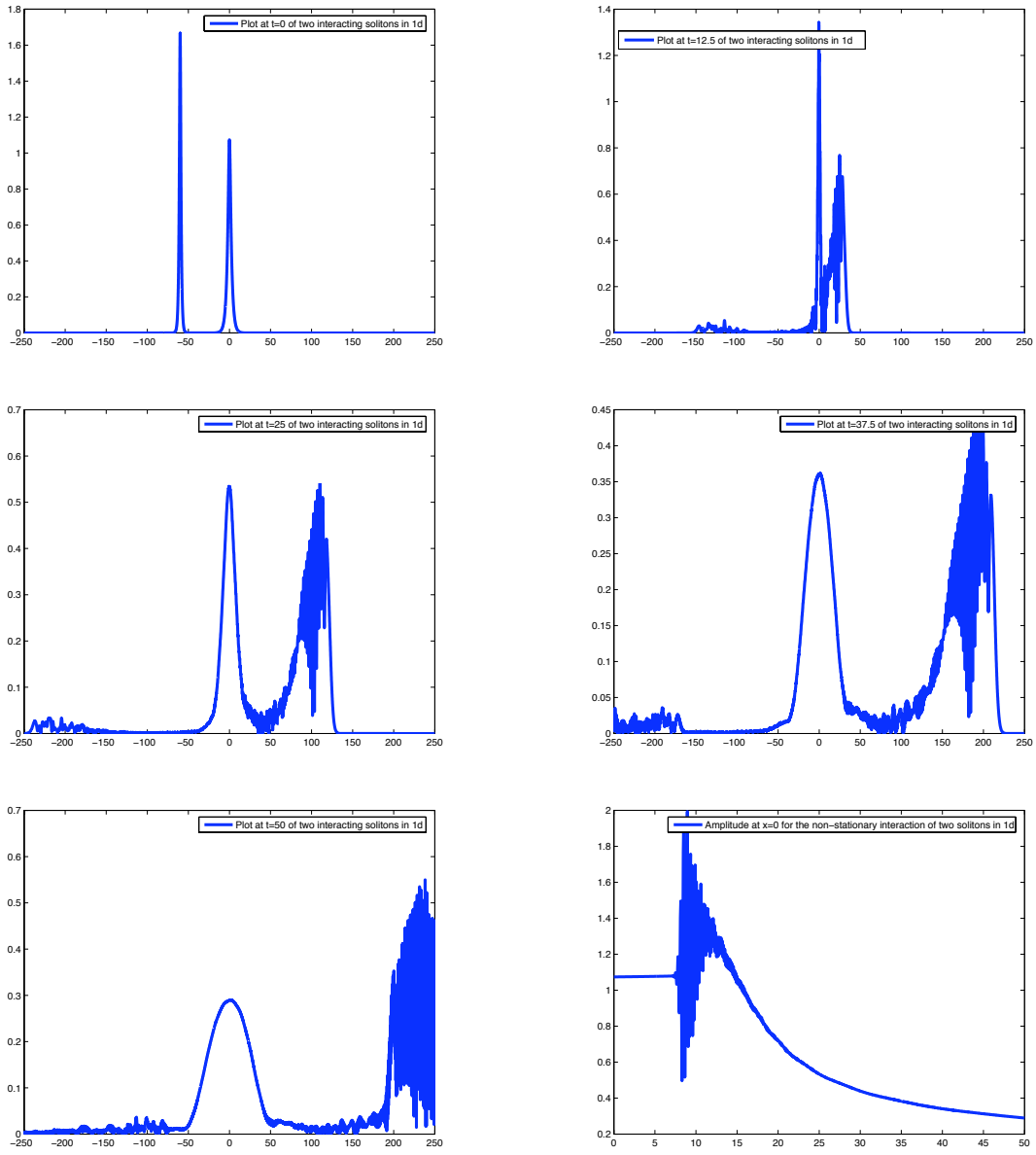

Figure 3.10. Numerical simulation of a stable soliton moving with velocity $v = 5$ interacting with a stationary soliton which is stable, yet close to the minimal mass soliton for saturated nonlinearities of type 1 in $\mathbb{R}$ with $p = 7$, $q = 3$. Presented here are plots of the absolute value of the solution with respect to the spatial grid for times $t = 0, 12.5, 25, 37.5, 50$ as well as a plot of the amplitude at 0 with respect to time.

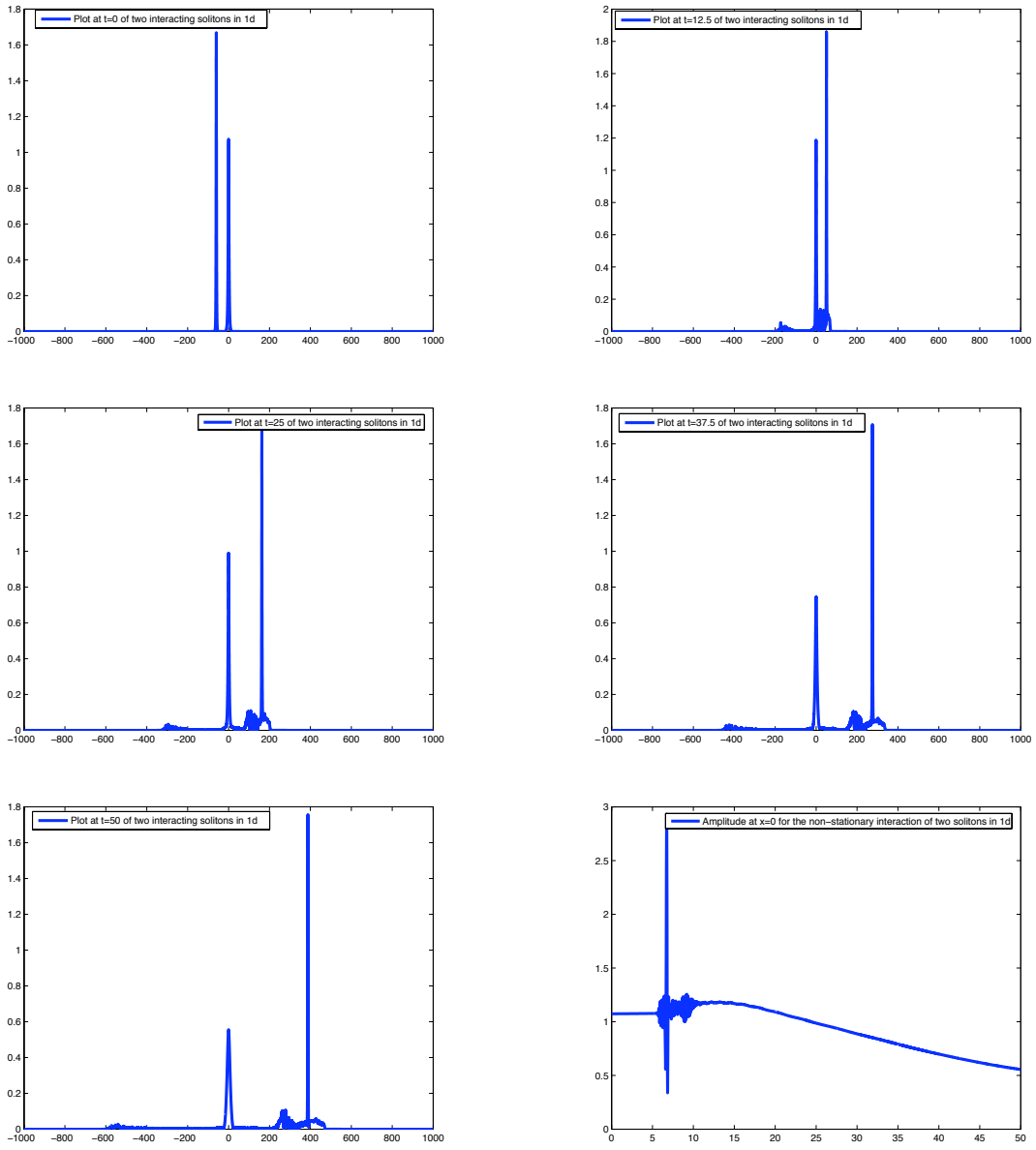

Figure 3.11. A more accurate numerical simulation of a stable soliton moving with velocity $v = 5$ interacting with a stationary soliton which is stable, yet close to the minimal mass soliton for saturated nonlinearities of type 1 in $\mathbb{R}$ with $p = 7$, $q = 3$. Presented here are plots of the absolute value of the solution with respect to the spatial grid for times $t = 0, 12.5, 25, 37.5, 50$ as well as a plot of the amplitude at 0 with respect to time.

Chapter 4

Spectral Results

In this chapter, we analytically and numerically describe the spectrum on a linearized Hamiltonian resulting from a saturated NLS equation. It will become clear that while much is still unknown about the spectrum, in general, one can see that they are in fact admissible as defined below in Section 1.4.1 and discussed in Chapter 1.

4.1 Spectral Properties of the Linearized Hamiltonian

To begin, let us write down the form of the linearized equation. First of all, let us assume we are looking for a solution $\psi = e^{i\lambda t}(R_\lambda + \phi(x,t))$. For simplicity, set $R = R_\lambda$. Inserting this into the equation we know that since ϕ is a soliton solution we have

$$i(\phi)_t + \Delta(\phi) \quad = \quad -\beta(R^2)\phi - 2\beta'(R^2)R^2\mathrm{Re}(\phi) + O(\phi^2), \tag{4.1}$$

by splitting ϕ up into its real and imaginary parts, then doing a Taylor Expansion. Hence, if $\phi = u + iv$, we get

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{H} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{4.2}$$

where

$$\mathcal{H} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}, \tag{4.3}$$

where

$$L_- = -\Delta + \lambda - \beta(R_\lambda)$$

and

$$L_+ = -\Delta + \lambda - \beta(R_\lambda) - 2\beta'(R_\lambda^2)R_\lambda^2.$$

There are many things we can immediately say about L_-, L_+ and $\mathcal{H}$. For a reference on the spectral theory involved, see Hislop-Sigal (HS) or Reed-Simon (RSv4). First of all, both L_- and L_+ are self-adjoint operators. Also, L_- is a non-negative definite operator and its null space is span$\{R\}$. Note also that the functions $\frac{\partial R}{\partial x_j}$ for $j = 1, 2, ..., n$ are in the null space of L_+. By comparison with the operator $\Delta + \lambda$ and using the fact that R decays exponentially, we see that the essential spectrum of $\mathcal{H}$ is the set $(-\infty, \lambda] \cup [\lambda, \infty)$ from Weyl's Theorem (see Appendix B.4). Finally, using the fact that L_- is non-negative definite and looking at eigenvalues $\mathcal{H}^2$, we see

$$L_- L_+ u = \nu^2 u. \tag{4.4}$$

However, this can be rewritten as

$$Tg = L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} g = \nu^2 g$$

for $g = L_-^{\frac{1}{2}} u$. Since the operator T is self-adjoint, we must have $\nu \in \mathbb{R} \cup i\mathbb{R}$.

We formalize this heuristic discussion with the following theorem from (ES1). Before we begin, note that we also have

$$\mathcal{H} = \mathcal{H}_0 + V = \begin{bmatrix} -\Delta + \lambda^2 & 0 \\ 0 & \Delta + \lambda^2 \end{bmatrix} + \begin{bmatrix} -V_1 & -V_2 \\ V_2 & V_1 \end{bmatrix},$$

when we act on u and $\bar{u}$ instead of u_1 and u_2 where $u = u_1 + iu_2$.

Theorem 11 (Erdogan-Schlag)**.** *Assume there are no embedded eigenvalues in the continuous spectrum of spec$(\mathcal{H})$. The essential spectrum of $\mathcal{H}$ equals $(-\infty, -\lambda^2] \cup [\lambda^2, \infty)$. Moreover, spec$(\mathcal{H}) = -$spec$(\mathcal{H} = \overline{\text{spec}(\mathcal{H})} =$ spec$(\mathcal{H}^*)$ and spec$(\mathcal{H}) \subset \mathbb{R} \cup i\mathbb{R}$. The discrete spectrum consists of eigenvalues $\{z_j\}_{j=1}^N$, $0 \leq N \leq \infty$, of finite multiplicity. For each $z_j \neq 0$, the algebraic and geometric multiplicities coincide and Ran$(\mathcal{H} - z_j)$ is closed. The zero eigenvalue has finite multiplicity.*

Proof. Using standard techniques, $\mathcal{H}$ is closed on $W^{2,2} \times W^{2,2}$. Since $\mathcal{H}_0^* = \mathcal{H}_0$, we have spec$\mathcal{H}_0 \subset \mathbb{R}$. For $\mathrm{Re}\,(z) \neq 0$, we have

$$(\mathcal{H}_0 - z)^{-1} = -(\mathcal{H}_0 + z) \begin{bmatrix} (H_0^2 - z^2)^{-1} & 0 \\ 0 & (H_0^2 - z^2)^{-1} \end{bmatrix} \tag{4.5}$$

$$= - \begin{bmatrix} (H_0^2 - z^2)^{-1} & 0 \\ 0 & (H_0^2 - z^2)^{-1} \end{bmatrix} (\mathcal{H}_0 + z), \tag{4.6}$$

and

$$(\mathcal{H} - z)^{-1} = (\mathcal{H}_0 - z)^{-1}[I - U_1[I - U_2 J(\mathcal{H}_0 - z)^{-1}U_1]^{-1}U_2 J(\mathcal{H}_0 - z)^{-1}U_1]^{-1}], \tag{4.7}$$

where

$$J = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},\ U_1 = \begin{bmatrix} |V_1|^{\frac{1}{2}} & 0 \\ 0 & |V_2|^{\frac{1}{2}} \end{bmatrix},\ U_2 = \begin{bmatrix} |V_1|^{\frac{1}{2}} sign(V_1) & 0 \\ 0 & |V_2|^{\frac{1}{2}} sign(V_2) \end{bmatrix}.$$

and $W_1 = V_1 - V_2$ and $W_2 = V_1 + V_2$. From (4.5), we see that $\sigma_{ess}\mathcal{H}_0 = (-\infty, -\lambda^2] \cup [\lambda^2, \infty)$ since the resolvent can be written in terms of $(H_0^2 - z^2)$. Then, using Weyl's Theorem from (RSv4) (see Appendix B.4), since we have sufficient decay on V_1 and V_2, $\mathcal{H}$ is relatively compact to $\mathcal{H}_0$, so $\sigma_{ess}(\mathcal{H}) = \sigma_{ess}\mathcal{H}_0$. By analytic Fredholm theory (see Appendix B.5), we have that $(\mathcal{H} - z)^{-1}$ is meromorphic on $\mathbb{C} \setminus (-\infty, -\lambda^2] \cup [\lambda^2, \infty)$. Further, the poles of the resolvent are eigenvalues of finite multiplicity and $\mathrm{Ran}(\mathcal{H} - z_j)$ is closed at each pole z_j.

For the matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},\ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

we have $\sigma_j \mathcal{H} \sigma_j = \mathcal{H}$ and $\sigma_j^2 = I$ for $j = 1, 2, 3$. Hence, the symmetries of the spectrum arise naturally by looking at

$$\sigma_j \mathcal{H} \sigma_j u = u.$$

Let us now look at $\mathcal{H}u = Eu$ for $E \neq 0$. This gives that $u_1,\ u_2 \neq 0$ and $u_1 \perp \mathrm{Ker}(L_-)$. Hence, for $g = L_-^{-\frac{1}{2}} u_1$, we get

$$L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} g = Eg.$$

Claim 4.1.1. *The operator* $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} : H^4(\mathbb{R}^d) \to H^4(\mathbb{R}^d)$ *is self-adjoint.*

Proof. Suppose that $\langle L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} f, g\rangle = \langle f, h\rangle$ for all $f \in H^4$ and fixed $g,\ h \in L^2$. Let $f \in \mathrm{Ker}(L_-)$, then we have $h \in \{\mathrm{Ker}(L_-)\}^{\perp})$. Applying the Fredholm alternative to $L_-^{\frac{1}{2}}$, we have $h = L_-^{\frac{1}{2}} h_1$ for some $h_1 \in D(L_-^{\frac{1}{2}}) = H^1$. Since h_1 is defined only modulo $\mathrm{Ker}(L_-)$, we have

$$\langle L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} f, g\rangle = \langle f, L_-^{\frac{1}{2}}(h_1 + c\phi)\rangle = \langle L_-^{\frac{1}{2}} f, h_1 + c\phi\rangle$$

for any $c \in \mathbb{C}$ and all $f \in H^4$. Set $f_1 = L_-^{\frac{1}{2}} f$, then

$$\langle L_-^{\frac{1}{2}} L_+ f_1, g\rangle = \langle f_1, h_1 + c\phi\rangle.$$

Note that we have $f_1 \in H^3$ and $f_1 \perp \phi$. In order to remove the latter restriction, take c to be such that

$$\langle L_-^{\frac{1}{2}} L_+ \phi, g\rangle = \langle \phi, h_1 + c\phi\rangle,$$

then

$$\langle L_-^{\frac{1}{2}} L_+ \phi, g\rangle = \langle f_1 + \lambda\phi, h_1 + c\phi\rangle.$$

Taking now $h_2 = h_1 + c\phi$, we have

$$\langle L_-^{\frac{1}{2}} L_+ f_1, g\rangle = \langle f_1, h_2\rangle$$

for all $f_1 \in H^3$. Since this now implies that $h_1 \perp \mathrm{Ker}(L_+)$, we have $h_1 = L_+(h_2 + \sum_{j=1}^d c_j\phi_j)$, $h_2 \in H^3$. Once again, choose the c_j's such that

$$\langle L_-^{\frac{1}{2}}(L_+ f_1 + \sum_{j=1}^d \lambda_j\phi_j), g\rangle = \langle L_+ f_1 + \sum_{j=1}^d \lambda_j\phi_j, h_2 + c\sum_{j=1}^d c_j\phi_j\rangle$$

for all λ_j. Finally, we have

$$\langle L_-^{\frac{1}{2}} f_2, g\rangle = \langle f_2, h_2\rangle$$

for all $f_2 \in H^1$ where $h = L^{\frac{1}{2}} L_+ h_2$, $h_2 \in H^3$. Since L_- is self-adjoint, $h_2 = L_-^{\frac{1}{2}} g$. Hence, $g \in H^4$ and $h = L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$. □

Let us discuss the generalized null space of a Hamiltonian resulting from linearizing about a minimal mass soliton. From the discussion in Section 2.1, we know that the vectors

$$\begin{bmatrix} 0 \\ R \end{bmatrix}, \begin{bmatrix} R_j \\ 0 \end{bmatrix},$$

for all $j = 1, \ldots, d$ are contained in $Ker(\mathcal{H})$. Now, as $Q(R_\lambda)$ is differentiable with respect to λ, we have by a simple calculation that $L_+ \partial_\lambda R = -R$ and $L_-(x\phi) = -2\nabla R$. Hence, the vectors

$$\begin{bmatrix} 0 \\ x_j R \end{bmatrix}, \begin{bmatrix} (\partial_\lambda R)_{\lambda_0} \\ 0 \end{bmatrix}$$

in the generalized null space of order 2. Notice that so far we have constructed at $2d + 2$ dimensional null space. Since we know the null spaces of L_- and L_+ exactly, these are unique. For the higher dimensional null spaces, we would have solutions of form

$$L_- L_+ L_- v = 0, \tag{4.8}$$

$$L_+ L_- L_+ u = 0. \tag{4.9}$$

From Equation (4.9), we would have $L_- L_+ u = a \cdot \nabla R$ or $L_+ u = Ca \cdot xR$, where $a \in \mathbb{R}^d$, $C \in \mathbb{R}$. Hence, we use an orthogonality argument by taking the inner product of (4.9) with u. Namely, we have

$$\langle u, 0 \rangle = 0,$$

but

$$\langle a \cdot \nabla R, Ca \cdot xR \rangle \neq 0.$$

For simplicity, let us say that the minimal mass occurs at 1 and say $R_1 = R$. Since we are at a minimal mass soliton, we have the chance to actually solve Equation (4.8) since

$$(\partial_\lambda Q)(\lambda_0) = 0,$$

or

$$\langle \partial_\lambda R, R \rangle = 0.$$

Hence, if we look at the equation

$$L_-\alpha = (\partial_\lambda \phi),$$

there exists a solution α. Note that we can choose α to be spherically symmetric. To move one step higher, we have the equations

$$L_+L_-L_+L_-v = 0, \tag{4.10}$$

$$L_-L_+L_-L_+u = 0. \tag{4.11}$$

Now, for Equation (4.10), we would have

$$L_-L_+L_-v = a \cdot \nabla\phi.$$

Then, we would have

$$L_+L_-v = \tilde{a} \cdot xR.$$

Hence, no such solution can exist by orthogonality. For Equation (4.11), we have

$$L_+u = \alpha.$$

Since the kernel of L_+ consists only of non-sperically symmetric functions, again there exists a β such that $L_+\beta = \alpha$. Using a comparable technique, it is easy to see that the generalized null space ends at dimension 4.

As a result, we have the following theorem

Theorem 12. *There exists a $2d+4$ dimensional null space for $\mathcal{H}$ consisting of the span of the vectors*

$$\left\{ \begin{bmatrix} 0 \\ R \end{bmatrix}, \begin{bmatrix} R_j \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x_jR \end{bmatrix}, \begin{bmatrix} (\partial_\lambda R)_{\lambda_0} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \alpha \end{bmatrix}, \begin{bmatrix} \beta \\ 0 \end{bmatrix} \right\}.$$

The generalized null space of the adjoint can be found by reversing the location of the non-zero elements in the above vectors.

Suppose that there exists a generalized eigenspace for the eigenvalue $E \neq 0$. Then, there exists $\chi \neq 0$ and $\psi \neq 0$ such that $(\mathcal{H} - E)\psi = \chi$ and $(\mathcal{H} - E)\chi = 0$. Then, note that

$$\begin{aligned}(\mathcal{H}^2 - E^2)\psi &= (A + E)\chi = 2E\chi, \\ (\mathcal{H}^2 - E^2)\chi &= 0.\end{aligned}$$

Hence, $\mathcal{H}^2$ has a generalized eigenspace at E^2. As a result, we see that $T = L_+L_-$ has a generalized eigenspace at E^2. Let $T\chi = E^2\chi$ and $(T - E^2)\psi = c\chi$, for some $c \neq 0$. Hence,

$$\begin{aligned}(L_-^{\frac{1}{2}}L_+L_-^{\frac{1}{2}} - E^2)L_-^{\frac{1}{2}}\psi_1 &= c\chi_1, \\ (L_-^{\frac{1}{2}}L_+L_-^{\frac{1}{2}} - E^2)^2L_-^{\frac{1}{2}}\psi_1 &= cL_-^{\frac{1}{2}}(L_+L_- - E^2)\chi_1 = 0,\end{aligned}$$

where given $P_R^c = I - P_R$, we have $\chi_1 = P_R^c\chi \neq 0$ since $T\chi = E^2\chi$ and $\psi_1 = P_R^c\psi \neq 0$ since $(T - E^2)\psi = c\chi$. However, this means that the self-adjoint operator $L_-^{\frac{1}{2}}L_+L_-^{\frac{1}{2}}$ has a generalized eigenvalue, which is impossible by an orthogonality argument.

Since we have assumed there are no eigenvalues at the endpoints of the continuous spectrum, there can be no accumulation and the number of discrete eigenvalues is finite. □

4.2 Large Eigenvalues

We seek to prove the absence of large L^2 eigenvalues imbedded in the continuous spectrum for Hamiltonian operators that result from linearizing a Nonlinear Schrödinger Equation about a soliton, ϕ. The operator is a non-selfadjoint matrix operator whose entries are second order differential operators. For simplicity, we look at the fourth order equation comes from squaring the operator. Therefore, we look for eigenvalues $u \in L^2$ of the differential operator:

$$L_-L_+u = \mu^4 u, \tag{4.12}$$

where

$$L_- = -\Delta + \lambda - f(x)$$

and

$$L_+ = -\Delta + \lambda - g(x).$$

where μ is in the continuous spectrum for L_-L_+, or $\mu \in (\lambda^2, \infty)$. The dispersive estimates that result from the matrix operator depend heavily on resolvent estimates which heavily rely on spectral properties. Hence, it is necessary to fully understand the spectral properties of such an operator in order to determine linear stability for solutions to the original NLS equation. From the properties of the soliton and the nonlinearity, we have that any solution to Equation 4.12 is locally smooth by using an iteration argument similar to that in Chapter 7, Theorem 18. Asymptotic analysis gives exponential decay as in Section 5.2. As a result, we have that an analysis of the possible range of frequencies limits us to smooth solutions that are bounded in frequency by μ or

$$\|\nabla u\|_{L^2} \leq \mu \|u\|_{L^2}. \tag{4.13}$$

See Appendix C.1 or (Tataru), (Stein) for references on microlocal analysis. Hence, we have the following theorem:

Theorem 13. *There exists a $\mu_0 > \lambda$ such that for all $\mu \geq \mu_0$, the eigenvalue equation* (4.12) *has only the trivial zero solution in L^2.*

Proof. Define the standard Mourre commutator to be:

$$M = x \cdot \nabla + \nabla \cdot x.$$

From the equation itself and the fact that M is skew-adjoint and L_-, L_+ are self-adjoint, we have the following two identities:

$$\begin{aligned} \langle [ML_-L_+ &- L_+L_-M]u, u\rangle = 0, \\ \langle [L_-L_+ &- \mu^4]u, u\rangle = 0, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the standard L^2 inner product.

By combining these two equations and using the frequency bound in (4.13), we see that for μ large enough, L^2 solutions are not possible. We have:

$$\begin{aligned} \langle (\Delta^2 - 2\lambda\Delta &+ \Delta(g(x)) + f(x)\Delta - \lambda^2(f+g) + (\lambda^4 - \mu^4) + fg)u, u\rangle = 0, \ (*) \\ ML_-L_+ &- L_+L_-M = [M, L_-]L_+ + L_-[M, L_+] + [L_-, L_+]M, \\ [M, -\Delta] &= 4\Delta, \ [M, f] = 2x \cdot \nabla f. \end{aligned}$$

Hence,

$$\begin{aligned}\langle(-8\Delta^2 &+ 8\lambda^2\Delta - 4(f+g)\Delta + 2(x\cdot\nabla f + x\cdot\nabla g)\Delta \\ &- 2\lambda^2(x\cdot\nabla f + x\cdot\nabla g) + 2(x\cdot\nabla f)g + 2(x\cdot\nabla g)f \ (**) \\ &- [(f-g)\Delta - \Delta(f-g)][d + 2x\cdot\nabla])u, u\rangle = 0.\end{aligned}$$

Note that the last term is a product of skew-adjoint operators and hence, they commute in the current setting. Therefore, by looking at $4(*) + (**)$ and using the fact that $\|\nabla u\|_{L^2} \leq \mu^2\|u\|_{L^2}$ from the frequency bound dictated by the operator, we have for $\mu > \mu_0$ a negative definite system and hence no eigenvalues. Note that by this estimate and standard Sobolev embeddings, we have $u \in H^k$ for any k and hence, u is smooth.

The system we deal with is:

$$\begin{aligned}\int[-4(\Delta u)^2 &- 4\lambda^2(f+g)u^2 - 4(\mu^4-\lambda^4)u^2] + [4fg - 2\lambda^2(x\cdot\nabla f + x\cdot\nabla g) \\ &+ 2(x\cdot\nabla f)g + 2(x\cdot\nabla g)f + \Delta(x\cdot\nabla f + x\cdot\nabla g) - x\cdot\nabla(\Delta f - \Delta g) \\ &- d(\Delta f - \Delta g)]u^2 + [4(x\cdot\nabla u)(\nabla(f-g)\cdot\nabla u) \\ &- 2(x\cdot\nabla f + x\cdot\nabla g)\nabla u\cdot\nabla u]\ dx = 0.\end{aligned}$$

Hence,

$$\begin{aligned}\int(-4(\Delta u)^2 &- 4\lambda^2(f+g)u^2)dx - 4(\mu^4-\lambda^4)\|u\|_{L^2}^2 + \|F\|_{L^\infty}\|u\|_{L^2}^2 \\ &+ (2\|x\cdot\nabla f + x\cdot\nabla g\|_{L^\infty} + C_d\max_{j,k}\|\partial_j(f-g)x_k\|)\|\nabla u\|_{L^2}^2 \\ &\leq \int(-4(\Delta u)^2 - 4\lambda^2(f+g)u^2)dx - 4(\mu^4 - \lambda^4 - C_1 - C_2\mu^2)\|u\|_{L^2}^2,\end{aligned}$$

where

$$\begin{aligned}F &= 4fg - 2\lambda^2(x\cdot\nabla f + x\cdot\nabla g) + 2(x\cdot\nabla f)g \\ &+ 2(x\cdot\nabla g)f + \Delta(x\cdot\nabla f + x\cdot\nabla g) \\ &- x\cdot\nabla(\Delta f - \Delta g) - d(\Delta f - \Delta g),\end{aligned}$$

and $C_j = C_j(f, g, \lambda, d)$ for $j = 1, 2$. Hence, for μ large, we have that:

$$\int(-4(\Delta u)^2 - 4\lambda^2(f+g)u^2 - C_3(\mu,\lambda,f,g)u^2)\ dx \leq 0,$$

for $C_3 > 0$ and u a smooth function, hence $u = 0$.

□

4.3 Spherical Harmonics

Since our potential functions are radially symmetric, we expand our solution into a sum of radially symmetric functions multiplied by eigenfunctions on the $(d-1)$-sphere. Note, this method only applies for dimension $d \geq 2$. Using separation of variables, we have that the spherical harmonic eigenfunctions have the form

$$\sum_k u_k(r)\phi_k(\theta),$$

where

$$\Delta_S \phi_k(\theta) = (k^2 + (d-1)d)\phi_k(\theta).$$

See Appendix B.1 for a description of the eigenspaces of the spherical Laplacian, Δ_S. Then, we have the following ODE eigenvalue problem:

$$\left[\left(-\frac{\partial^2}{\partial r} - \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{\alpha^2}{r^2} + \lambda^2 - f(r)\right) \times \right. \tag{4.14}$$

$$\left.\left(-\frac{\partial^2}{\partial r} - \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{\alpha^2}{r^2} + \lambda^2 - g(r)\right) - \mu^4\right] u(r) = 0, \tag{4.15}$$

where $\alpha^2 = k^2 + (d-1)d$ for $k = 0, 1, 2, \ldots$.

From this framework, we have following theorem:

Theorem 14. *There exists some $\alpha_0 > 0$ such that for all $\alpha \geq \alpha_0$, the eigenvalue equation* (4.14) *has only the trivial zero solution in L^2.*

Proof. So, for the radial inner product defined by:

$$\langle u, v\rangle_r = \int uvr^{d-1}dr,$$

we define the operators:

$$\begin{aligned} \Delta_r &= r^{1-d}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\right), \\ P_r &= d + 2r\frac{\partial}{\partial r}. \end{aligned}$$

Using the same commutator approach from Section 4.2, we have:

$$\begin{aligned}\langle[4\Delta_r &- \frac{4\alpha^2}{r^2} - 2rf'(r)][-\Delta_r + \frac{\alpha^2}{r^2} + \lambda^2 - g(r)] \\ &+ [-\Delta_r + \frac{\alpha^2}{r^2} + \lambda^2 - f(r)][4\Delta_r - \frac{4\alpha^2}{r^2} - 2rg'(r)] \\ &+ [(f-g)\Delta_r - \Delta_r(f-g)][d + 2r\frac{\partial}{\partial r}]u, u\rangle_r = 0.\end{aligned}$$

Thus, we have

$$\begin{aligned}\big\langle -8(\Delta_r)^2 u &+ 16\Delta_r(\frac{\alpha^2}{r^2}u) + 8\lambda^2\Delta_r u - 4(f+g)\Delta_r u - \frac{8\alpha^4}{r^4}u - \frac{8\lambda^2\alpha^2}{r^2}u \\ &+ \frac{4\alpha^2}{r^2}(f+g)u + 2(rf'(r) + rg'(r))\Delta_r u - 2\frac{\alpha^2}{r}(f'(r) + g'(r))u \\ &- 2\lambda^2 r(f'(r) + g'(r))u + 2rf'(r)g(r)u + 2rf(r)g'(r)u \\ &+ (d + 2r\frac{\partial}{\partial r})((f-g)\Delta_r - \Delta_r(f-g))u, u\rangle_r = 0.\end{aligned}$$

This implies:

$$\begin{aligned}\int [-8(\Delta_r u)^2 &- (16\frac{\alpha^2}{r^2} + 8\lambda^2)(u_r)^2 - (\frac{8\alpha^4}{r^4} + \frac{8\lambda^2\alpha^2}{r^2})]u^2 r^{d-1}dr \\ &+ \int [4(f+g) + 2rf_r - 6rg_r](u_r)^2 r^{d-1}dr \\ &+ \int [\frac{4\alpha^2}{r^2}(f+g) - (d-4)\frac{\alpha^2}{r^4} - \frac{d(d-1) + 2\alpha^2}{r}f_r \\ &+ \frac{(d-1)(d-2) - 2\alpha^2}{r}(g_r) - (d-2)f_{rr} + 3dg_{rr} + 2rg_{rrr} \\ &+ 2r(g_r f + fg_r) - 2\lambda^2 r(f_r + g_r)]u^2 r^{d-1}dr \leq 0.\end{aligned}$$

Note that we have integrated by parts several times above. To justify this, u must be 0 at $r = 0$ to sufficiently high order to compensate for the singular terms. However, spherical harmonics result from eigenvalues of the spherical Laplacian. These take the values

$$\nu_k = k^2 + (d-2)k,$$

and for each k, the eigenfunctions (and hence the spherical harmonics) are traces of harmonic polynomials of degree k. Hence, in order to give a smooth solution as guaranteed above, u_k must be 0 of order k at the origin. Hence, for $k \geq \max 0, 5 - d$, the behavior of u is sufficient to allow the above calculations to be rigorous. Since all of the eigenvalues are non-negative, we have used the notation α^2 for simplicity and will continue to do so. For a brief note,

see Appendix B.1 for a discussion of spherical harmonics. See (Taylor2), Chapter 8 for a detailed description of eigenfunctions for the Laplacian on the sphere.

Remark 4.1. *Note that in the Section 1, we had to take advantage both of the frequency bounds on the solution as well as the operator equation itself in order to take advantage of the large negative parameter. However, due to the nature of the operator, the large negative parameter appeared in the commutator and all other terms could be viewed as lower order. Regularity is not an issue in the above analysis by an argument involving resolvents which is a variant of that in Chapter 6.*

In the commutator expression, the parameter that must dominate is α^4. Since f, g are smooth, exponentially decaying functions by assumption, all functions involving f, g and derivatives thereof are nicely bounded at 0 and exponentially decaying. Hence, for $0 \leq r \leq 1$, all of the functions above are easily controlled by $\frac{\alpha^4}{r^4}$ for α large. Similarly, for $r > R$, R large, the exponential decay of f, g and their derivatives imply that any function above is dominated again on this region by $\frac{\alpha^4}{r^4}$ for α large. In the intermediate region, again using the smoothness of the potential functions, we can find α^4 large enough to bound the lower order terms. In order to determine α exactly, a careful analysis must be done involving all of the potential functions, but there is certainly an α_0 such that for all $\alpha \geq \alpha_0$, the operator when conjugated by the radial Mourre operator gives a negative definite system. □

Remark 4.2. *Ideally, the potentials will be non-trivial only on a very small region, then the R and hence the α can be taken fairly small in all regions. There will definitely balancing that must be done in each special case. Hence, any embedded eigenvalue can be expressed purely as a finite sum of spherical harmonics with radial coefficients. Combining this with the result in Section 4.2 gives a limited number of calculations to do in order to determine whether or not an operator has embedded eigenvalues.*

4.4 Absence of Embedded Resonances

This result is developed in the earlier work of Erdogan-Schlag (ES1) and Agmon (Agmon). We present the argument here for the sake of completeness. Let

$$H_0 = -\Delta + \lambda,$$

$$\mathcal{H}_0 = \begin{bmatrix} H_0 & 0 \\ 0 & -H_0 \end{bmatrix},$$

and

$$V = \begin{bmatrix} -V_1 & -V_2 \\ V_2 & V_1 \end{bmatrix}.$$

Also, define the space

$$X_\sigma = L^{2,\sigma} \times L^{2,\sigma},$$

where

$$L^{2,\sigma} = \{f || x|^\sigma f \in L^2\}.$$

Theorem 15. *Let V_1, V_2 have sufficient decay at ∞. Then for any μ such that $|\mu| > \lambda^2$, $(\mathcal{H}_0 - (\mu \pm i0))^{-1} V : X_{-\frac{1}{2}-} \to X_{-\frac{1}{2}-}$ is a compact operator, and*

$$I + (\mathcal{H}_0 - (\mu \pm i0))^{-1} V$$

is invertible on these spaces.

Proof. The compactness follows from an argument similar to the one below for the square of $\mathcal{H}$. We refer the reader to this proof for the main idea. From Fredholm theory, we see that we must exclude solutions $\Psi = (\psi_1, \psi_2) \in X_{-\frac{1}{2}-}$ such that

$$[I + (\mathcal{H}_0 - (\mu \pm i0))^{-1} V]\Psi = 0,$$

of from standard results about the spectral theory of Schrödinger equations

$$0 = \psi_1 - R_0(\mu - \lambda + i0)(V_1\psi_1 + V_2\psi_2) \tag{4.16}$$

$$0 = \psi_2 - R_0(-\mu - \lambda)(V_2\psi_1 + V_1\psi_2), \tag{4.17}$$

where $R_0(z) = (-\Delta - z)^{-1}$ is the standard resolvent from the linear Schrödinger operator. Using the decay of V_j, $j = 1, 2$ and the fact that$R_0(-\mu - \lambda) : L^2 \to L^2$, we see immediately that $\psi_2 \in L^2$.

By taking the inner products of (4.16) with

$$V_1\psi_1 + V_2\psi_2,\ V_2\psi_1,\ \text{and } V_1\psi_2,$$

then combining necessary terms, we see

$$\mathrm{Im}\langle R_0(\mu - \lambda + i0)V_1\psi_1 + V_2\psi_2, V_1\psi_1 + V_2\psi_2\rangle = 0.$$

Lemma 4.4.1 (Agmon)**.** *Let $\lambda > 0$ and f sufficiently regular. Then,*

$$Im\langle R_0(\lambda \pm i0)f, f\rangle = \pm\frac{\pi}{2\sqrt{\lambda}}\int_{|\xi|=\sqrt{\lambda}} |(\tau\hat{f})(\xi)|^2 d\sigma,$$

where $\tau\hat{f}$ denotes the trace of $\hat{f}$ on the sphere $|\xi| = \sqrt{\lambda}$.

This is similar to a restriction theorem from harmonic analysis. The proof follows from a calculation using the the specific structure of R_0. This allows us to use the following bootstrapping theorem, also proved in Agmon (Agmon).

Theorem 16 (Agmon)**.** *Let $f \in H^s(\mathbb{R}^d)$ for some $s > \frac{1}{2}$. Suppose that $f(x) = 0$ for all x such that $|x| = k$ in the trace sense, and let $K^{-1} \le k \le K$, $K > 1$. For any multi-index α with $0 \le |\alpha| \le 2$, set*

$$v_\alpha(x) = \frac{x^\alpha f(x)}{|x|^2 - k^2}.$$

Then, $v_\alpha \in H^s \cap L^1_{loc}$ and

$$\|v_\alpha\|_{H^{s-1}} \le C\|f\|_{H^s}.$$

From the Agmon bootstrapping theorem we have $\hat{\psi}_1 \in L^2$, hence $\psi_1 \in L^2$. Thus, any imbedded resonance is actually an embedded eigenvalue, which we have ruled out by assumption.

□

Chapter 5

Spectral Numerical Methods and Results

5.1 Discrete Spectrum

In (Schlag1), it is shown that the discrete spectrum for supercritical exponents is determined by the discrete spectrum of $L_{\pm}$. We present here a version for linearizations about a minimal mass soliton, $R = R_{min}$, in saturated nonlinearities.

Theorem 17 (Schlag)**.** *Assume that L_- has no discrete eigenvalues on the interval $(0, \lambda]$ and $\mathcal{H}$ is an admissible Hamiltonian resulting from linearizing about a minimal mass soliton. Then, the only discrete eigenvalue for $\mathcal{H}$ in the interval $[-\lambda, \lambda]$ is 0.*

Proof. We argue by contradiction. To this end, assume $\mathcal{H}$ has an eigenvalue away from 0, say at E. Let $\lambda = 1$ for simplicity. Then $\mathcal{H}^2$ has an eigenvalue at some value $E^2 \in (0, 1]$. Hence, we have

$$L_- L_+ u_E = E^2 u_E,$$

for $E^2 < 1$. Since L_- is self-adjoint, we see that $u_E \perp \phi$. By elliptic regularity, we have that $u_E \in H^4_{loc}$. Let P be the projection orthogonal to ϕ. Let $A = PL_+P$. Using that

$$\mathrm{Ker}(L_+) = \mathrm{span}\{\partial_j \phi | 1 \leq j \leq d\},$$

and

$$\langle R, \partial_\lambda R \rangle = 0,$$

we have

$$\mathrm{Ker}(L_+) = \mathrm{span}\{\partial_\lambda R,\ \partial_j R | 1 \leq j \leq d\}.$$

Take E_0 to be the unique negative eigenvalue for L_+. Then, define

$$g(\alpha) = \langle (L_+ - \alpha)^{-1} R, R \rangle,$$

which is well-defined and differentiable on $(E_0, 1)$ since ϕ is orthogonal to the kernel of L_+. We have

$$g'(\alpha) = \langle (L_+ - \alpha)^{-1} R, R \rangle > 0,\ g(0) = \frac{1}{2} \langle R, \partial_\lambda R \rangle = 0.$$

Hence, $g(0) = 0$ is the only 0 for g in the interval $(E_0, 1)$ since $\lim_{\alpha \to E_0} g(\alpha) \to -\infty$.

Conversely, if $Af = \alpha f$ for some $-\infty < \alpha < 1$, $\alpha \neq 0$ and $f \in L^2$, then $f \perp R$ and

$$(PL_+P - \lambda) f = (A - \lambda) f = 0.$$

Since

$$E_0 \langle f, f \rangle \leq \langle L_+ f, f \rangle = \lambda \langle f, f \rangle,$$

we know that $\lambda \geq E_0$. If $\lambda = E_0$, then f is a ground state of L_+ and hence not orthogonal to R. However, $g(\lambda) = 0$, hence $\lambda = 0$. So, A has a collection of eigenvalues at 0. Define

$$\mathcal{G} = \mathrm{span}\{R, R_j, R_\lambda, u_E\}.$$

We would like to show that $\dim(\mathcal{G}) = d + 3$. Since ϕ is orthogonal to all the other functions, we need only show that the equation

$$c_1 u_E + c_2 R_\lambda + \sum_{j=1}^{d} c_{j+2} R_j = 0 \tag{5.1}$$

has only the trivial solution $c_j = 0$ for all j. By applying L_+ to (5.1), we see

$$c_1 L_+ u_E + c_2 R = 0.$$

By taking the inner product with u_E, we have that $c_1 = 0$. This then implies that $c_2 = 0$. As a result, $c_j = 0$ for $j = 2, \ldots, d+2$. Now, if we can show that

$$\sup_{\|f\|_{L^2}, f \in \mathcal{G}} \langle Af, f \rangle < 1, \tag{5.2}$$

then by the Courant minimax principle, there would be at least $d+3$ eigenvalues less than 1 for A. However, we have shown there are exactly $d+2$ of them. Note that neither the minimal eigenfunction for L_+ nor ϕ itself are eigenvalues of A due to orthogonality arguments. Hence, if we can prove (5.2), we have proved the result.

Since $\langle PL_-^{-1}Pf, f \rangle < \langle f, f \rangle$ given $f \neq 0$ by our assumption on the spectrum of L_-. Since $E \leq 1$ by assumption, we can actually prove the stronger result that

$$\langle Af, f \rangle \leq E^2 \langle PL_-^{-1}Pf, f \rangle$$

for all $f = au_E + b\phi + \vec{c} \cdot \nabla R + dR_\lambda$. To this end, we have

$$\begin{aligned}
\langle Af, f \rangle &= \langle L_+(au_E), au_E + \vec{c} \cdot \nabla \phi + d\phi_\lambda \rangle \\
&+ \langle L_+ \vec{c} \cdot \nabla R, au_E + \vec{c} \cdot \nabla R + dR_\lambda \rangle \\
&= E^2 \langle L_-^{-1}(au_E), au_E + \vec{c} \cdot \nabla R + dR_\lambda \rangle \\
&+ E^2 \langle dR_\lambda, L_-^{-1}(au_E) \rangle \\
&\leq E^2 \langle L_-^{-1}(au_E), au_E + \vec{c} \cdot \nabla R + dR_\lambda \rangle + E^2 \langle dR_\lambda, L_-^{-1}(au_E) \rangle \\
&+ E^2 \langle L_-^{-1}(\vec{c} \cdot \nabla R + dR_\lambda), (\vec{c} \cdot \nabla R + dR_\lambda) \rangle \\
&\leq E^2 \langle PL_-^{-1}Pf, f \rangle,
\end{aligned}$$

since $L_-^{-1}u_E \perp \nabla R$ and L_- is positive definite on $\mathcal{G} \setminus R$. □

In order to test the discrete spectrum of our linearized equation, we first review the Birman-Schwinger method. In particular, we present the formulation given in (DS). Let $H = L_- - \lambda^2 = -\Delta - V$ for $V > 0$. Since we are looking for small, positive eigenvalues of L_-, so take $Hf = -\alpha^2 f$ for $0 < \alpha < \lambda$ so we have $L_- f = (\lambda^2 - \alpha^2) f$. Set $U = \sqrt{V}$ and $g = Uf$, then

$$g = U(-\Delta + \alpha^2)^{-1} U g,$$

or $g \in L^2$ is an eigenfunction for

$$K(\alpha) = U(-\Delta + \alpha^2)^{-1} U g$$

with eigenvalue 1, where $K > 0$ and K compact. Conversely, if $g \in L^2$ satisfies $K(\alpha)g = g$, then

$$f = U^{-1} g = (-\Delta + \alpha^2)^{-1} U g \in L^2$$

and $Hf = -\alpha^2 f$. The eigenvalues of $K(\alpha)$ are seen to be strictly increasing as $\alpha \to 0$ since

$$K'(\alpha) = -2\lambda U(-\Delta + \alpha^2)^{-2} U.$$

This implies that

$$\#\left\{\alpha : \mathrm{Ker}(H - \alpha^2) \neq \{0\}\right\} = \#\left\{E > 1 : \mathrm{Ker}(K(0) - E) \neq \{0\}\right\}$$

counted with multiplicity.

Finally, use the symmetric resolvent identity to see

$$(H - z)^{-1} = (-\Delta - z)^{-1} + (-\Delta - z)^{-1} U \left[I - U(-\Delta - z)^{-1} U\right]^{-1} U(-\Delta - z)^{-1}.$$

Hence, the Laurent expansion about $z = 0$ does not require negative powers for z iff $I + U(-\Delta - z)^{-1} U$ is invertible at $z = 0$, i.e.

$$\mathrm{Ker}(I - U(-\Delta)^{-1} U) = \{0\}$$

by the Fredholm alternative since V has exponential decay. Hence, if H has no resonance or eigenvalue at 0, then $K(0)$ will not have an eigenvalue at 1. If we then count the eigenvalues α_j in decreasing order for $K(0)$, then H has exactly N negative eigenvalues and neither an eigenvalue or resonance at 0 iff $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_N > 1$ and $\alpha_{N+1} < 1$. Hence, we can study numerically study K for a soliton of the saturated nonlinear Schrödinger equation. To do so, we must accurately find a soliton, then use it as potential in the truncation and discretization scheme presented in (DS). As a result of this study, we have numerically verified that for standard nonlinearities of types 1 and 2, the gap condition holds for Hamiltonians resulting from linearizing about the minimal mass soliton. Using these results, we also see numerical

verification of the fact that the kernel of L_+ is 3 dimensional. For future research, the author will explore where along the soliton curve the gap condition breaks down. It is the hope of the author that this will somehow correlate to point along the soliton curve where the energy $H_\lambda = 0$ or to another interesting transition region.

5.2 Asymptotic Analysis

We seek an asymptotic expansion for R, using Equation (1.4)

$$(-\Delta + \lambda^2 - f(R))R = 0.$$

As $r \to \infty$, we look for an asymptotic expansion

$$e^{-\lambda r} r^\beta \sum_{n=0}^{\infty} \alpha_n r^{-n}. \tag{5.3}$$

To find simply the leading order behavior, we wish to find β. To this end, we have

$$\begin{aligned}
\frac{\partial}{\partial_r}(e^{-\lambda r} r^\beta) &= -\lambda e^{-\lambda r} r^\beta + \beta e^{-\lambda r} r^{\beta-1}, \\
\frac{\partial^2}{\partial^2 r}(e^{-\lambda r} r^\beta) &= \lambda^2 e^{-\lambda r} r^\beta - 2\lambda\beta e^{-\lambda r} r^{\beta-1} + \beta(\beta-1) e^{-\lambda r} r^{\beta-2}.
\end{aligned}$$

Thus, plugging (5.3) in to (1.4), we see

$$[(-\lambda^2 + \lambda^2) + \frac{(2\lambda\beta + \lambda(n-1))}{r} + l.o.t.] = 0.$$

Hence, $\beta = -(d-1)/2$ and the leading order behavior is

$$r^{-\frac{n-1}{2}} e^{-\lambda r}. \tag{5.4}$$

In terms of the linearized equation, we once again look at the square of the operator to get

$$(-\Delta + \lambda^2 - f(R^2))(-\Delta + \lambda^2 - f(R^2) - 2f'(R^2)R).$$

Then, since we desire a non-spherically symmetric solution eigenvalue embedded in the continuous spectrum, we have an ODE of the form

$$\begin{aligned}
\mu^4 u &= (-\frac{\partial^2}{\partial^2 r} - \frac{(n-1)}{r}\frac{\partial}{\partial r} + \lambda^2 + \frac{\alpha^2}{r^2} - f(R^2)) \\
&\times (-\frac{\partial^2}{\partial^2 r} - \frac{(n-1)}{r}\frac{\partial}{\partial r} + \lambda^2 + \frac{\alpha^2}{r^2} - f(R^2) - 2f'(R^2)R)u.
\end{aligned}$$

Using similar techniques to those for (5.4), we see the leading order behavior to be

$$u \approx e^{-\sqrt{\mu^2+\lambda^2}r^{-\frac{n-1}{2}}}(1+\frac{2\alpha^2}{\mu^2}r^{-2}),$$

as $r \to \infty$.

5.3 Small eigenvalues in the continuous spectrum

We desire to show that via numerical methods, one can rule out the possibility of embedded eigenvalues. Note that from Section 5.2, we have the required asymptotics as $r \to \infty$ and since by the commutator argument, there are only a finite number of spherical harmonics allowed, one can use a shooting method similar to that in finding the soliton to calculate numerical approximations to the radial eigenvalue problem

$$\partial_r A(r) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \beta_0(r) & \beta_1(r) & \beta_2(r) & \beta_3(r) \end{bmatrix} A(r),$$

where

$$A(r) = \begin{bmatrix} u(r) \\ u'(r) \\ u''(r) \\ u'''(r) \end{bmatrix},$$

and

$$\begin{aligned}
\beta_0(r) &= -\left[\mu^2(\beta(R^2)+2\beta'(R^2)R^2)+\left[\frac{d^2}{dr^2}(\beta(R^2)+2\beta'(R^2)R^2)\right]\right.\\
&+ 2\beta(R^2)(\beta(R^2)+2\beta'(R^2)R^2)-2\left[\frac{d}{dr}(\beta(R^2)+2\beta'(R^2)R^2)\right](\lambda+\mu^2)^{\frac{1}{2}}\\
&- \left(\frac{(\lambda+\mu^2)}{2}+\frac{(2\beta(R^2)+2\beta'(R^2)R^2)}{4}\right)\frac{(d-1)(d-3)+4\alpha^2}{r^2}\\
&- \left.\frac{(\lambda+\mu^2)((d-1)(d-3)+4\alpha^2)}{r^3}+\frac{(d^2-9)(d-7)}{16r^4}\right]\\
\beta_1(r) &= -\left[2\left[\frac{d}{dr}(\beta(R^2)+2\beta'(R^2)R^2)\right]+4\lambda(\lambda+\mu^2)^{\frac{1}{2}}\right.\\
&- 2(2\beta(R^2)+2\beta'(R^2)R^2)(\lambda+\mu^2)^{\frac{1}{2}}-4(\lambda+\mu^2)^{\frac{3}{2}}\\
&+ \left.\frac{(\lambda+\mu^2)^{\frac{1}{2}}((d-1)(d-3)+4\alpha^2)}{r^2}+\frac{((d-1)(d-3)+4\alpha^2)}{r^3}\right],\\
\beta_2(r) &= -\left[2\beta(R^2)+2\beta'(R^2)R^2+2\mu^2+4\sqrt{\lambda+\mu^2}-\frac{2\alpha^2+\frac{(d-1)(d-3)}{2}}{r^2}\right],\\
\beta_3(r) &= 4\sqrt{\lambda+\mu^2},
\end{aligned}$$

where $\alpha = \sqrt{k^2+(d-2)k}$, $k = 0,1,2,\ldots$. The above formula is attained by taking $P = L_-L_+$ and linearizing the fourth order ODE into a first order matrix ODE.

In shooting, we then analyze the growth near the origin. For there to be a possible eigenvalue, we must see that the growth is sufficiently small as to be bounded in L^2_r. Then, we can take a discretize the allowed μ interval (λ^2, μ_0^2) and test each grid point. As the grid is refined, a continuity argument can be used to show that the behavior will be similar in a neighborhood of that endpoint. The drawback to this method is that it is incredibly computationally intensive.

A slightly more robust method would look for the spectrum in the range of allowed μ for the fourth order operator that depends on α. Then, for the discrete, finite set of α's, one discretization would work. Most eigenfunction analysis is done using spectral methods, however, the continuous spectrum should be precisely where these break down heuristically. A recent approach suggested by Michael Weinstein looks for eigenvalues as zeros of an Evans Function that is formed from the operator and the desired boundary conditions on the half-line. The author hopes to explore this approach in forthcoming works.

Chapter 6

Linear Theory

6.1 General Distorted Fourier Basis Theory

We present here the combined results of (Agmon) and (Ho2), Chapter 14.

Both presentations are valid for operators of the form

$$(P(D) + V(x, D))u = 0,$$

where $P(D)$ is a self-adjoint, constant coefficient differential operator and $V(x, D)$ is a *short range*, symmetric differential operator. The perturbation $V(x, D)$ is defined to be *short range* in order to say that

$$\lim_{z \to \lambda, \pm \operatorname{Im} z > 0} R(z) = R^{\pm}(z)$$

exists in the uniform operator topology of $B(L^{2,s}, \mathcal{H}_{2,-s})$, where

$$L^{2,s}(\mathbb{R}^d) = \{u(x) | (1 + |x|^2)^{\frac{s}{2}} u(x) \in L^2\}$$

and

$$\mathcal{H}_{m,s} = \{u(x) | D^{\alpha} u \in L^{2,s},\ 0 \leq |\alpha| \leq m\}.$$

Also, for any $f \in L^{2,s}$,

$$R^{\pm}(\lambda) f = R_0^{\pm}(\lambda) f - R_0^{\pm}(\lambda) V R^{\pm}(\lambda) f,$$

where R_0 is the resolvent for the constant coefficient operator. As the notion of short range deals with compactness of the operator $Z(u) = R(Vu)$, being short range requires sufficient decay assumptions at ∞. Heuristically, it is required that the coefficients of V decrease as fast as an integrable function in $|x|$ and for each fixed x_0, we have

$$\frac{V(x_0, \xi)}{P_0(x, \xi)} \to 0 \text{ as } \xi \to \infty.$$

The reasons why these heuristics hold true are explored in this chapter, hence we forego this analysis here and move on with the fact that $V(x, D)$ is a short range perturbation as an assumption. Note that in the case explored below, V is Schwartz in x and is dominated by $P(\xi)$ as $|\xi| \to \infty$. It is also important to note that while our contour integration works out nicely in $3d$, the results presented here hold in any dimension where R_0^+ and R_0^- are arrived at through a limiting procedure.

The Agmon approach to the distorted Fourier transform is equivalent to the approach taken by the author. Namely, define

$$\phi_\pm(x, \xi) = e^{ix\xi} - R^\mp(|\xi|)[Ve^{ix\cdot\xi}](x).$$

Then, we define the distorted Fourier transform to be $\mathcal{F}_\pm : L^2 \to L^2$ such that
(i) $Ker(\mathcal{F}_\pm) = L^2_d$. The restriction of $\mathcal{F}_\pm$ is a unitary operator from L^2_c onto L^2,
(ii) for any $f \in L^2$

$$(\mathcal{F}_\pm f)(\xi) = (2\pi)^{-\frac{d}{2}} \lim_{N\to\infty} \int_{|x|<N} f(x)\overline{\phi_\pm(x, \xi)}dx \text{ in } L^2_\xi,$$

and

$$(\mathcal{F}^*{}_\pm f)(x) = (2\pi)^{-\frac{d}{2}} \lim_{j\to\infty} \int_{K_j} f(\xi)\phi_\pm(x, \xi)d\xi \text{ in } L^2_x$$

where K_j is an increasing sequence of compact sets such that $\cup_j K_j = \mathbb{R}^d \setminus \mathcal{N}$ for

$$\mathcal{N}(H) = \{\xi \in \mathbb{R}^d | |\xi|^2 \text{ is an eigenvalue for } H\} \cup 0,$$

and
(iii) If P_c is the projection of L^2 onto L^2_c, then

$$(P_c H)f = (\mathcal{F}^*_\pm M_{P(\xi)} \mathcal{F}_\pm)f$$

for any $f \in D(H)$ where $M_{P(\xi)}$ denotes multiplication by $P(\xi)$.

In addition to all of this, we have $\|P_c f\|_{L^2} = \|\mathcal{F}_\pm f\|_{L^2}$. In other words, we have a Plancherel theorem for our distorted Fourier basis.

Now, (Ho2), Chapter 14 arrives at the same conclusions using

$$(\mathcal{F}_\pm f)(\xi) = \mathcal{F}(I + V R_0^\pm)^{-1} f(\xi).$$

However, using the resolvent identity

$$R(z) = R_0(z)(I + V R_0(z))^{-1},$$

we see that a formal iteration similar to that in Chapter 7 shows equivalence between these definitions for ξ large. It is precisely this iteration we use to get uniform bounds in ξ.

6.2 Distorted Fourier Basis

Note that in the sequel, we take the convention that the soliton parameter is λ^2 instead of λ. This serves to remind the reader of the positivity of this parameter. The convention of λ slightly simplifies the variational formulation, but has no impact on the linear analysis presented in this Chapter.

We seek to understand the functions in the continuous spectrum of $\mathcal{H}$ by decomposing them using a Distorted Fourier Basis given by

$$(-\Delta + \lambda^2 - V_1)(-\Delta + \lambda^2 - V_2)u_{\xi_0} = (\lambda^2 + |\xi_0|^2)^2 u_{\xi_0}, \tag{6.1}$$

where $u_{\xi_0} = e^{ix\xi_0} + g_{\xi_0}$ and g_{ξ_0} is yet to be determined.

From (6.1),

$$[(-\Delta + \lambda^2)^2 - (\lambda^2 + |\xi_0|^2)^2]u_{\xi_0} = (-\Delta + \lambda^2)V_2 u_{\xi_0} + V_1(-\Delta + \lambda^2 - V_2)u_{\xi_0}.$$

Hence,

$$\begin{aligned}[(-\Delta + \lambda^2)^2 - (\lambda^2 + |\xi_0|^2)^2]g_{\xi_0} &= F(x;\xi_0) + (-\Delta + \lambda^2)V_2 g_{\xi_0} + V_1(-\Delta + \lambda^2 - V_2)g_{\xi_0} \quad (6.2)\\ &= F_{\xi_0}(x)e^{ix\xi_0} + \tilde{V}(x,D)g_{\xi_0}, \quad (6.3)\end{aligned}$$

where $F_{\xi_0}(x)$ is a Schwartz function. Then, taking the Fourier Transform, we have

$$[(|\xi|^2+\lambda^2)^2-(|\xi_0|^2+\lambda^2)^2]\hat{g}_{\xi_0}=\hat{F}(\xi;\xi_0)+(\tilde{V}_{\mathcal{F}}\hat{g}_{\xi_0})(\xi),$$

where

$$\begin{aligned}(\tilde{V}_{\mathcal{F}}g)(\xi) &= \lambda^2(\hat{V}_2+\hat{V}_1)*(g)+(|\xi|^2\hat{V}_2)*(g)+(\hat{V}_2+\hat{V}_1)*(|\xi|^2g)\\ &+ (\xi\hat{V}_2)*(\xi g)-(\widehat{V_1V_2})*(g).\end{aligned}$$

Given

$$\begin{aligned}L_{\xi_0} &= [(|\xi|^2+\lambda^2)^2-(|\xi_0|^2+\lambda^2)^2]\\ &= [(|\xi|+|\xi_0|)(|\xi|-|\xi_0|)(|\xi|^2+2\lambda^2+|\xi_0|^2)],\end{aligned}$$

we have

$$\begin{aligned}g_{\xi_0} &= \mathcal{F}^{-1}\left\{\{L^{\pm}_{\xi_0}\}^{-1}(\hat{F}+\tilde{V}_{\mathcal{F}}\hat{g}_{\xi_0})\right\}\\ &= K^{\pm}_{\xi_0}*F_{\xi_0}+K^{\pm}_{\xi_0}*(\tilde{V}(x,D)g_{\xi_0}),\end{aligned}$$

where

$$K^{\pm}_{\xi_0}(x)=(\mathcal{F}^{-1}\{L^{\pm}_{\xi_0}\}^{-1})(x)$$

and

$$L^{\pm}_{\xi_0}=[(|\xi|+|\xi_0|\pm i0)(|\xi|-|\xi_0|\mp i0)(|\xi|^2+2\lambda^2+|\xi_0|^2)].$$

Note that for simplicity we have omitted a small complex perturbation in the elliptic term $(|\xi|^2+2\lambda^2+|\xi_0|^2)$ since it does not effect the analysis.

To explore $K^{\pm}_{\xi_0}$ further, we see in $\mathbb{R}^3$

$$\int_{\xi}\frac{e^{i\xi\cdot x}}{(|\xi|+|\xi_0|\pm i0)(|\xi|-|\xi_0|\mp i0)(|\xi|^2+2\lambda^2+|\xi_0|^2)}d\xi=$$
$$\int_{\mathbb{R}^3}\frac{e^{i\xi_1|x|}}{(|\xi|+|\xi_0|\pm i0)(|\xi|-|\xi_0|\mp i0)(|\xi|+|\xi_0|)(|\xi|^2+2\lambda^2+|\xi_0|^2)}d\xi,$$

using the change of variables $\xi_1\rightarrow\frac{x}{|x|}$. Then, we have

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{e^{r\cos(\theta)|x|}}{(r+|\xi_0|\pm i0)(r-|\xi_0|\mp i0)(r^2+2\lambda^2+|\xi_0|^2)}r\sin(\theta)drd\theta d\phi.$$

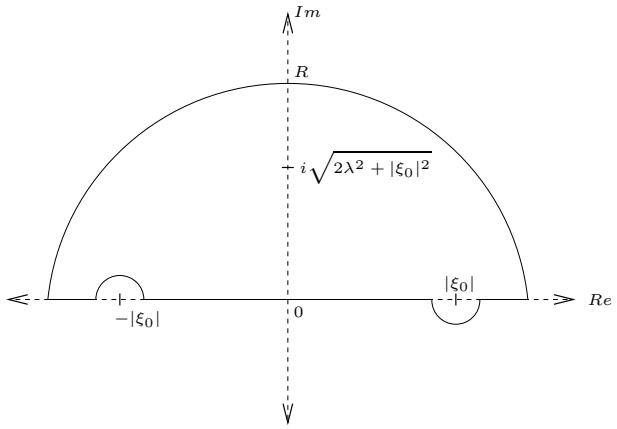

Figure 6.1. The contour for computing the behavior of the fundamental solution in the limiting case.

Doing integration first in θ, then a contour integral, we have as in B.7 that

$$K^{\pm}_{\xi_0} = \hat{L^{-1}_{\xi_0}} = \frac{\pi^2}{|\xi_0|^2+\lambda^2}\left[\frac{e^{\pm i|x||\xi_0|} - e^{-|x|\sqrt{|\xi_0|^2+2\lambda^2}}}{|x|}\right]. \tag{6.4}$$

For simplicity, we take $K(x) = K^{+}_{\xi_0}(x)$ as the analysis for $K^{-}_{\xi_0}$ will be similar. Then, we want to use an iterative argument to show that for mid to high range frequencies, these distorted Fourier bases exist in L^4. It will become clear in the sequel why L^4 is chosen. Note that since near 0, K is bounded, we have $K \in L^{3+s}$ for any $s > 0$. In particular we show the following:

Lemma 6.2.1. *For the operator $K^{\pm}$ defined in Equation* (6.4)*, we have*

$$K^{\pm} : L^{\frac{4}{3}} \to L^4 \ (O((|\xi_0|^2+\lambda^2)^{-1}|\xi_0|^{-\frac{1}{2}})).$$

Proof. We actually prove the result for

$$\tilde{K^a_{\xi_0}}(x) \quad = \quad \mathcal{F}^{-1}(\frac{1}{(|\xi|^2-(|\xi_0|+i0)^2)^a}).$$

The proof for K will be essentially the same.

Using distribution theory and Appendix B.7, we have

$$\begin{aligned}
\tilde{K^0_{\xi_0}}(x) &= \delta(x), \\
\tilde{K^1_{\xi_0}}(x) &= \frac{4\pi^2}{|x|}[e^{i|x||\xi_0|}], \\
\tilde{K^2_{\xi_0}}(x) &= \frac{i2\pi^2 e^{i|x||\xi_0|}}{|\xi_0|}.
\end{aligned}$$

As convolution operators,

$$\tilde{K}^0 : L^2 \to L^2 \ (O(1))$$

and

$$\tilde{K}^2 : L^1 \to L^\infty \ (O(|\xi_0|^{-1})),$$

hence using complex interpolation (see Appendix B.2)

$$\tilde{K}^1 : L^{\frac{4}{3}} \to L^4 \ (O(|\xi_0|^{-\frac{1}{2}})).$$

□

For simplicity, we from now on write $\tilde{K}$ instead of $\tilde{K}^1_{\xi_0}$. Now, we seek to analyze the equation

$$g_{\xi_0} = K^\pm_{\xi_0} * F_{\xi_0} + K^\pm_{\xi_0} * (\tilde{V}(x, D) g_{\xi_0}), \tag{6.5}$$

In particular, we have the following:

Theorem 18. *Let $P(x, D)$ be a differential operator of the form*

$$P(x, D) = (-\Delta + \lambda^2 - V_1)(-\Delta + \lambda^2 - V_2),$$

where V_1, $V_2 \in \mathcal{S}$. Assuming that there are no eigenvalues embedded in the continuous spectrum $[\lambda^4, \infty)$, there exists $g^\pm_{\xi_0} \in L^4$ such that Equation (6.5) is satisfied for $u_{\xi_0} = e^{ix\xi_0} + g^\pm_{\xi_0}(x)$. We have

$$g^\pm_{\xi_0}(x) = K^\pm * [f_0(\cdot, \xi_0, |\xi_0|)],$$

where f_0 is smooth in x, ξ_0 and $|\xi_0|$,

$$|\langle x \rangle^N \partial^\alpha_x f_0| \lesssim 1.$$

Moreover, there exists a value M such that for $\xi_0 \geq M$,

$$f_0(x, \xi_0) = e^{i(x,\xi_0)} f(x, \xi_0),$$

where

$$|\langle x\rangle^N \partial_{\xi_0}^{\alpha}\partial_x^{\beta} f(x,\xi_0)| \lesssim |\xi_0|^{2-|\alpha|}, \tag{6.6}$$

for any multi-indices α and β, $N > 0$.

Proof. The solution to (6.5) will be solved differently for large and small values of ξ_0. In particular, we use a Fredholm theory approach for the small frequencies and an iterative approach for the large frequencies.

To begin, let us take $|\xi_0| > M$, where M will be determined in the exposition. Then, we solve Equation (6.5) using Picard iteration. For simplicity, let $g_{\xi_0} = v$. Setting $v^0 = 0$ and $Tu = \tilde{V}(K * u)$, we have

$$\begin{aligned}
v^1 &= K(x) * [F_{\xi_0}(x)e^{ix\cdot\xi_0}] \\
v^2 &= K(x) * [(F_{\xi_0}(x)e^{ix\cdot\xi_0}) + ((\tilde{V}(x,D)K(x) * (F_{\xi_0}(x)e^{ix\cdot\xi_0})) \\
&= K(x) * [(F_{\xi_0}(x)e^{ix\cdot\xi_0}) - (V_1+V_2)(\lambda^2+|\xi_0|^2)K(x) * (F_{\xi_0}(x)e^{ix\cdot\xi_0}) \\
&+ (V_1+V_2)\tilde{K}(x) * (F_{\xi_0}(x)e^{ix\cdot\xi_0}) - (\nabla V_2 \cdot \nabla K(x) * (F_{\xi_0}(x)e^{ix\cdot\xi_0})) \\
&- (V_1(x)V_2(x) + \Delta V_2)K(x) * (F_{\xi_0}(x)e^{ix\cdot\xi_0})] \\
&\vdots \\
v^n &= K(x) * [F_{\xi_0}(x)e^{ix\cdot\xi_0} + \tilde{V}(x,D)v^{n-1}] \\
&= K(x) * [\sum_{m=0}^{n-1} T^m F_{\xi_0}(x)e^{ix\cdot\xi_0}] \\
&\vdots \quad .
\end{aligned}$$

We wish to show that this iteration converges in L^4. To see this, let $u \in L^4$. Note that

$$\|K * \tilde{V}(x,D)u\|_{L^4} \lesssim \|K * Vu\|_{L^4} + \|\nabla K * \bar{V}u\|_{L^4} + \|\Delta K * \bar{\bar{V}}u\|_{L^4},$$

where V, $\bar{V}$, $\bar{\bar{V}} \in \mathcal{S}$. Then,

$$\begin{aligned}
\|K * \tilde{V}(x,D)u\|_{L^4} &\lesssim \frac{1}{\xi_0^2+\lambda^2}\frac{1}{|\xi_0|^{\frac{1}{2}}}\|Vu\|_{L^{\frac{4}{3}}} \\
&+ \frac{|\xi_0|}{\xi_0^2+\lambda^2}\|(|y|^{-1}) * \bar{V}u\|_{L^4} + \frac{\xi_0^2}{\xi_0^2+\lambda^2}\|K * \bar{\bar{V}}u\|_{L^4},
\end{aligned}$$

so using the Hardy-Littlewood-Sobolev inequality and the bounds on K, we have

$$\begin{aligned}\|K * \tilde{V}(x,D)u\|_{L^4} &\lesssim |\xi_0|^{-\frac{1}{2}} \|Vu\|_{L^{\frac{4}{3}}} \\ &\lesssim |\xi_0|^{-\frac{1}{2}} \|V\|_{L^2} \|u\|_{L^4},\end{aligned}$$

for some $V \in \mathcal{S}$. As a result,

$$\|K * \tilde{V}(x,D)\|_{L^4 \to L^4} \leq C|\xi_0|^{-\frac{1}{2}},$$

where C is determined by V_1, V_2. If $|\xi_0| > C^2$, then

$$\|K * \tilde{V}(x,D)\|_{L^4 \to L^4} \leq 1,$$

and the existence of $g_\xi \in L^4$ for

$$(I - K * \tilde{V}(x,D))g_\xi = g_\xi$$

follows from a contraction argument. In the notation from the theorem, we have $C^2 = M$.

Now, for the smaller frequencies, we wish to use Fredholm theory. This approach also works for large $|\xi_0|$, however the iterative approach gives us uniform bounds for all ξ_0 such that $|\xi_0| > M$. Once differentiability in ξ_0 has been obtained, we will then have uniform bounds for all ξ_0. However, we must be particularly careful near $\xi_0 = 0$ as K has a particularly challenging dependence upon $|\xi_0|$. We explore this shortly, but first let us finish the existence argument for low frequencies.

To begin, Equation (6.5) shows that

$$g_{\xi_0} = K * (\tilde{V}(x,\xi_0)e^{ix\cdot\xi_0}) + K * (\tilde{V}(x,D)g_{\xi_0}), \tag{6.7}$$

where

$$\tilde{V}(x,D) = (-\Delta + \lambda^2 - V_1)V_2 + V_1(-\Delta + \lambda^2)$$

is a second order operator.

Now, if $K * (\tilde{V}(x,D)\cdot)$ is a compact operator, we may use Fredholm Theory (see Appendix B.5) to say that either there is a unique solution to (6.7) or there exists a nontrivial $u \in L^4$ such that

$$(I - K * \tilde{V})u = 0.$$

However, expanding the equation for u, we see this u is an embedded resonance and hence an embedded eigenvalue from Section 4.4. As our spectral assumptions preclude the existence of embedded eigenvalues, the solution to (6.7) is unique.

Let us now discuss the compactness. The operator itself is of the form

$$\begin{aligned} K * (\tilde{V}v) &= \int \pi^2 \frac{[e^{i|x-y||\xi_0|} - e^{-|x-y|\sqrt{|\xi_0|^2+2\lambda^2}}]}{|x-y|(|\xi_0|^2+\lambda^2)} \tilde{V}(y, D_y)v(y)dy \\ &= \int \pi^2 \frac{[e^{i|x-y||\xi_0|} - e^{-|x-y|\sqrt{|\xi_0|^2+2\lambda^2}}]}{|x-y|(|\xi_0|^2+\lambda^2)} \\ &\times \quad [(-\Delta_y + \lambda^2 - V_1(y))V_2(y) + V_1(y)(-\Delta_y + \lambda^2)]v(y)dy. \end{aligned}$$

Hence, using integration by parts, we are concerned about the following two types of operators

$$\begin{aligned} (1)\ P_1 u &= \int K(x-y)V(y)u(y)dy \\ (2)\ P_2 u &= \int \tilde{K}(x-y)V(y)u(y)dy, \end{aligned}$$

where $V \in \mathcal{S}$. Of course, technically there will be terms with derivatives falling on K and V, however a brief calculation shows that these fall into the same class of operators as P_2. Note, to see this it is useful to remember that by construction

$$(-\Delta - |\xi_0|^2)\tilde{K} = 0$$

and

$$(-\Delta - |\xi_0|^2)K = \frac{4\pi^2}{|x|}[e^{-|x|\sqrt{|\xi_0|^2+2\lambda^2}}],$$

hence when all derivatives fall on K, simply by looking at $-\Delta - |\xi_0|^2 + |\xi_0|^2$, we get reduction back to P_1 or P_2 since K is a convolution kernel for an exact solution.

We now need to prove

$$P_i : L^4 \to L^4,$$

for $i = 1, 2$.

Assume that $u_j \to^w 0$ in L^4. Since we are working in $\mathbb{R}^3$, using duality and the properties of V, we have

$$P_i u_j(x) \to 0 \text{ as } j \to \infty$$

for almost every x, where $i = 1, 2$. By the uniform boundedness of weakly convergent sequences, the Hardy-Littlewood-Sobolev Inequality, and Hölder we have,

$$\begin{aligned} \|P_i u_j\|_{L^4} &\leq \|V\|_{L^{\frac{3}{2}}} \|u_j\|_{L^4} \\ &\leq C, \end{aligned}$$

for $i = 1, 2$. Hence, there is a subsequence j_k such that $\|P_i u_{j_k}\|_{L^4}$ converges. Therefore, it must converge to 0. As a result, the operator $K * (\tilde{V} \cdot) : L^4 \to L^4$ is compact and there exists a unique g_{ξ_0} for all ξ_0. Note that $\tilde{V}K$ is compact from $L^{\frac{4}{3}} \to L^{\frac{4}{3}}$ using similar arguments.

To discuss the continuous dependence upon ξ_0, we need to study the functions g_{ξ_0} in more detail. From the expression for g_{ξ_0}, we know that

$$\begin{aligned} (I - K * (\tilde{V}_{\xi_0} \cdot)) g_{\xi_0} &= (I - P_{\xi_0}) g_{\xi_0} \\ &= K * \tilde{V}_{\xi_0} e^{ix\xi_0}, \end{aligned}$$

so

$$g_{\xi_0} = (I - P_{\xi_0})^{-1} (K * \tilde{V}_{\xi_0} e^{ix\xi_0}),$$

where

$$K(\xi) = [(-\Delta - \xi^2)(-\Delta + 2\lambda^2 + \xi^2)]^{-1}.$$

From Fredholm Theory and the spectral assumptions, $(I - P_{\xi_0})^{-1}$ is a resolvent which is uniquely defined. However, using the decay of $\tilde{V}$, we can write

$$\tilde{V} = \tilde{V}_1 \tilde{V}_2,$$

where $|e^{c|x|} \tilde{V}_1| \lesssim 1$, $|e^{c|x|} \tilde{V}_2 f| \lesssim \|f\|_{L^{2,\infty}}$ given $0 < c < c_0$. The constant c_0 is determined by the decay of $\tilde{V}$. Hence, we have

$$g_{\xi_0} = \tilde{V}_1 (I - \tilde{V}_2 K \tilde{V}_1)^{-1} (\tilde{V}_2 e^{i(\cdot, \xi_0)}).$$

Using the decay properties of $\tilde{V}_i$ for $i = 1, 2$ and the differentiability of K, for any ξ_0 we have $\tilde{V}_2 K \tilde{V}_1(z)$ is well-defined for $z \in \mathbb{C}$ in a small neighborhood of $|\xi_0|$. As a result,

$$(I - \tilde{V}_2 K \tilde{V}_1)^{-1}$$

is analytic with respect to z. Also, K is analytic with respect to $|\xi|$, $V_\xi e^{ix\xi}$ is analytic with respect to ξ and we see that g_{ξ_0} depends smoothly on $|\xi|$ and ξ. Using the resolvent identity

$$f_0(x,\xi) = \tilde{V}e^{ix\cdot\xi} + \tilde{V}(1-K\tilde{V})^{-1}K*(\tilde{V}e^{ix\cdot\xi}),$$

the decay in x for f_0 follows.

For $|\xi_0| \geq M$, let us return to the iteration scheme

$$\begin{aligned} g^0_{\xi_0} &= K*[\tilde{V}(\cdot,\xi_0)e^{i(\cdot,\xi_0)}], \\ &\vdots \\ g^n_{\xi_0} &= K*[\tilde{V}(\cdot,\xi_0)e^{i(\cdot,\xi_0)} + \tilde{V}(\cdot,\xi_0)g^{n-1}_{\xi_0}], \end{aligned}$$

for $n \geq 1$. Assuming $g_\xi = e^{ix\cdot\xi_0}f_0(x,\xi_0,|\xi_0|)$, we have

$$f_0 = \tilde{V}(x,\xi_0) + e^{-ix\xi_0}\tilde{V}K*(e^{ix\xi_0}f_0),$$

where by the mapping properties of K, choosing M large enough, this expression is valid in $L^{\frac{4}{3}}_x$ for all $|\xi_0| \geq M$.

We would like to better understand the regularity in x and ξ. To begin, let $u = K*[e^{i(\cdot,\xi_0)}\phi(\cdot,\xi_0)]$. Then, we see

$$\begin{aligned} (\partial_x - i\xi_0)u(x) &= (\partial_x - i\xi_0)(K*[\phi(\cdot,\xi_0)e^{i(\cdot,\xi_0)}])(x) \\ &= i\xi_0\int K(y)e^{i(x-y)\xi_0}\phi(x-y,\xi_0)dy - i\xi_0\int K(y)e^{i(x-y)\xi_0}\phi(x-y,\xi_0)dy \\ &+ \int K(y)e^{i(x-y)\xi_0}\phi_x(x-y,\xi_0)dy \\ &= \int K(y)e^{i(x-y)\xi_0}\phi_x(x-y,\xi_0)dy. \end{aligned}$$

From here, recognizing that $e^{-ix\xi_0}$ cancels from

$$e^{-ix\xi_0}\tilde{V}K*(e^{ix\xi_0}\cdot)$$

and again using the mapping properties of K, we have

$$\|\partial^\alpha_x f_0\|_{L^{\frac{4}{3}}_x} \leq C_\alpha,$$

for all multi-indices α. Hence, $f_0 \in C_x^\infty \cap L_x^\infty$. Similarly,

$$\|\langle x\rangle^N \partial_x^\alpha f_0\|_{L_x^{\frac{4}{3}}} \leq C_{N,\alpha},$$

for any $N \geq 0$ using the decay in x of the operator $\tilde{V}$.

For the regularity in ξ, note that taking once again $u = K * [e^{i(\cdot,\xi_0)}\phi(\cdot,\xi_0)]$, we have

$$\begin{aligned}
(\partial_{\xi_0} - ix)u &= (\partial_{\xi_0} - ix)(K * [\phi(\cdot,\xi_0)e^{i(\cdot,\xi_0)}])(x) \\
&= \frac{4\pi^2}{(\xi_0^2+\lambda^2)}\left(i\frac{\xi_0}{|\xi_0|}\right)\int e^{i|x-y||\xi_0|}e^{i(y)\xi_0}\phi(y,\xi_0)dy \\
&+ i\frac{\xi_0}{\sqrt{\xi_0^2+2\lambda^2}}\int e^{-|x-y|\sqrt{\xi_0^2+2\lambda^2}}e^{i(y)\xi_0}\phi(y,\xi_0)dy \\
&- ix\int K(y)e^{i(x-y)\xi_0}\phi(x-y)dy + i\int K(y)e^{i(y)\xi_0}y\phi(y,\xi_0)dy \\
&+ \int K(y)e^{i(x-y)\xi_0}\phi(x-y,\xi_0)dy \\
&= \left(\frac{1}{\sqrt{\xi_0^2+2\lambda^2}} - \frac{1}{|\xi_0|}\right)\int K(y)e^{iy\xi_0}y\phi(y,\xi_0)dy \\
&+ \int K(y)e^{i(x-y)\xi_0}\phi_{\xi_0}(x-y,\xi_0)dy,
\end{aligned}$$

where we have used $i\xi_0 e^{iy\xi_0} = \partial_y e^{iy\xi_0}$ and integrated by parts. As a result,

$$\|\partial_{\xi_0}^\beta f_0\|_{L^{\frac{4}{3}}} \leq |\xi_0|^{2-|\beta|}C_\beta,$$

for any multi-index β, $|\beta| = 0, 1, 2, \ldots$. Combining the above results, we have

$$|\partial_\xi^\alpha \partial_x^\beta f_0(x,\xi)| \leq C_{\alpha,\beta}|\xi|^{2-|\alpha|},$$

or $f_0 \in S^2$, which gives (6.6).

For the spatial regularity result, we once again use that the distorted Fourier basis satisfies the equation

$$g_{\xi_0} = K * (Fe^{ix\cdot\xi_0}) + K * (\tilde{V}g_{\xi_0}).$$

We have existence for g_{ξ_0} in L^4, but we can take advantage of the structure of $K * P$ in order to show improved regularity. We have

$$\nabla g_{\xi_0} = (\nabla K) * (Fe^{ix\cdot\xi_0}) + (\nabla K) * (\tilde{V}g_{\xi_0}).$$

Hence, we must explore the nature of $(\nabla K) * (\tilde{V})$. Upon differentiating, we see

$$(\nabla K)(x-y) = O(|x-y|^{-1}),$$

which means by a similar approach to Section 6.2, we get

$$\|\nabla g_{\xi_0}\|_{L^4} \leq C(\|F\|_{L^{\frac{12}{11}}} + \|V\|_{L^{\frac{3}{2}}} \|g_{\xi_0}\|_{L^4}).$$

To see this, we first use the Hardy-Littlewood-Sobolev inequality (see B.3 with $\gamma = 1$ so

$$\frac{1}{p} = \frac{2}{3} + \frac{1}{4} = \frac{11}{12},$$

then Hölders inequality such that

$$\|Vg\|_{L^{\frac{12}{11}}} \leq \|V\|_{L^{\frac{3}{2}}} \|g\|_{L^4}.$$

Then, we can iterate this for all derivatives and using Sobolev embeddings, get continuity of all derivatives and hence smoothness.

To prove existence for $\partial_{\xi_0} g_{\xi_0}$ in Sobolev spaces, we must show that $\partial_{\xi_0} g_{\xi_0}$ is defined and bounded in some space of functions. In this direction, we look at

$$[(-\Delta + 2\lambda^2 + \xi_0^2)(-\Delta - \xi_0^2)]g_{\xi_0} \quad = \quad F_{\xi_0} e^{ix\xi_0} + \tilde{V} g_{\xi_0}$$

and

$$[(-\Delta + 2\lambda^2 + (\xi_0 + h_j)^2)(-\Delta - (\xi_0 + h_j)^2)]g_{\xi_0 + h_j} \quad = \quad F_{\xi_0 + h_j} e^{ix(\xi_0 + h_j)} + \tilde{V} g_{\xi_0 + h_j},$$

where $h_j = he_j$ and e_j is the unit vector in the j-th coordinate. Hence, if we define

$$v_h = g_{\xi_0 + h_j} - g_{\xi_0},$$

then we must solve

$$\begin{aligned} L_{\xi_0}(v_h) &= (F_{\xi_0 + h_j} e^{ix\cdot(\xi_0 + h_j)} - F_{\xi_0} e^{ix\xi_0}) + O(h) u_{\xi_0} + \tilde{V}(v_h) \\ &= O(h)(\tilde{F}_{\xi_0} + F_{\xi_0} + K * \tilde{V} u_{\xi_0}) + \tilde{V}(v_h). \end{aligned}$$

We can write this as

$$L_{\xi_0}[v_h - O(h) K * (K * (\tilde{V} g_{\xi_0}))] = O(h)(G) + \tilde{V}[v_h - O(h) K * (K * (\tilde{V} g_{\xi_0}))],$$

where we have

$$G = \tilde{F}_{\xi_0} + F_{\xi_0} - \tilde{V}K * (K * (\tilde{V}g_{\xi_0})).$$

To see that $G \in L^4$, we need only see that

$$\|\tilde{V}K * (K * (\tilde{V}g_{\xi_0}))\|_{L^4} < \infty$$

since the other terms are dealt with above in the spatial regularity analysis. However, we have

$$K * (K * \cdot) : L^1 \to L^\infty,$$

following analysis similar to the complex interpolation argument. Also, by moving all of the derivatives onto Pu, we see this is smooth. All we lack is nice decay, hence

$$\|\tilde{V}K * (K * (\tilde{V}g_{\xi_0}))\|_{L^4} < \|K * (K * (\tilde{V}g_{\xi_0}))\|_{L^\infty}\|V\|_{L^4},$$

for $V \in \mathcal{S}$ as given in the description of P. Hence, from the Fredholm Theory, we know

$$\|\frac{v_h}{h} - O(1)K * (K * (\tilde{V}g_{\xi_0}))\|_{L^4} \leq C,$$

for $C = C(\xi_0)$. However, given $w \in C_0^\infty \cup L^4$ a sufficiently decaying, smooth function, we have

$$\begin{aligned} \|w\frac{v_h}{h}\|_{L^4} &\leq C(1 + \|wK * (K * (\tilde{V}g_{\xi_0}))\|_{L^4}) \\ &\leq C, \end{aligned}$$

from Section 6.2, where C is independent of h. In this case, we have

$$K * (K * (\tilde{V}g_{\xi_0})) \in L^\infty$$

using Hölder's inequality, so we can take $w = \langle x \rangle^{-1}$. Thus, we can take the limit as $h \to 0$ to see that derivatives in ξ_0 are bounded in weighted L^4 spaces. Iterating this process involves taking stonger weight functions at each step of the iteration.

Hence, since $\tilde{V}$ has exponentially decaying terms in x and $\tilde{V}g_{\xi_0}$ is well-defined in L^4 from the spatial regularity, we have the desired regularity in ξ_0.

Now that we have differentiability with respect to ξ_0,

$$\partial_{(\xi_0)_j}\left([(-\Delta+2\lambda^2+\xi_0^2)(-\Delta-\xi_0^2)]g_{\xi_0}=Fe^{ix\xi_0}+\tilde{V}g_{\xi_0}\right)$$

which implies

$$\begin{aligned}L_{\xi_0}\partial_{(\xi_0)_j}g_{\xi_0} &= \partial_{(\xi_0)_j}(Fe^{ix\xi_0})+P\partial_{(\xi_0)_j}g_{\xi_0}\\ &- 2(\xi_0)_j(-\Delta-\xi_0^2)g_{\xi_0}-(\xi_0)(-\Delta+2\lambda^2+\xi_0^2)g_{\xi_0}.\end{aligned}$$

For higher derivatives in ξ_0, we iterate this procedure.

□

Remark 6.1. *Note that the above analysis can also be done in the case where instead of L^4 we use $L^2(\langle x\rangle^{-s})$ as in (Agmon). To see this, note that*

$$\|\phi\|_{L^1}\lesssim\|\phi\|_{L^2(\langle x\rangle^s)},$$

where $s>d$, and

$$\|\phi\|_{L^2(\langle x\rangle^{-s})}\lesssim\|\phi\|_{L^\infty},$$

where $s>d$. Then, we can go to the Sobolev norms to apply Hardy-Littlewood-Sobolev and use Hölder's inequality in weighted spaces and the boundedness of V_1 and V_2 in weighted L^2 spaces to complete the argument.

Remark 6.2. *As $x\to\infty$, note that since V_1, $V_2\in\mathcal{S}$, using Equation* (6.2)*, we have*

$$u_{\xi_0}\to\frac{\pi^2}{|\xi_0|^2+\lambda^2}\left[\frac{e^{\pm i|x||\xi_0|}-e^{-|x|\sqrt{|\xi_0|^2+2\lambda^2}}}{|x|}\right],$$

which explains the choice of spaces $L^{2,s}$ for $x>\frac{1}{2}$ in (Agmon).

6.3 Representation of the solution

We present here a slightly different approach to the distorted Fourier transform.

Theorem 19. *For $V \in \mathcal{S}$, there exists a distorted Fourier basis $\tilde{\phi}_\xi$ and correspondingly a distorted Fourier transform $\mathcal{G}$ for the nonselfadjoint operator $\mathcal{H}$, where*

$$\mathcal{G}_\pm f = \int \tilde{\phi}_\xi^\pm(x) f(x) dx.$$

Similarly, there exists an inverse Fourier basis $\tilde{\phi}_\xi^{-1}(x)$ and correspondingly an inverse Fourier transform $\mathcal{G}^{-1}$ for the nonselfadjoint operator $\mathcal{H}$, where

$$\mathcal{G}_\pm^{-1} f = \int \{\tilde{\phi}_\xi^\pm\}^{-1}(x) f(\xi) d\xi.$$

It follows that

$$\begin{aligned} \|\mathcal{G}_\pm\|_{L^2 \to L^2} &\lesssim 1, \\ \|\mathcal{G}_\pm^{-1}\|_{L^2 \to L^2} &\lesssim 1. \end{aligned}$$

These operators are not unitary, however

$$\|\mathcal{G}_\pm^{-1}\mathcal{G}\|_{L^2 \to L^2} \lesssim 1$$

and

$$\mathcal{G}_\pm^{-1}\mathcal{G}_\pm \phi = P_c \phi.$$

Before we prove the theorem, look at the operator

$$\mathcal{H}^2 = \begin{bmatrix} L_- L_+ & 0 \\ 0 & L_+ L_- \end{bmatrix},$$

for which we have the following self-adjoint realization

$$\tilde{\mathcal{H}} = \begin{bmatrix} L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} \end{bmatrix}.$$

Since

$$\begin{aligned} L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} &= (-\Delta + \lambda^2 - V_1)^{\frac{1}{2}} (-\Delta + \lambda^2 - V_1 - V_2)(-\Delta + \lambda^2 - V_1)^{\frac{1}{2}} \\ &= (-\Delta + \lambda^2 - V_1)^2 - (-\Delta + \lambda^2 - V_1)^{\frac{1}{2}} V_2 (-\Delta + \lambda^2 - V_1)^{\frac{1}{2}} \\ &= L_-^2 - L_-^{\frac{1}{2}} V_2 L_-^{\frac{1}{2}}. \end{aligned}$$

This is a 4th order constant coefficient operator with a lower order perturbation. However, the perturbation is no longer a differential operator. Ideally, by a similar analysis to that in (Agmon), there exists a distorted Fourier basis, say $\tilde{u}_\xi$ such that

$$L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} \tilde{u}_\xi = (\lambda^2 + \xi^2)^2 \tilde{u}_\xi.$$

To prove this, we need to show $L_-^{\frac{1}{2}}$ is a pseudodifferential operator of strong enough class, which we explore in the sequel.

From Theorem 18, we have $u_\xi = e^{ix\xi} + f_\xi(x)$, $v_\xi = e^{ix\xi} + g_\xi(x)$ such that

$$\mathcal{H}^2 \begin{bmatrix} u_\xi \\ v_\xi \end{bmatrix} = (\lambda^2 + \xi^2)^2 \begin{bmatrix} u_\xi \\ v_\xi \end{bmatrix},$$

where $f_\xi(x)$, $g_\xi(x) \in L^4_x$, smooth in x and ξ, and

$$f_\xi, g_\xi \sim \frac{\pi^2}{|\xi_0|^2 + \lambda^2} \left[\frac{e^{\pm i|x||\xi_0|} - e^{-|x|\sqrt{|\xi_0|^2 + 2\lambda^2}}}{|x|} \right]$$

as $x \to \infty$.

Formally, we would like to say

$$\begin{bmatrix} L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} L_-^{-\frac{1}{2}} u_\xi \\ L_-^{\frac{1}{2}} v_\xi \end{bmatrix} = (\lambda^2 + \xi^2)^2 \begin{bmatrix} L_-^{-\frac{1}{2}} u_\xi \\ L_-^{\frac{1}{2}} v_\xi \end{bmatrix},$$

however as u_ξ, $v_\xi \notin L^2$, we must investige further.

Before we begin, let us analyze the connection between u_ξ and v_ξ. For instance,

$$\begin{aligned} L_+(L_- L_+ u_\xi) &= L_+ L_- (L_+ u_\xi) \\ &= L_+(\lambda^2 + \xi^2)^2 u_\xi, \\ L_-(L_+ L_- v_\xi) &= L_- L_+ (L_- v_\xi) \\ &= L_-(\lambda^2 + \xi^2)^2 v_\xi. \end{aligned}$$

Hence

$$L_+ u_\xi = C v_\xi$$

and

$$L_- v_\xi = C u_\xi.$$

In particular, we are interested in

$$\begin{aligned} L_- v_\xi &= (-\Delta + \lambda^2 - V_1)(e^{ix\xi} + g_\xi) \\ &= (\xi^2 + \lambda^2)e^{ix\xi} + L_- g_\xi - V_1 e^{ix\xi}, \\ Cu_\xi &= C(e^{ix\xi} + f_\xi). \end{aligned}$$

Then, $C = (\lambda^2 + \xi^2)$, so

$$L_- v_\xi = (\lambda^2 + \xi^2)u_\xi,$$

$$L_-^{-1} u_\xi = (\lambda^2 + \xi^2)^{-1} v_\xi,$$

and

$$f_\xi = \frac{1}{\lambda^2 + \xi^2}(L_- g_\xi - V_1 e^{ix\xi}).$$

A similar calculation holds for $L_+ u_\xi = C v_\xi$.

Note also that if we look at the vector

$$\vec{\phi}_\xi = \begin{bmatrix} iu_\xi \\ v_\xi \end{bmatrix},$$

then we have

$$\mathcal{H}\vec{\phi}_\xi = (\lambda^2 + \xi^2)\vec{\phi}_\xi.$$

To be more precise, we say that the operator $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$ has a distorted Fourier basis given by $\tilde{u}_\xi$, then find an expression for the distorted Fourier transform of $\mathcal{H}P_c$. This distorted Fourier transform will be defined via a distorted Fourier basis that will give the relationship between $\tilde{u}_\xi$, u_ξ and v_ξ. The existence of $\tilde{u}_\xi$ must be proved since there is a lower order PDO perturbation instead of a differential operator. See (Ho2).

To prove $L_-^{\pm\frac{1}{2}}$ is a PDO, we must use a result similar to one from (Ho4), Chapter 29 as presented with proof in Appendix C.3.

To this end, we refer to the following theorem given in (Ho4):

Theorem 20. *Let X be a compact manifold, Ψ a space of pseudo-differential operators and $\Omega^{\frac{1}{2}}$ be the space of half-densities on X. Let $P \in \Psi^m_{phg}(X;\Omega^{\frac{1}{2}},\Omega^{\frac{1}{2}})$ be a positive, elliptic, symmetric operator. Then, P defines a positive, self-adjoint operator $\mathcal{P}$ in $L^2(X,\Omega^{\frac{1}{2}}$. If $m > 0$ and $a \in \mathbb{R}$, then $\mathcal{P}^a$ is also defined by a pseudodifferential operator in $\Psi^{am}_{phg}(X;\Omega^{\frac{1}{2}},\Omega^{\frac{1}{2}})$, with principal and subprincipal symbols p^a and $ap^{a-1}p^s$ if p and p^s are those for P.*

We seek to prove a slightly different version here:

Theorem 21. *Let P be a positive, symmetric, self-adjoint operator in $\Psi^{m,(2)}_{\rho,\delta}(\mathbb{R}^d)$. Then, P defines a postive, self-adjoint operator $\mathcal{P}$ in $L^2(\mathbb{R}^d,\mathbb{R}^d)$. If $m > 0$ and $a \in \mathbb{R}$, then $\mathcal{P}^a$ is also defined by a pseudodifferential operator in $\Psi^{am,(2)}_{\rho,\delta}(\mathbb{R}^d,\mathbb{R}^d)$, with principal and subprincipal symbols p^a and $ap^{a-1}p^s$ if p and p^s are those for P.*

Note that since $R \in \mathcal{S}$, $F(R) \in \mathcal{S}$ by the properties of the nonlinearity. Hence, we have the following:

Lemma 6.3.1. *The perturbation V_1 is short-range.*

We seek to prove that given the operator,

$$L_- = -\Delta + \lambda^2 - V_1 \in S^2,$$

the new operator L_-^a is a pseudodifferential operator for $a \in \mathbb{R}$.

Lemma 6.3.2. *For an operator P, the resolvent $R(z) = (P-z)^{-1}$ exists and is analytic for all z except the eigenvalues of P. Also, $\|R(z)\|_{L^2\to L^2}$ is bounded by the inverse of the distance from z to the nearest eigenvalue.*

Proof. This follows from basic facts from spectral theory as discussed in Appendix B.4. □

Theorem 22. *The operator L_-^a is pseudodifferential operator in the class S^{2a} for $a \in \mathbb{R}$.*

Before we prove the theorem, let us prove the following lemma from (Ho3).

Lemma 6.3.3. *Let $a \in S^m$. If*

$$|a(x,\xi)| > c|\xi|^m \tag{6.8}$$

for $|\xi| > C$, then there exists $b \in S^{-m}$ such that

$$(i)\ a(x,\xi)b(x,\xi) - 1 \in S^{-1},$$

$$(ii)\ a(x,D)b(x,D) - I \in OpS^{-\infty},$$

and

$$(iii)\ b(x,D)a(x,D) - I \in OpS^{-\infty}.$$

Proof of Lemma. First, let us prove that (6.8) implies (i). We can reduce this to the case where $m = 0$ by looking at $a(x,\xi)(1+|\xi|^2)^{-m/2}$ and $b(x,\xi)(1+|\xi|^2)^{m/2}$.

Claim 6.3.4. *If $a_1,\ a_2 \in S^0$ and $F \in C^\infty(\mathbb{C}^2), then F(a_1,a_2) \in S^0$.*

Proof. Since the $\operatorname{Re} a_j,\ \operatorname{Im} a_j \in S^0$ for $j = 1,2$, we may assume that a_j is real and $F \in C^\infty(\mathbb{R}^2)$. Then,

$$\begin{aligned} \frac{\partial F(a)}{\partial x_j} &= \sum_k \frac{\partial F}{\partial a_k}\partial_{x_j} a_k, \\ \frac{\partial F(a)}{\partial \xi_j} &= \sum_k \frac{\partial F}{\partial a_k}\partial_{\xi_j} a_k, \end{aligned}$$

where $\partial_{x_j} a_k \in S^0$, $\partial_{\xi_j} a_k \in S^{-1}$. Hence, it is clear the derivatives of $F(a)$ decay as necessary for $F(a)$ to be in S^0. □

Hence, for $m = 0$, choose $F \in C^\infty$ so that $F(z) = \frac{1}{z}$ for $|z| > c$. Set $b = F(a) \in S^0$ so $a(x,\xi)b(x,\xi) = 1$ for $|\xi| > C$. This proves (i).

Using (i), we have that

$$a(x,D)b(x,D) = I - r(x,D),\ r \in S^{-1}.$$

Set

$$b(x,D)r(x,D)^k = b_k(x,D),\ b_k \in S^{-m-k},$$

so we can iterate out the error. Let b' be the asymptotic sum of the b_k's, so

$$a(x,D)b'(x,D) - I = a(x,D)(b'(x,D) - \sum_{j<k} b_j(x,D)) - r(x,D)^k \in OpS^{-k},$$

for every k. Then, we have (ii) replacing b with b'. Similarly, we can find a b'' which satisfies (iii). Note also that

$$b' - b'' = b'(I - ab') + (b''a - I)b',$$

hence b' and b'' are equivalent modulo $S^{-\infty}$. □

Proof of Theorem 22. Since L_- is self-adjoint, we have that $R(z)$ is defined and analytic for all z except at the eigenvalues of L_-. The L^2 norm of the resolvent can be estimated by the inverse of the distance to the set of eigenvalues. Now, since $a < 0$, we have by the spectral theorem

$$\tilde{L}^a u = -(2\pi i)^{-1} \int_{-i\infty}^{i\infty} z^a R(z) u dz,$$

where the contour is slightly deformed near the origin to avoid $z = 0$ and z^a is analytic in the right half plane and equal to 1 when $z = 1$. Since $L_-^{a+1} = L_-^1 L_-^a$, the distribution kernel of L_-^a is an entire analytic function of a.

To understand the behavior of the singularities, we construct a parametrix. Namely, since $|L_-(x,\xi)| > c|\xi|^2$ for $|\xi| > C$, we have the existence of an inverse modulo S^{-1}. Then, we can iterate that error, to find an inverse modulo $S^{-\infty}$.

In particular, we have B_z such that

$$(P - z)B_z = I - Q_z,$$

where $b_z = F(P(x,\xi) - z)$, $F(z) \sim 1/|z|$ for z large and $Q_z \in Op(S^{-1})$. Then, there is an E_z given by the asymptotic sum

$$\sum_{j=0}^{\infty} B_z(x,D)(Q_z(x,D))^j,$$

such that

$$(P - z)E_z = I - W_z,$$

where $W_z \in Op(S^{-\infty})$. So, we have

$$R(z) = E_z + R(z)W_z.$$

Then, for $a < 0$, we have

$$\tilde{L}^a = -(2\pi i)^{-1} \int_{-i\infty}^{i\infty} z^a E_z dz + T(a)u.$$

Here, $T(a)$ should be analytic in a for $a < 1$. In particular, this remainder will be a well-behaved pseudo-differential operator using Beals' Theorem as discussed in Appendix C.2. From the composition of pseudodifferential operators, we have that

$$Q_z = \sum_{\alpha>0} \partial_\xi^\alpha L_-(x,\xi)\partial_x^\alpha F(L_- - z)/\alpha!.$$

Hence, the terms of E_z outside of compact set in phase space look like

$$(P-z)^{-k-1}q$$

where $q \in S^{mk-\kappa}$ for some $\kappa \geq 0$.

Hence, there is a pseudodifferential operator representation of $L_-^{-\frac{1}{2}}$ and thus $L_-^{\frac{1}{2}}$ by multiplication by the operator. If p is the principal symbol of L_-, the principal symbol of L_-^a will be $F(p)$ where $F(z) = z^a$ for $|z| > C$.

□

Lemma 6.3.5. *The pseudodifferential operator $L_-V_1 + V_1(-\Delta + \lambda^2) + L_-^{\frac{1}{2}} V_2 L_-^{\frac{1}{2}}$ is a short range perturbation.*

Proof. This proof should be similar to that in Lemma 6.3.1. The argument for the differential operator $L_-V_1 + V_1(-\Delta + \lambda^2)$ follows precisely as above. Hence, we focus only on the compactness and iteration arguments for the pseudodifferential operator, $L_-^{\frac{1}{2}} V_2 L_-^{\frac{1}{2}}$. In what follows, let $Tu = L_-^{\frac{1}{2}} V L_-^{\frac{1}{2}} K * (u)$. In particular, we need to prove:

$$L_-^{\frac{1}{2}} V L_-^{\frac{1}{2}} e^{ix\cdot\xi_0} = e^{ix\cdot\xi_0} V_{\xi_0}, \text{ for } V_{\xi_0} \in \mathcal{S}, \ \|V_{\xi_0}\|_L^\infty \sim |\xi_0|^2, \tag{6.9}$$

$$\|K * T^n(e^{ix\cdot\xi_0} V_{\xi_0})\|_{L^4} = O(|\xi_0|^{-\frac{n+1}{2}}), \tag{6.10}$$

$$K * (L_-^{\frac{1}{2}} V L_-^{\frac{1}{2}} \cdot) : W^{2,4} \hookrightarrow W^{2,4}. \tag{6.11}$$

For (6.9), we have in the sense of distributions that

$$\mathcal{F} e^{ix\xi_0} = \delta_{\xi_0}(\xi).$$

Hence, since $V \in \mathcal{S}$,

$$\begin{aligned} L_-^{\frac{1}{2}} V L_-^{\frac{1}{2}} &= \int P(x,\xi) e^{i(x-x_1)\xi} V(x_1) \int P(x_1,\xi_1) e^{ix_1\xi_1} \delta_{\xi_0}(\xi_1) d\xi_1 dx_1 d\xi \\ &= \int P(x,\xi) e^{i(x-x_1)\xi} V(x_1) \int P(x_1,\xi_0) e^{ix_1\xi_0} dx_1 d\xi \\ &= e^{ix\xi_0} \tilde{V}(x,\xi_0) + l.o.t., \end{aligned}$$

where $\tilde{V} \in \mathcal{S}(x)$ and $|\tilde{V}| \lesssim \xi_0^2$ precisely as in Section 6.2. This comes in particular from realizing that the principal symbol of $L_-^{\frac{1}{2}} V L_-^{\frac{1}{2}}$ is

$$(\xi^2 + \lambda^2 - V_1(x))V_2(x).$$

The results (6.10) and (6.11) follow from the following theorem proved in (Stein), Chapter VI.

Theorem 23 (Stein)**.** *Suppose T_a is a pseudo-differential operator whose symbol a belongs to S^m. If m is an integer and $k \geq m$, then T_a is a bounded mapping from $W^{k,p} \to W^{k-m,p}$, whenever $1 < p < \infty$.*

Since $L_-^{\frac{1}{2}} \in S^1$ and $V \in S^0$, we have $L_-^{\frac{1}{2}} V L_-^{\frac{1}{2}} \in S^2$, hence

$$L_-^{\frac{1}{2}} V L_-^{\frac{1}{2}} : W^{2,4} \to L^4.$$

As $V \in \mathcal{S}$, we in fact have more than this. Define the symbol class

$$S_r^m = \{p(|p \in S^m,\ |x^\alpha \partial_x^\beta \partial_\xi^\gamma p(x,\xi)| \leq C_{\alpha,\beta} |\xi|^{m-\gamma}.$$

In other words, we have the standard symbol class S^m, where the symbol has rapid decay in x. Here, $V \in S_r^0$. Note that due to the properties of Schwarz class functions, we have for $p \in S^{m_1}$ and $q \in S^{m_2}$,

$$pq, qp \in S_r^{m_1+m_2}$$

and

$$qu : W^{m_2,p} \to L^q,$$

where $1 < p, q < \infty$.

For (6.10), from the analysis in Theorem 18 we have

$$(K * \cdot) : L^{\frac{4}{3}} \to W^{2,4}.$$

We have from (6.9)

$$\| \int K(x-y) e^{iy\cdot\xi_0} V_{\xi_0}(y) dy \|_{L^4} \lesssim |\xi_0|^{-\frac{1}{2}}.$$

Then,

$$\begin{aligned} \| \int K(x-y) L_-^{\frac{1}{2}} V(y) L_-^{\frac{1}{2}} \int K(y-z) e^{iz\cdot\xi_0} V_{\xi_0}(z) dz dy \|_{L^4} \\ = |\xi_0|^{-\frac{1}{2}} \| L_-^{\frac{1}{2}} V(y) L_-^{\frac{1}{2}} \int K(y-z) e^{iz\cdot\xi_0} V_{\xi_0}(z) dz dy \|_{L^4 3} \\ \lesssim |\xi_0|^{-\frac{1}{2}} \| \int K(y-z) e^{iz\cdot\xi_0} V_{\xi_0}(z) dz dy \|_{W^{2,4}} \\ \lesssim |\xi_0|^{-1} \| e^{iz\cdot\xi_0} V_{\xi_0}(z) \|_{L^4 3}, \end{aligned}$$

using the fact that $V \in S^0_r$ and the mapping properties of K described in Theorem 18. Iterating this procedure, we get the result.

For (6.11), if $u \in W^{2,4}$,

$$\| L_-^{\frac{1}{2}} V L_-^{\frac{1}{2}} u \|_{L^{\frac{4}{3}}} \lesssim \| u \|_{W^{2,4}}.$$

By decay properties of V, we have

$$\| x L_-^{\frac{1}{2}} V L_-^{\frac{1}{2}} u \|_{L^{\frac{4}{3}}_x} \lesssim \| u \|_{W^{2,4}} + \| u \|_{W^{1,4}} \lesssim \| u \|_{W^{2,4}}.$$

The inherent integration by parts is justified as $V \in \mathcal{S}$. Hence, by iterating this procedure and properties of convolutions,

$$[K * (L_-^{\frac{1}{2}} V L_-^{\frac{1}{2}} \cdot)] : W^{2,4} \to W^{2,4}(\langle\cdot\rangle^N),$$

for any $N \in \mathbb{N}$. However, $W^{2,4}(\langle\cdot\rangle^N)$ is compactly embedded in $W^{2,4}$, so (6.11) holds.

□

Lemma 6.3.6. *There exists a distorted Fourier basis, $\tilde{u}_\xi$, for $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$ with the aforementioned smoothness properties.*

Proof. Apply the techniques from the proof of Theorem 18, applying (6.9), (6.10), and (6.11) when necessary. Once the compactness is established, the standard self-adjoint techniques are available to give

$$\begin{aligned} \|P_c\phi\|_{L^2}^2 &= (2\pi)^{-d}\int |\mathcal{F}_\pm\phi|^2 dx, \\ \mathcal{F}_\pm^{-1}P_0\mathcal{F}_\pm\phi &= P_c\phi, \end{aligned}$$

where $\mathcal{F}_\pm$ is the distorted Fourier transform associated to $\tilde{u}_{\xi_0}^\pm$ and $P_0(\xi_0) = (\xi_0^2+\lambda)^2$ is the symbol for the leading order constant coefficient operator.

□

Since

$$\begin{bmatrix} L_-^{\frac{1}{2}}L_+L_-^{\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}}L_+L_-^{\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} L_-^{-\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} L_-L_+ & 0 \\ 0 & L_+L_- \end{bmatrix} \begin{bmatrix} L_-^{\frac{1}{2}} & 0 \\ 0 & L_-^{-\frac{1}{2}} \end{bmatrix},$$

we have

$$\begin{bmatrix} L_-^{-\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}} \end{bmatrix} \mathcal{H}^2 \begin{bmatrix} L_-^{\frac{1}{2}} & 0 \\ 0 & L_-^{-\frac{1}{2}} \end{bmatrix} P_c f = \tilde{\mathcal{F}}_\pm^*|(\xi^2+\lambda^2)^2|\tilde{\mathcal{F}}_\pm f,$$

where $\tilde{\mathcal{F}}_\pm$ is the distorted Fourier transform with respect to $\tilde{u}_\xi$. Setting

$$f = \begin{bmatrix} L_-^{-\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}} \end{bmatrix} \tilde{f},$$

we see

$$\begin{bmatrix} L_-^{-\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}} \end{bmatrix} \mathcal{H}^2 \begin{bmatrix} L_-^{\frac{1}{2}} & 0 \\ 0 & L_-^{-\frac{1}{2}} \end{bmatrix} P_c f = \tilde{\mathcal{F}}_\pm^*|(\xi^2+\lambda^2)^2|\tilde{\mathcal{F}}_\pm \begin{bmatrix} L_-^{-\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}} \end{bmatrix} \tilde{f}.$$

Hence,

$$\mathcal{H}^2(P_c\tilde{f}) = \begin{bmatrix} L_-^{\frac{1}{2}} & 0 \\ 0 & L_-^{-\frac{1}{2}} \end{bmatrix} \tilde{\mathcal{F}}_\pm^*|(\xi^2+\lambda^2)^2|(\tilde{\mathcal{F}}_\pm \begin{bmatrix} L_-^{-\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}} \end{bmatrix})\tilde{f},$$

or

$$\mathcal{H}^2(P_c\tilde{f})(x) = \begin{bmatrix} \int (L_-^{\frac{1}{2}}\bar{\tilde{u}}_\xi)(x)|(\xi^2+\lambda^2)^2|\int \tilde{u}_\xi(y)(L_-^{-\frac{1}{2}}\tilde{f}_1)(y)dyd\xi \\ \int (L_-^{-\frac{1}{2}}\bar{\tilde{u}}_\xi)(x)|(\xi^2+\lambda^2)^2|\int \tilde{u}_\xi(y)(L_-^{\frac{1}{2}}\tilde{f}_2)(y)dyd\xi. \end{bmatrix}$$

The inverse operations in these arguments are justified by the fact that

$$L^2 = \mathrm{Ker}(\mathcal{H}) \oplus \mathrm{Ker}(\mathcal{H}^*)^{\perp}.$$

We desire an oscillatory integral formulation for $\mathcal{H}P_c$. The continuous spectrum is spanned by the values $\pm(\lambda^2+\xi^2)$ for all $|\xi| \geq 0$. Hence, we seek a diagonalization of the form

$$\mathcal{H}P_c = Q^{-1} \begin{bmatrix} (\lambda^2+\xi^2) & 0 \\ 0 & -(\lambda^2+\xi^2) \end{bmatrix} Q.$$

Using the above analysis for $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$, we see that

$$\begin{aligned} Q &= \frac{1}{\sqrt{2}} \begin{bmatrix} i(\lambda^2+\xi^2)^{\frac{1}{2}} \mathcal{F} L_-^{-\frac{1}{2}} & (\lambda^2+\xi^2)^{-\frac{1}{2}} \mathcal{F} L_-^{\frac{1}{2}} \\ -i(\lambda^2+\xi^2)^{\frac{1}{2}} \mathcal{F} L_-^{-\frac{1}{2}} & (\lambda^2+\xi^2)^{-\frac{1}{2}} \mathcal{F} L_-^{\frac{1}{2}} \end{bmatrix} \\ Q^{-1} &= \frac{1}{\sqrt{2}} \begin{bmatrix} -iL_-^{\frac{1}{2}} \mathcal{F}^* (\lambda^2+\xi^2)^{-\frac{1}{2}} & iL_-^{\frac{1}{2}} \mathcal{F}^* (\lambda^2+\xi^2)^{-\frac{1}{2}} \\ L_-^{-\frac{1}{2}} \mathcal{F}^* (\lambda^2+\xi^2)^{\frac{1}{2}} & L_-^{-\frac{1}{2}} \mathcal{F}^* (\lambda^2+\xi^2)^{\frac{1}{2}} \end{bmatrix} \end{aligned}$$

Note that we have for $\vec{f} = P_c \vec{f}$

$$\mathcal{H} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} iL_- f_2 \\ -iL_+ f_1 \end{bmatrix},$$

which is exactly what results from the decomposition. The resulting integral equation is

$$\mathcal{H}P_c \vec{f} = \begin{bmatrix} -iL_-^{\frac{1}{2}} \mathcal{F}^* \mathcal{F} L_-^{\frac{1}{2}} f_2 \\ iL_-^{-\frac{1}{2}} \mathcal{F}^* (\lambda^2+\xi^2)^2 \mathcal{F} L_-^{-\frac{1}{2}} \end{bmatrix}.$$

So, since we have a pseudo-differential operator representation of $L_-^{\frac{1}{2}}$, we could write $\mathcal{H}P_c$ in terms of an oscillatory integral.

Remark 6.3. *Note that we have now made precise the definition*

$$\tilde{\phi}_\xi \asymp \begin{bmatrix} iu_\xi & v_\xi \\ -iu_\xi & v_\xi \end{bmatrix} \tag{6.12}$$

$$= \begin{bmatrix} i(\xi^2+\lambda^2) L_-^{-\frac{1}{2}} \tilde{u}_\xi & (\xi^2+\lambda^2)^{-1} L_-^{\frac{1}{2}} \tilde{u}_\xi \\ -i(\xi^2+\lambda^2) L_-^{-\frac{1}{2}} \tilde{u}_\xi & (\xi^2+\lambda^2)^{-1} L_-^{\frac{1}{2}} \tilde{u}_\xi \end{bmatrix}, \tag{6.13}$$

where using the pseudo-differential analysis above, $L_-^{\pm\frac{1}{2}} \tilde{u}_\xi$ is well-defined.

Proof of Theorem 19. If

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \sigma_{ac}(\mathcal{H}),$$

then

$$Pf = \begin{bmatrix} L_-^{\frac{1}{2}} & 0 \\ 0 & L_-^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \sigma_{ac}(\tilde{\mathcal{H}}),$$

where

$$\tilde{\mathcal{H}} = \begin{bmatrix} L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} & 0 \\ 0 & L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} \end{bmatrix}.$$

Assume $f \in \mathcal{S}$, which we will relax later. Let ψ the PDO representation of P and $\Phi_\xi(x)$ is the vector where both elements are the distorted Fourier basis function ϕ_ξ for the self-adjoint operator $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$. Then, we have

$$\begin{aligned} (\mathcal{G}\psi f)(\xi) &= \langle \psi f, \Phi_\xi \rangle \\ &= \langle f, T\psi^* \Phi_\xi \rangle \\ &= \langle f, \tilde{\Phi}_\xi \rangle, \end{aligned}$$

where

$$T = \begin{bmatrix} i(\lambda^2 + \xi^2)^{\frac{1}{2}} & (\lambda^2 + \xi^2)^{-\frac{1}{2}} \\ -i(\lambda^2 + \xi^2)^{\frac{1}{2}} & (\lambda^2 + \xi^2)^{-\frac{1}{2}} \end{bmatrix},$$

and $\tilde{\Phi}_\xi$ is uniquely defined in the sense of distributions as

$$P(x, \xi) e^{ix\xi} + \tilde{u}_\xi(x, \xi),$$

where

$$\tilde{u} = P(x, D) u_\xi(x),$$

and $\tilde{u} \in \mathcal{S}$. Then,

$$(\mathcal{G}f)(\xi) = \int f \tilde{\Phi}_\xi dx.$$

Similarly, we have

$$(\mathcal{G}^{-1}f)(x) = \int f(\xi)\tilde{\Phi}_{\xi}^{-1}d\xi,$$

where

$$\tilde{\Phi}_{\xi}^{-1} = PT^{*}\Phi_{\xi}^{*}(x).$$

The modified Fourier transforms are in fact variations on the expansion involving the matrix Q. In fact, the Fourier transforms are arrived at by $\square$

Corollary 6.3.7. *As a result of the decomposition, we have a new proof of the fact that*

$$\|P_c e^{it\mathcal{H}}f\|_{L^2} \lesssim \|f\|_{L^2}.$$

Proof. This follows simply from mapping properties of pseudodifferential operators and the fact that the self-adjoint distorted Fourier transform is an L^2 isometry. $\square$

Remark 6.4. *Note that for convenience in terms of defining the resolvent, our result has been proved here only in $\mathbb{R}^3$. However, using similar bounds developed in (Agmon) for higher dimensional resolvents, we expect that a result similar to that of 19 holds in all dimensions and as a result similar estimates will follow below. The main difficulties presented would be a thorough discussion of the spectrum of $\mathcal{H}$ as some of the numerical techniques are unique to $\mathbb{R}^3$.*

6.4 Time Decay

Using our distorted Fourier basis, we have that a solution to the problem

$$e^{i\mathcal{H}t}P_c\phi = Q^{-1}e^{itW}Q\phi, \tag{6.14}$$

for

$$W = \begin{bmatrix} (\lambda^2+\xi^2) & 0 \\ 0 & -(\lambda^2+\xi^2) \end{bmatrix}.$$

The structure on Q allows us to do oscillatory integration in order to study the properties of $e^{i\mathcal{H}t}$. First of all, we show the following:

Theorem 24. *Let $\mathcal{H}$ be an admissible Hamiltonian as defined above. Then,*

$$\|e^{it\mathcal{H}}P_c\vec{\psi}\|_{L^\infty} \leq Ct^{-\frac{d}{2}}\|\vec{\psi}\|_{L^1}. \tag{6.15}$$

Proof of 24. Using matrix notation, we have

$$\{\mathcal{G}\vec{\psi}\}(\xi) = \int \tilde{\phi}_\xi(x)\vec{\psi}(x)dx,$$

where

$$\vec{\psi}(x) = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix},$$

and $\tilde{\phi}_\xi$ is given by (6.12).

Looking at the integral representation, we have

$$e^{it\mathcal{H}}P_c\vec{\psi}(x) = \int_\xi \tilde{\phi}_\xi^{-1}(x)e^{itW}\int_y \tilde{\phi}_\xi(y)\vec{\psi}(y)dyd\xi.$$

Let $\chi \in C_c^\infty$, be a smooth, cut-off function chosen such that the iteration techniques in Theorem 18 hold for $\xi \in \mathbb{R}^d \setminus \text{spt}(\chi)$. Then, take

$$\begin{aligned} e^{it\mathcal{H}}P_c\vec{\psi}(x) &= \int_\xi \chi(\xi)\tilde{\phi}_\xi^{-1}(x)e^{itW}\int_y \tilde{\phi}_\xi(y)\vec{\psi}(y)dyd\xi & (6.16)\\ &+ \int_\xi [1-\chi(\xi)]\tilde{\phi}_\xi^{-1}(x)e^{itW}\int_y \tilde{\phi}_\xi(y)\vec{\psi}(y)dyd\xi. & (6.17)\end{aligned}$$

Hence, we must bound

$$\begin{aligned} I &= \int_\xi e^{ix\xi}e^{\pm it(\xi^2+\lambda^2)}\int_y e^{-iy\xi}\psi(y)dyd\xi,\\ II &= \int_\xi [\chi(\xi)+(1-\chi(\xi)]g_\xi^{-1}(x)e^{\pm it(\xi^2+\lambda^2)}\int_y e^{-iy\xi}\psi(y)dyd\xi,\\ III &= \int_\xi [\chi(\xi)+(1-\chi(\xi)]e^{ix\xi}e^{\pm it(\xi^2+\lambda^2)}\int_y g_\xi(y)\psi(y)dyd\xi,\\ IV &= \int_\xi [\chi(\xi)+(1-\chi(\xi)]g_\xi^{-1}(x)e^{\pm it(\xi^2+\lambda^2)}\int_y g_\xi(y)\psi(y)dyd\xi. \end{aligned}$$

From henceforward, we work only with the terms

$$\frac{e^{\pm i|x||\xi|}}{|x|}$$

from g_ξ, as the analysis for the exponentially decaying term will follow using simpler versions of the methods for this case. Many of the techniques used are developed from the presentation in (Schlag1). The challenge lies mostly in that $\partial_\xi^\alpha |\xi|$ is not bounded near 0 for $|\alpha| \geq 2$. Thus, we must be careful near the origin using stationary phase arguments since error terms require a minimum of two derivatives. A discussion of stationary phase complete with proofs is given in Appendix C.4. Take the integral,

$$\mathcal{I} = \int g(x) e^{i\tau P(x)} dx,$$

where $g(x) \in C_c^\infty$, $P(x) \in C^\infty$. Assume that $\partial_x P(0) = 0$ and $\partial_x^2 P(0) \neq 0$. Then, the principle of stationary phase gives

$$\mathcal{I} \sim \tau^{-\frac{d}{2}} \sum_{j=0}^{\infty} a_j \tau^{-j},$$

where the asymptotic terms in the stationary phase expansion are given by

$$a_j = L^j g(0),$$

for L^j an order $2j$ differential operator as discussed in Appendix C.4.

Equation I is bounded using standard techniques of contour integration from the Linear Schrödinger equation. In particular, we have

$$\|I\|_{L^\infty} \lesssim t^{-\frac{d}{2}} \|\vec{\psi}\|_{L^1}.$$

Before we investigate further, we recall some properties of the functions $\partial_{\xi_0}^\alpha g_{\xi_0}$. From the expression for g_{ξ_0}, we know that

$$\begin{aligned} [I - K * (V_{\xi_0} \cdot)] g_{\xi_0} &= [I - P_{\xi_0})] g_{\xi_0} \\ &= K * (V_{\xi_0} e^{ix\xi_0}), \end{aligned}$$

so

$$g_{\xi_0} = [I - P_{\xi_0}]^{-1} (K * V_{\xi_0} e^{ix\xi_0}),$$

where

$$K(\xi) = [(-\Delta - \xi^2)(-\Delta + 2\lambda^2 + \xi^2)]^{-1}.$$

From Fredholm Theory and the spectral assumptions on $\mathcal{H}$, $(I - P_{\xi_0})^{-1}$ is well-defined, hence we can show that g_{ξ_0} is smooth in $|\xi_0|$ and ξ_0. Also, K is smooth with respect to the variables $|\xi|$, $V_\xi e^{ix\xi}$ is smooth with respect to ξ. As a result, g_{ξ_0} depends smoothly on $|\xi|$ and ξ. Therefore, for ξ near 0, we can take up to 3 derivatives before we lose integrability in g_ξ. For ξ_0 large enough, from Theorem 18, we have

$$g_{\xi_0} = K * f,$$

where

$$f = e^{ix\cdot\xi_0} f_0(x, \xi_0),$$

where $f_0(x, \xi_0)$ behaves like a symbol in S^2.

For (6.17), we use the principle of nonstationary phase and the principle of stationary phase in different regions. We have

$$\int_\xi [1 - \chi(\xi)] \tilde{\phi}_\xi^{-1}(x) e^{itW} \int_y \tilde{\phi}_\xi(y) \vec{\psi}(y) dy d\xi,$$

where $1 - \chi$ is supported away from 0. In particular, we have integrals of the type

$$\int_\xi [1 - \chi(\xi)](e^{ix\xi} + \tilde{g}_\xi(x)) e^{it(\xi^2+\lambda^2)} \int_y (e^{iy\xi} + g_\xi(y))\psi(y) dy d\xi,$$

where g and $\tilde{g}$ are of the same form described above. Hence, we must bound the following

$$\begin{aligned} II &= \int_\xi [1 - \chi(\xi)] g_\xi(x) e^{it(\xi^2+\lambda^2)} e^{-iy\xi} \psi(y) d\xi, \\ III &= \int_\xi [1 - \chi(\xi)] e^{ix\xi} e^{it(\xi^2+\lambda^2)} g_\xi(y) \psi(y) d\xi, \\ IV &= \int_\xi [1 - \chi(\xi)] g_\xi(x) e^{it(\xi^2+\lambda^2)} g_\xi(y) \psi(y) d\xi. \end{aligned}$$

For integrals of type II and III, we have oscillatory integrals of the form

$$\int e^{i(|z||\xi_0| + (x-z)\xi_0 + t\xi_0^2 - y\xi_0)} \frac{f_0(x - z, \xi_0)}{|z|} d\xi_0. \tag{6.18}$$

Looking at the phase function, we have

$$\begin{aligned} \phi(\xi_0) &= |z||\xi_0| + (x - z)\xi_0 + t\xi_0^2 - y\xi_0, \\ \nabla_{\xi_0}\phi(\xi_0) &= 2t\xi_0 + (x - z - y) + |z|\frac{\xi_0}{|\xi_0|}, \\ \nabla^2_{\xi_0}\phi(\xi_0) &= 2tI_d + \frac{|z|}{|\xi_0|}(I_d - \frac{\xi_0 \otimes \xi_0}{|\xi_0|^2}). \end{aligned}$$

If we restrict ξ_0 to a region such that

$$|\xi_0| \geq \frac{\max|z+y-x|-|z|,0}{2t}+1,$$

then $\phi(\xi_0)$ has no critical points. As a result, we can use the principle of non-stationary phase on this region with the decay properties of the function f_0 to see we have decay like t^{-N} for any N.

Let us hence assume that we are restricted a region

$$|\xi_0| \leq \frac{\max|z+y-x|-|z|,0}{2t}+1,$$

so ϕ has at least one critical point. In fact, the critical point occurs where

$$\xi_0\left(1+\frac{|z|}{|\xi_0|}\right)=z+y-x. \tag{6.19}$$

Also, for $|z+y-x|-|z|\neq 0$, we have

$$|\xi_0|=\frac{|z+y-x|-|z|}{2t}. \tag{6.20}$$

As a result, all critical points occur on the same sphere. Using (6.20), we have that if z, y and x are such that a critical point exists, that critical point is unique. Hence, we can define a cut-off function $\chi_{x,y,z}\in C_c^\infty(\mathbb{R}^d)$ such that

$$\chi_{x,y,z}(\xi)=\begin{cases} 1 \text{ for } |\xi_0|\leq \frac{\max|z+y-x|-|z|,0}{2t}+\frac{1}{2}, \\ 0 \text{ for } |\xi_0|\geq \frac{\max|z+y-x|-|z|,0}{2t}+1. \end{cases}$$

Let us assume that a critical point exists, say ξ_0^c. If $|\xi_0^c|<\frac{|z|}{2t}$, the Hessian matrix is at least of rank 1 as $\xi\otimes\xi$ is a rank 1 matrix. So, there is at least one nondegenerate direction for ξ. After making an orthogonal change of coordinates bringing that nondegenerate direction to ξ_1, using stationary phase on $\mathbb{R}$, we have decay of the form

$$\|(6.18)\|_{L^\infty}\lesssim t^{-\frac{1}{2}}.$$

However, in the integral, we have $\frac{1}{|z|}<\frac{1}{|\xi_0|t}$, so using the decay of f_0 in z, the overall decay is once again

$$\|(6.18)\|_{L^\infty}\lesssim t^{-\frac{d}{2}},$$

where the integral is bounded in the remaining directions. For $|\xi_0^c| > \frac{|z|}{2t}$, the Hessian is nondegenerate. We can thus apply stationary phase in ξ to get decay of the form

$$\|(6.18)\|_{L^\infty} \lesssim t^{-\frac{d}{2}},$$

where we have once again used the regularity of f_0 is x and ξ. Then, given the uniform decay of f_0 and boundedness in y and x, we have uniform boundedness with decay of type $t^{-\frac{d}{2}}$.

The analysis for oscillatory integrals of type IV is similar in that the phase function becomes

$$\begin{aligned}
\phi(\xi_0) &= |z||\xi_0| + (x-z)\xi_0 + t\xi_0^2 - |z_0||\xi_0| + (y-z_0)\xi_0, \\
\nabla_{\xi_0}\phi(\xi_0) &= 2t\xi_0 + (x-z) + (y-z_0) + (|z|-|z_0|)\frac{\xi_0}{|\xi_0|}, \\
\nabla^2_{\xi_0}\phi(\xi_0) &= 2tI_d + \frac{|z|-|z_0|}{|\xi_0|}(I_d - \frac{\xi_0 \otimes \xi_0}{|\xi_0|^2}).
\end{aligned}$$

Hence, where critical points exist, we split up the regions of integration into $|\xi_0| > \frac{|z|-|z_0|}{2t}$ and $|\xi_0| < \frac{|z|-|z_0|}{2t}$. Once again, we have stationary phase in full on the first region and stationary phase in at least one direction, coupled with the fact that $\frac{1}{|z|} < \frac{1}{2|\xi_0|t}$. Away from the critical points, we once again apply non-stationary phase.

Let us now analyze (6.16). In particular, we have integrals of the type

$$\int_\xi [\chi(\xi)](e^{ix\xi} + g_\xi(x))e^{it(\xi^2+\lambda^2)} \int_y (e^{iy\xi} + g_\xi(y))\psi(y)dyd\xi.$$

Thus, we have to bound

$$\begin{aligned}
II &= \int_\xi [\chi(\xi)]g_\xi^{-1}(x)e^{\pm it(\xi^2+\lambda^2)}e^{-iy\xi}d\xi, \\
III &= \int_\xi [\chi(\xi)]e^{ix\xi}e^{\pm it(\xi^2+\lambda^2)}g_\xi(y)d\xi, \\
IV &= \int_\xi [\chi(\xi)]g_\xi^{-1}(x)e^{\pm it(\xi^2+\lambda^2)}g_\xi(y)d\xi.
\end{aligned}$$

For integrals of type II and III, we have an oscillatory integral of the form

$$\int e^{i(|x||\xi_0|+t\xi_0^2-y\xi_0)}\frac{f_0(x-z)}{|x|}d\xi_0. \tag{6.21}$$

The phase function is

$$\begin{aligned}\phi(\xi_0) &= |x||\xi_0| + t\xi_0^2 - y\xi_0,\\ \nabla_{\xi_0}\phi(\xi_0) &= 2t\xi_0 - y + |x|\frac{\xi_0}{|\xi_0|},\\ \nabla^2_{\xi_0}\phi(\xi_0) &= 2tI_d + \frac{|x|}{|\xi_0|}(I_d - \frac{\xi_0\otimes\xi_0}{|\xi_0|^2}).\end{aligned}$$

Let us begin with an integral of type II. After making the orthogonal change of coordinates $\xi_1 \to \frac{y}{|y|}$ and moving to polar coordinates in ξ, we need to bound

$$\begin{aligned}\|II\|_{L^\infty} &\lesssim \|\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\int \chi(r)\frac{e^{-ir|z_0|}}{|z_0|}e^{itr^2}e^{-ir|y|\cos(\theta)}\\ &\times f_0(x-z_0, r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta), r)r^2\sin(\theta)drd\theta d\phi\|_{L^\infty}.\end{aligned}$$

If we Taylor expand f_0 in terms of $(r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta), r)$, then we can integrate in ϕ. In which case, all terms in the expansion with odd powers of $\cos(\phi)$ or $\sin(\phi)$ vanish under integrate out, leaving us with a function of the form

$$\tilde{f_0}(r^2\cos^2(\theta), r).$$

Integrating by parts in θ, we have

$$\begin{aligned}\|II\|_{L^\infty} &\lesssim \|\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\chi(r)\frac{e^{-ir|z_0|}}{|z_0|}e^{itr^2}e^{-ir|y|\cos(\theta)}\\ &\times \tilde{f_0}(x-z_0, r\cos^(\theta), r)r^2\sin(\theta)drd\theta d\phi\|_{L^\infty}\\ &\lesssim \|\int_0^{\infty}\chi(r)\frac{e^{-ir|z_0|}}{|z_0|}e^{itr^2}\frac{\sin(r|y|)}{|y|}\\ &\times \tilde{f_0}(x-z_0, r)rdr\|_{L^\infty}\\ &+ \|\int_0^{\pi}\int_0^{\infty}\chi(r)\frac{e^{-ir|z_0|}}{|z_0|}e^{itr^2}\frac{e^{-ir|y|\cos(\theta)}}{|y|}\\ &\times \partial_\theta\tilde{f_0}(x-z_0, r\cos^2(\theta), r)rdydz_0drd\theta\|_{L^\infty}\\ &= \|\int_0^{\infty}\chi(r)\frac{e^{-ir|z_0|}}{|z_0|}e^{itr^2}\frac{\sin(r|y|)}{|y|}\\ &\times \tilde{f_0}(x-z_0, r)rdr\|_{L^\infty}\\ &+ \|\int_0^{\pi}\int_0^{\infty}\chi(r)\frac{e^{-ir|z_0|}}{|z_0|}e^{itr^2}\frac{\sin(-ir|y|\cos(\theta))}{|y|}\\ &\times \partial_\theta\tilde{f_0}(x-z_0, r\cos^2(\theta), r)rdrd\theta\|_{L^\infty},\end{aligned}$$

since for n odd,

$$\int_{-1}^{1}e^{i\mu x}x^n dx = i\int_{-1}^{1}\sin(\mu x)x^n dx.$$

Note that the boundedness in y and θ is hence maintained after the integration by parts.

Let us extend the region of integration in r to $\mathbb{R}$. Due to the nature of the oscillatory functions involved, we experience no loss in doing so. Then, using the linear Schrödinger equation dispersion, we have

$$\begin{aligned}
\|II\|_{L^\infty} &\lesssim \frac{1}{t}\int_{-\infty}^{\infty} e^{itr^2} \\
&\times \partial_r\left[\frac{e^{-ir|z_0|}}{|z_0|}\frac{e^{-ir|y|}-e^{ir|y|}}{|y|}\chi(r)f_0(x-z_0,r)r\right]drd\theta d\phi\|_{L^\infty} \\
&\lesssim \frac{1}{t^{\frac{3}{2}}}\int_{-\infty}^{\infty}|\int\frac{1}{|z_0||y|} \\
&\times \left[\mathcal{F}^{-1}\left[\chi(r)f_0(x-z_0,r)r\right](u+|z_0|+|y|)(x-z_0)\right. \\
&- \left.\mathcal{F}^{-1}\left[\chi(r)f_0(x-z_0,r)r\right](u+|z_0|-|y|)(x-z_0)|du\right\|_{L^\infty}.
\end{aligned}$$

From the estimate

$$\|\hat{u}\|_{L^1} \lesssim \sup_{|\alpha|\leq d+1}\|\partial^\alpha u\|_{L^1},$$

coupled with the facts that $\chi \in C_0^\infty$, $f_0 \in C_r^\infty$, and f_0 is rapidly decaying in x, we have

$$\|II\|_{L^\infty} \lesssim \frac{1}{t^{\frac{3}{2}}}\|f\|_{L^1}.$$

For the integrals of type III, we immediately apply the linear Schrödinger estimate to get

$$\begin{aligned}
\|III\|_{L^\infty} &\lesssim \frac{1}{t^{\frac{3}{2}}}\int|\int\chi(\xi)\frac{e^{i|y-x_1||\xi|}}{|y-x_1|}e^{iy\xi}f_0(x_1,\xi,|\xi|)f(y)dx_1d\xi|dy \\
&\lesssim \frac{1}{t^{\frac{3}{2}}},
\end{aligned}$$

using once again the smoothness and decay of χ, f_0.

The analysis for oscillatory integrals of type IV is similar to that for type II, except now we have no θ dependence in the phase. Thus, we have phase functions of the form

$$\begin{aligned}
\phi(\xi_0) &= |x||\xi_0| + t\xi_0^2 - |y||\xi_0|, \\
\nabla_{\xi_0}\phi(\xi_0) &= 2t\xi_0 + (|x|-|y|)\frac{\xi_0}{|\xi_0|}, \\
\nabla^2_{\xi_0}\phi(\xi_0) &= 2tI_d + \frac{|x|-|y|}{|\xi_0|}(I_d - \frac{\xi_0\otimes\xi_0}{|\xi_0|^2}).
\end{aligned}$$

At this point, it becomes convenient to move to polar coordinates in ξ. As a result, we have

$$\begin{aligned}\|IV\|_{L^\infty} &\lesssim \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \chi(r)\frac{e^{-ir|z_0|}}{|z_0|}e^{itr^2}\frac{e^{-ir|z_1|}}{|z_1|} \\ &\times f_0(y-z_0, r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta), r) \\ &\times \tilde{f}_0(x-z_1, r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta), r) \\ &\times r^2\sin(\theta)drd\theta d\phi.\end{aligned}$$

Hence, we can first extend the interval of integration in r to $\mathbb{R}$, then immediately integrate by parts in r to gain a factor of $\frac{1}{t}$. We once again apply the linear Schrödinger dispersive estimate to get

$$\|IV\|_{L^\infty} \lesssim \frac{1}{t^{\frac{3}{2}}}\|f\|_{L^1}.$$

Combining the above results, we have

$$\|(6.16)\|_{L^\infty} \leq t^{-\frac{d}{2}}\|\psi\|_{L^1}$$

and

$$\|(6.17)\|_{L^\infty} \leq t^{-\frac{d}{2}}\|\psi\|_{L^1}.$$

Hence, the theorem follows.

□

Before we begin, let us define the space

$$L^{1,M} = \{f \in L^1 | \|\langle\cdot\rangle^N f(\cdot)\|_{L^1} \leq \infty,\ N = 0, 1, \ldots, 2M\},$$

with norm $\|\cdot\|_{L^{1,M}}$ defined standardly. As necessary for the contraction argument presented in Section 7.2, we now show:

Theorem 25. *Let $\mathcal{H}$ be an admissible Hamiltonian as defined above. Assume $\vec{\psi} \in L^{1,M}$ and*

$$\partial_\xi^\alpha \partial_{|\xi|}^\beta \vec{\Psi}(0) = 0, \tag{6.22}$$

for multi-indices α, β such that $|\alpha| + |\beta| = 0, 1, 2, \ldots, 2M$, where

$$\vec{\Psi}(\xi) = \int_y \tilde{\phi}_\xi(y)\vec{\psi}(y)dy.$$

Then,

$$\|e^{-c|x|}e^{it\mathcal{H}}P_c\vec{\psi}\|_{L^\infty} \leq Ct^{-\frac{d}{2}-M}\|\vec{\psi}\|_{L^{1,M}}, \tag{6.23}$$

for any $c > 0$.

Proof of 25. We proceed similarly to the proof of Theorem 24, except now we must bound the following:

$$\begin{aligned} I &= |e^{-c|x|}\int_\xi \chi(\xi)\phi_\xi^{-1}(x)e^{\pm it(\xi^2+\lambda^2)}\int_y \phi_\xi(y)\psi(y)dyd\xi|, \\ II &= |e^{-c|x|}\int_\xi [1-\chi(\xi)]e^{ix\xi}e^{\pm it(\xi^2+\lambda^2)}\int_y g_\xi(y)\psi(y)dyd\xi|. \end{aligned}$$

For II, we look at oscillatory integrals of the form

$$\int_\xi [1-\chi(\xi)](e^{ix\cdot\xi} + g_\xi^{-1}(x))e^{it(\xi^2+\lambda^2)}\int_y (e^{iy\cdot\xi} + g_\xi(y))\psi(y)dyd\xi,$$

considering the phase function as $\phi(\xi) = it\xi^2$. By applying the principle of non-stationary phase from Appendix C.4, then using that

$$\begin{aligned} |\partial_\xi g_\xi| &\leq |\int \frac{\xi}{|\xi|}e^{i|x-y||\xi|}e^{iy\xi}f_0(y,\xi)dy| \\ &+ |\int \frac{\xi}{\sqrt{\xi^2+\lambda^2}}e^{-|x-y|\sqrt{\xi^2+\lambda^2}}e^{iy\xi}f_0(y,\xi)dy| \\ &+ |\int \frac{e^{i|x-y||\xi|} - e^{-|x-y|\sqrt{\xi^2+\lambda^2}}}{|x-y|}e^{iy\cdot\xi}yf_0(y,\xi)dy| \\ &+ |\int \frac{e^{i|x-y||\xi|} - e^{-|x-y|\sqrt{\xi^2+\lambda^2}}}{|x-y|}e^{iy\cdot\xi}\partial_\xi f_0(y,\xi)dy| \\ &\lesssim \int (\langle x\rangle + \langle y\rangle)|f_0(y,\xi)|dy + \int |\partial_\xi f_0(y,\xi)|dy \end{aligned}$$

and the regularity of f_0 in y and ξ_0, we have

$$\|e^{-c|x|}\int_\xi (1-\chi(\xi))\tilde{\phi}_\xi^{-1}(x)e^{itW}\int_y \tilde{\phi}_\xi(y)\vec{\psi}(y)dyd\xi\|_{L^\infty} \lesssim t^{-\frac{d}{2}-M}\|\psi\|_{L^{1,M}}.$$

Now, for I, we need to bound

$$\int_{\xi}[\chi(\xi)](e^{ix\cdot\xi}+g_{\xi}^{-1}(x))e^{it(\xi^2+\lambda^2)}(e^{iy\cdot\xi}+g_{\xi}(y))d\xi,$$

we use stationary phase with the phase function

$$\phi(\xi)=t\xi^2.$$

As a result, we arrive at the moments condition (6.22). This condition serves a dual purpose. First of all, we have a gain in time decay using the stationary phase argument

$$L^j\vec{g}(0)=0,$$

for $j=1,2,\ldots,2M$ is well-defined, where

$$\vec{g}(\xi)=\chi(\xi)\tilde{\phi}_{\xi}^{-1}(x)\int_y\tilde{\phi}_{\xi}(y)\vec{f}(y)dy, \tag{6.24}$$

and L_j is the order $2j$ differential operator resulting from stationary phase. Second of all, we have control near $\xi_0=0$ for the error bound from stationary phase

$$\sum_{|\alpha|\leq 2M+2}|D_{\xi}^{\alpha}\vec{u}|\lesssim 1,$$

where $\xi\in\text{spt}(\chi)$ and

$$u=[\chi(\xi)](e^{ix\cdot\xi}+g_{\xi}^{-1}(x))(e^{iy\cdot\xi}+g_{\xi}(y)).$$

In turn, (6.22) becomes our moments condition for the function space $\mathcal{P}_2^A$ as defined in Chapter 1, Section 1.5, where N is chosen to give a gain of t^{-A} in time decay. Specifically, $A=1$ implies $N=1$ and $A=2$ implies $N=4$. Note that to gain t^{-A}, we must have $N=2A$ at least. Such decay comes at the cost of polynomial growth in x and y from when the ∂_{ξ} terms fall on either $e^{i(\cdot,\xi)}$ or $g_{\xi}(\cdot)$. Hence, under our assumptions, we have

$$\|e^{-c|x|}\int_{\xi}\chi(\xi)\tilde{\phi}_{\xi}^{-1}(x)e^{itW}\int_y\tilde{\phi}_{\xi}(y)\vec{f}(y)dyd\xi\|_{L^{\infty}}\lesssim t^{-\frac{d}{2}-M}.$$

□

6.5 Dispersive Estimates

From Chapter 4, we have $H^1 = M \otimes S$ where M is $2d+4$ dimensional set of functions that span the 4th order generalized null space at 0 and S is the continuous spectrum.

Since M is spanned by functions with exponential decay, we have for $\phi \in M$

$$\|e^{it\mathcal{H}}\phi\|_{H^1} \leq C(1+|t|^3)\int e^{-c|x|}|\phi(x)|dx,$$

where c is determined by the exponential decay of all functions in M.

Now, from (ES1) and 6.3 we have for $\phi \in S$,

$$\|e^{it\mathcal{H}}\phi\|_{L^2} \leq C\|\phi\|_{L^2}. \tag{6.25}$$

Lemma 6.5.1. *Given Equation* (6.25)*, we have*

$$\|e^{it\mathcal{H}}\phi\|_{H^1} \leq C\|\phi\|_{H^1}.$$

Proof. For $\phi \in S$, we have

$$\begin{aligned}
\|e^{it\mathcal{H}}\phi\|_{H^2} &\leq \|\mathcal{H}e^{it\mathcal{H}}\phi\|_{L^2} + C\|e^{it\mathcal{H}}\phi\|_{L^2} \\
&\leq \|e^{it\mathcal{H}}\mathcal{H}\phi\|_{L^2} + C\|e^{it\mathcal{H}}\phi\|_{L^2} \\
&\leq \|\mathcal{H}\phi\|_{L^2} + C\|e^{it\mathcal{H}}\phi\|_{L^2} \\
&\leq \|\phi\|_{H^2} + C\|\phi\|_{L^2} \\
&\leq C\|\phi\|_{H^2}.
\end{aligned}$$

Hence, the result follows from interpolation. □

In order to push through the contraction argument, we need various dispersive estimates from (BouWa). We present the proofs here.

Theorem 26 (Schlag-Erdogan,Bourgain)**.** *Let P_c and P_d be projections onto the continuous*

and discrete spectrum of $\mathcal{H}$ respectively. Then,

$$\begin{aligned}
(i)\ \|e^{it\mathcal{H}}P_c\phi\|_{H^1} &\leq C\|\phi\|_{H^1} \\
(ii)\ \|e^{it\mathcal{H}}(P_c\phi)\|_{H^s} &\leq C\|\phi\|_{H^s} \\
(iii)\ \|e^{it\mathcal{H}}(P_d\phi)\|_{H^s} &\leq C(1+|t|^3)\int e^{-c|x|}|\phi(x)|dx \\
(iv)\ \||x|^\alpha e^{it\mathcal{H}}(P_c\phi)\|_{L^2} &\leq C(\||x|^\alpha\phi\|_{L^2}+(1+|t|^\alpha)\|\phi\|_{H^\alpha} \\
(v)\ \||x|^\alpha e^{it\mathcal{H}}(P_d\phi)\|_{L^2} &\leq C(1+|t|^3)\int |\phi|e^{-c|x|}dx.
\end{aligned}$$

Proof. Estimate (iii) follows from the discrete spectral decomposition into a 4 dimensional generalized null space. The exponential decay is apparent from the properties of the eigenfunctions. Estimate (v) follows similarly.

For $\phi \in \sigma_{ac}(\mathcal{H})$, we have from (ES1) or the analysis in Chapter 7 that

$$\|e^{it\mathcal{H}}P_c\phi\|_{L^2} \leq C\|\phi\|_{L^2}.$$

For $\phi \in \sigma_{ac}(\mathcal{H})$, we have

$$\begin{aligned}
\|e^{it\mathcal{H}}\phi\|_{H^2} &\leq \|\mathcal{H}e^{it\mathcal{H}}\phi\|_{L^2}+C\|e^{it\mathcal{H}}\phi\|_{L^2} \\
&\leq \|e^{it\mathcal{H}}\mathcal{H}\phi\|_{L^2}+C\|e^{it\mathcal{H}}\phi\|_{L^2} \\
&\leq \|\mathcal{H}\phi\|_{L^2}+C\|e^{it\mathcal{H}}\phi\|_{L^2} \\
&\leq \|\phi\|_{H^2}+C\|\phi\|_{L^2} \\
&\leq C\|\phi\|_{H^2}.
\end{aligned}$$

This gives (i). A similar argument shows

$$\|e^{it\mathcal{H}}\phi\|_{H^{2s+1}} \lesssim \|\phi\|_{H^{2s+1}}+\|e^{it\mathcal{H}}\phi\|_{H^{2s-1}}.$$

Thus, by induction, we have (ii) for all positive integers s and hence by interpolation all $s > 0$.

Let $\phi \in \sigma_{ac}(\mathcal{H})$ and $u = e^{it\mathcal{H}}\phi$. Then, since

$$iv_t - \mathcal{H}v = 0,$$

then

$$\begin{aligned}\frac{d}{dt}\int |x|^{2\alpha|}|v(x,t)|^2dx &= 2\,\mathrm{Re}\,\langle |x|^{2\alpha}v, v_t\rangle \\ &= 2\,\mathrm{Im}\,\langle |x|^{2\alpha}v, \mathcal{H}v\rangle \\ &= 2\,\mathrm{Im}\,\langle |x|^{2\alpha}v, \Delta v\rangle + O\left(\int |v|^2 e^{-c|x|}\right) \\ &\lesssim \int |x|^{2\alpha-1}|v||\nabla v|dx + \|v\|_2^2.\end{aligned}$$

Using the following interpolation inequality

$$\||x|^{\alpha-\gamma}|D^\gamma v|\|_{L^2} \le \||x|^\alpha v\|_{L^2}^{1-\frac{\gamma}{\alpha}}\|v\|_{\dot{H}^\alpha}^{\frac{\gamma}{\alpha}},$$

we have

$$\begin{aligned}\int |x|^{2\alpha-1}|v||\nabla v|dx &\le \||x|^\alpha|v|\|_{L^2}\||x|^{\alpha-1}|\nabla v|\|_{L^2} \\ &\le \||x|^\alpha|v|\|_{L^2}^{2-\frac{1}{\alpha}}\|v\|_{\dot{H}^\alpha}^{\frac{1}{\alpha}}.\end{aligned}$$

Hence, using (ii)

$$\frac{d}{dt}[\||x|^\alpha|v(t)|\|_2^2] \lesssim \||x|^\alpha|v|\|_{L^2}^{2-\frac{1}{\alpha}}\|\phi\|_{\dot{H}^\alpha}^{\frac{1}{\alpha}} + \|v\|_{L^2}^2.$$

Integrating, we have

$$\begin{aligned}\||x|^\alpha|v|\|_{L^2L^\infty([0,t])}^2 &\lesssim \||x|^\alpha|\phi|\|_2^2 + \int_0^t [\||x|^\alpha v(s)\|_{L^2}^{2-\frac{1}{\alpha}}\|\phi\|_{\dot{H}^\alpha}^{\frac{1}{\alpha}} + \|v(s)\|_{L^2}^2]ds \\ &\lesssim \||x|^\alpha|\phi|\|_2^2 + \||x|^\alpha|v|\|_{L^2L^\infty([0,t])}^{2-\frac{1}{\alpha}}\int_0^t [\|\phi\|_{\dot{H}^\alpha}^{\frac{1}{\alpha}} + \|\phi\|_{L^2}^2]ds \\ &\lesssim \||x|^\alpha|\phi|\|_2^2 + \epsilon\||x|^\alpha|v|\|_{L^2L^\infty([0,t])}^2 + C(\epsilon)(t^{2\alpha}+t)\||x|^\alpha|\phi|\|_2^2.\end{aligned}$$

Hence, estimate (iv) follows. □

6.6 Strichartz Estimates

From the above time decay, we can also prove the standard space-time Strichartz estimates for $e^{i\mathcal{H}t}\phi$ where $\phi \in P_c\mathcal{H}$. We review the standard methods here as seen in (SulSul). From henceforward, let us assume that we work on the subspace of functions contained in $P_c\mathcal{H}$.

Theorem 27. *For p and p' such that $\frac{1}{p}+\frac{1}{p'}=1$, with $2\le p\le\infty$, and $t\neq 0$, the transformation $e^{i\mathcal{H}t}$ maps continuously $L^{p'}(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ and*

$$\|e^{i\mathcal{H}t}\phi\|_{L^p}\lesssim\frac{1}{|t|^{d(\frac{1}{2}-\frac{1}{p})}}\|\phi\|_{L^{p'}}. \tag{6.26}$$

Proof. This result follows from the interpolation result presented in Appendix B.2. □

Definition 6.6.1. *The pair (q,r) of real numbers is called admissible if $\frac{2}{q}=\frac{d}{2}-\frac{d}{r}$ with $2\le r<\frac{2d}{d-2}$ when $d>2$, or $2\le r\le\infty$ when $d=1$ or $d=2$.*

The following result proving Strichartz estimates is from (Schlag1).

Theorem 28 (Schlag)**.** *For every $\phi\in L^2$ and every admissible pair (q,r), the function $t\to e^{i\mathcal{H}t}\phi$ belongs to $L^q(\mathbb{R},L^r(\mathbb{R}^d))\cap C(\mathbb{R},L^2(\mathbb{R}^d))$, and there exists a constant C depending only on q such that*

$$\|e^{i\mathcal{H}t}\phi\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))}\le C\|\phi\|_{L^2}. \tag{6.27}$$

Proof. Typically, one uses a duality argument when the operator $e^{i\mathcal{H}t}$ is unitary. Namely,

$$|\langle e^{i\mathcal{H}t}\phi,G\rangle_{L^2(\mathbb{R}^{d+1})}|\lesssim\|\phi\|_{L^2}\|G\|_{L^{q'}L^{r'}}.$$

To this end, write

$$\begin{aligned}|\int_{-\infty}^{\infty}\langle e^{i\mathcal{H}t}\phi,G\rangle_{L^2(\mathbb{R}^d)}ds| &= |\left\langle\phi,\int_{-\infty}^{\infty}e^{i\mathcal{H}-s}G(s)ds\right\rangle_{L^2(\mathbb{R}^d)}|\\ &\le \|\phi\|_{L^2(\mathbb{R}^d)}\left\|\int_{-\infty}^{\infty}e^{i\mathcal{H}-s}G(s)ds\right\|_{L^2(\mathbb{R}^d)},\end{aligned}$$

where

$$\begin{aligned}\left\|\int_{-\infty}^{\infty}e^{i\mathcal{H}-s}G(s)ds\right\|^2_{L^2(\mathbb{R}^d)} &= \left\langle\int_{-\infty}^{\infty}e^{i\mathcal{H}-s}G(s)ds,\int_{-\infty}^{\infty}e^{i\mathcal{H}-t}G(t)dt\right\rangle_{L^2(\mathbb{R}^d)}\\ &= \left\langle\int_{-\infty}^{\infty}G(t)dt,\int_{-\infty}^{\infty}e^{i\mathcal{H}t-s}G(s)ds\right\rangle_{L^2(\mathbb{R}^d)}\\ &\le \|G\|_{L^{q'}L^{r'}}\left\|\int_{-\infty}^{\infty}e^{i\mathcal{H}\cdot-s}G(s)ds\right\|_{L^qL^r}.\end{aligned}$$

Using Equation 6.26, we have

$$\begin{aligned}\left\|\int_{-\infty}^{\infty} e^{i\mathcal{H}t-s}G(s)ds\right\|_{L^r} &\leq \int_{-\infty}^{\infty} \left\|e^{i\mathcal{H}t-s}G(s)\right\|_{L^r} ds \\ &\leq \int_{-\infty}^{\infty} \frac{1}{|t-s|^{d(\frac{1}{2}-\frac{1}{r})}} \|G(s)\|_{L^{r'}} ds \\ &\leq \int_{-\infty}^{\infty} \frac{1}{|t-s|^{\frac{2}{q}}} \|G(s)\|_{L^{r'}} ds.\end{aligned}$$

Hence, using the Hardy-Littlewood-Sobolev Theorem with $\gamma = -\frac{2}{q}$,

$$\left\|\int_{-\infty}^{\infty} e^{i\mathcal{H}t-s}G(s)ds\right\|_{L^qL^r} \lesssim \|G\|_{L^{q'}L^{r'}}.$$

However, for systems, this is not applicable. Hence, we must use the Christ-Kiselev Lemma (ChKi).

Lemma 6.6.2. *Let X, Y be Banach spaces and let $K(t,s)$ be the kernel of the operator*

$$K : L^p([0,T];X) \to L^q([0,T];Y).$$

Denote by $\|K\|$ the operator norm of K. Define the lower diagonal operator

$$\tilde{K} : L^p([0,T];X) \to L^q([0,T];Y)$$

to be

$$\tilde{K}f(t) = \int_0^t K(t,s)f(s)ds.$$

Then, the operator $\tilde{K}$ is bounded from $L^p([0,T];X) \to L^q([0,T];Y)$ and it norm $\|\tilde{K}\| \leq c\|K\|$, provided $p < q$.

A perturbative approach originated by Kato is used. Define

$$(SF)(t,x) = \int_0^t (e^{-i(t-s)\mathcal{H}}P_cF(s,\cdot))(x)ds.$$

Then,

$$\|SF\|_{L_t^\infty L_x^2} \lesssim \|F\|_{L_t^1 L_x^2}.$$

Using the fractional integration argument from the unitary case, we have

$$\|SF\|_{L_t^r L_s^p} \lesssim \|F\|_{L_t^{r'} L_x^{q'}},$$

where (r,p) is admissible. By Duhamel, we have

$$e^{-it\mathcal{H}}P_c = e^{-it\mathcal{H}_0}P_c - i\int_0^t e^{-i(t-s)\mathcal{H}_0}Ve^{-is\mathcal{H}}P_c ds.$$

Set $V = \tilde{M}\tilde{M}^{-1}V$, where

$$\tilde{M} = \begin{bmatrix} \langle x\rangle^{-1-} & 0 \\ 0 & \langle x\rangle^{-1-} \end{bmatrix}.$$

Then,

$$\left\| \int_0^\infty e^{-i(t-s)\mathcal{H}_0}\tilde{M}g(s)ds \right\|_{L_t^r L_x^p} \lesssim \left\| \int_0^\infty e^{is\mathcal{H}_0}\tilde{M}g(s) \right\|_{L^2} \lesssim \|g\|_{L_t^2 L_x^2},$$

where the last inequality follows from local smoothing. Applying the Christ-Kiselev lemma, for any Strichartz pair (r,p), we have

$$\left\| \int_0^t e^{-i(t-s)\mathcal{H}_0}\tilde{M}g(s)ds \right\|_{L_t^r L_x^p} \lesssim \|g\|_{L_t^2 L_x^2}.$$

Then,

$$\left\|e^{-it\mathcal{H}}P_c f\right\|_{L_t^r L_x^p} \lesssim \|f\|_{L^2} + \left\|\tilde{M}^{-1}Ve^{-is\mathcal{H}}P_c f\right\|_{L_s^2 L_x^2},$$

so we need

$$\left\|\tilde{M}^{-1}Ve^{-is\mathcal{H}}P_c f\right\|_{L_s^2 L_x^2} \lesssim \|f\|_{L^2}.$$

Taking a Fourier transform in s gives

$$\int_{-\infty}^\infty \|\tilde{M}^{-1}V[P_c(\mathcal{H}-\lambda-i0)P_c]^{-1}P_c f\|_{L^2}^2 d\lambda \lesssim \|f\|_{L^2}^2.$$

However, this follows from the smoothing estimate on $\mathcal{H}_0$, plus the standard resolvent identity under the spectral assumptions on $\mathcal{H}$. Hence,

$$\|e^{-it\mathcal{H}}P_c f\|_{L_t^r L_x^p} \lesssim \|f\|_{L^2}.$$

□

Chapter 7

Nonlinear Theory

7.1 Preliminaries

We wish to construct a contraction argument similar to that presented in (BouWa) in the case where we have a more general nonlinearity. In particular, we have the equation

$$\begin{cases} iu_t + \Delta u + F(|u|^2)u = 0, \\ \qquad u(0,x) = u_0(x), \end{cases}$$

where F is chosen to be of type 1 or type 2.

Many estimates that hold for the L^2 critical equation hold at that soliton because they share the property that $\partial_\lambda Q(u_\lambda) = 0$ where λ is the soliton parameter and Q is the L^2 mass. As this is a minimal energy soliton, there are many possible perturbations. One could perturb onto the manifold of stable solitons, onto the manifold of unstable solitons, or in fact, reduce the L^2 energy so that solitons no longer formed. Unfortunately, due to a lack of scaling and general difficulties, very little is known about stable perturbations to such a soliton. Also, it is a major question whether or not we have dispersion and scattering for initial data with L^2 mass below the minimal soliton energy. We hope to answer this at some point, but for now, we wish to prove the existence of stable solutions to the minimal mass soliton. We may assume that the minimal mass soliton occurs at $\lambda_0 = 1$. In other

words, if R is the desired soliton, we seek a solution of the form

$$u = Re^{it} + z_\phi e^{it} + we^{it},$$

where $w \in C([\frac{1}{\delta}, \infty]; X)$ and $\|w\|_X \leq \frac{1}{t^N}$ for some normed space X and some large N to be determined. The goal is to solve this problem for z solving both

$$\begin{aligned} iz_t + \Delta z &= 0, \\ z(0,x) &= \phi(x), \end{aligned}$$

as well as

$$\begin{aligned} iz_t + Hz &= 0, \\ z(0,x) &= \phi(x), \end{aligned}$$

for H the matrix Hamiltonian that results from linearizing about the minimal mass soliton.

To begin, we run through the contraction argument assuming that we are using the linear Schrodinger operator, $e^{i\Delta t}$, and the space $X = X_A$ defined by

$$X_A = \{\phi | \|\phi\|_{H^A} + \|(1+|x|)^A \phi\|_{L^2} < \infty\}.$$

Let $v_0 = z_\phi e^{-it}$ and let $u(x,t) = e^{it}(R+v)$ for $v = w + v_0$. Then, v must satisfy

$$iv_t + \Delta v - v + [F(|R+v|^2)(R+v) - F(R^2)R] = 0,$$

or

$$iv_t + \Delta v - v + (F(R^2) + F'(R^2)R^2)v + (F'(R^2)R^2)\bar{v} + O(|v|^2) = 0.$$

Since $i(v_0)_t + \Delta(v_0) - v_0 = 0$, we have

$$iw_t + \Delta w - w + [F(|R+v_0+w|^2)(R+v_0+w) - F(R^2)R] = 0.$$

Let

$$\begin{aligned} f_0 &= F(|R+v_0|^2)(R+v_0) - F(R^2)R, \\ a &= [F(|R+v_0|^2) + F'(|R+v_0|^2)|R+v_0|^2] - [F(R^2) + F'(R^2)R^2], \\ b &= F'(|R+v_0|^2)(R+v_0)^2 - F'(R^2)R^2, \\ G(w) &= F(|R+v_0+w|^2)(R+v_0+w) - F(|R+v_0|^2)(R+v_0) \\ &- [F(|R+v_0|^2) + F'(|R+v_0|^2)|R+v_0|^2]w - F'(|R+v_0|^2)(R+v_0)^2\bar{w}. \end{aligned}$$

Then, we have

$$iw_t + \Delta w - w + (F(R^2) + F'(R^2)R^2)w + F'(R^2)R^2\bar{w} + f_0 + aw + b\bar{w} + G(w) = 0.$$

In other words, we have

$$iw_t - Hw + aw + b\bar{w} + f_0 + G(w) = 0,$$

where G is at least quadratic in w and f_0 is linear in v_0.

To see this, note that for nonlinearities of type 1, we have

$$\begin{aligned}
F(x) &= \frac{x^{\frac{p}{2}}}{1 + x^{\frac{p-q}{2}}} \\
F'(x) &= \frac{x^{\frac{p}{2}-1}(\frac{p}{2} + \frac{q}{2}x^{\frac{p-q}{2}})}{(1 + x^{\frac{p-q}{2}})^2} \\
F''(x) &= \frac{x^{\frac{p}{2}-2}(\frac{p}{2}(\frac{p}{2} - 1) + (pq - \frac{q^2}{4} - \frac{q}{2} - \frac{p^2}{2})x^{\frac{p-q}{2}} + (\frac{q^2}{4} - \frac{q}{2})x^{p-q})}{(1 + x^{\frac{p-q}{2}})^3},
\end{aligned}$$

and for type 2,

$$\begin{aligned}
F(x) &= \frac{x}{(1+x)^{\frac{2-q}{2}}} \\
F'(x) &= \frac{1 + \frac{q}{2}x}{(1+x)^{2-\frac{q}{2}}} \\
F''(x) &= \frac{(q-2) + \left(\frac{q^2}{4} - \frac{q}{2}\right)x}{(1+x)^{3-\frac{q}{2}}}.
\end{aligned}$$

Note that in both cases, $F \in C^1$ and in the second case, $F \in C^\infty$. However, we can define $G(z, \bar{z}) = F(|R + z|^2)(R + z)$. This is C^2 at $z = 0$ in both cases. To see this, note

$$\begin{aligned}
\partial_z G &= F'(|R+z|^2)(R+z)(R+\bar{z}) + F(|R+z|^2) \\
\partial_{\bar{z}} G &= F'(|R+z|^2)(R+z)^2 \\
\partial_{zz} G &= 2F'(|R+z|^2)(R+\bar{z}) + F''(|R+z|^2)(R+z)(R+\bar{z})^2 \\
\partial_{\bar{z}\bar{z}} G &= F''(|R+z|^2)(R+z)^3 \\
\partial_{z\bar{z}} G &= 2F'(|R+z|^2)(R+z) + F''(|R+z|^2)(R+z)^2(R+\bar{z}),
\end{aligned}$$

hence at $z = 0$, the terms resulting in exponential growth from F'' are controlled. In the resulting Taylor expansion, we see

$$G(z, z') = F(R^2)R + F'(R^2)R^2\bar{z} + (F'(R^2)R^2 + F(R^2))z + O(|R + z|^{p-1}|z|^2).$$

Let us make the assumption that we are working with type 1 nonlinearities with $\frac{10}{3} < p < \infty$. The author believes that similar results should hold even if p is not restricted to allow more regularity of the nonlinearity, however for the expansions in the sequel to be accurate, we must restrict the nonlinearities to have sufficient regularity, as well as to provide sufficient decay in t.

Let us explore the behaviors of the the above functions. For simplicity, let $v_0 = v_1 + iv_2$. To begin,

$$\begin{aligned} |f_0| &= |F(R^2 + 2Rv_1 + v_1^2 + v_2^2)(R + v_0) - F(R^2)R| \\ &\lesssim |[F(R^2)R + O(|R + v_0|^p|v_0|) - F(R^2)R| \\ &\lesssim O(R^p|v_0|) + O(|v_0|^{p+1})|. \end{aligned}$$

A similar calculation gives that

$$|a| \lesssim O(R^{p-1}|v_0|) + O(|v_0|^p),$$

and similarly for b, we have

$$|b| \lesssim O(R^{p-1}|v_0|) + O(|v_0|^p).$$

Finally, we have

$$|G| \lesssim O(|w|^2).$$

Hence, we solve the integral equation for w

$$w(t) = -i \int_t^\infty e^{i(\tau - t)H}[f_0 + aw + b\bar{w} + G(w)]d\tau.$$

We would like to see that $\|w(t)\|_{X_A} \leq \frac{1}{t^M}$ for some M to be determined in order to show that we have a stable manifold of perturbations on the function space X^A. However, we are actually only able to prove $\|w(t)\|_{X^0} \leq \frac{1}{t^M}$, where $X^A \subset X^0$. The resulting effects of this will appear in Section 7.6.

7.2 Contraction Argument

Making the assumption that

$$\int x^{\beta}\phi(x)dx = 0,$$

for the multi-index $|\beta| = 0, 1, 2, ..., 2M$ for M to be determined, we have by Taylor expanding the exponential in the fundamental solution that

$$z_{\phi}(x,t) = O\left(\frac{1}{t^{M+\frac{d}{2}}}\right)$$

where $|x| \lesssim 1$.

We have that

$$\begin{aligned} \|D^{\alpha}v_0\|_{L^{\infty}} &\leq \frac{C}{t^{\frac{d}{2}p}}, \\ |e^{-c|x|}D^{\alpha}v_0(x,t)| &< \frac{C}{t^{N}}, \end{aligned}$$

for t and N large and the range of α's to be determined. In general, we explore low regularity perturbations but are in need of L^{∞} bounds, so α will be small, but positive.

Then, we have for s in some range to be determined

$$\|(Rv_0)(t)\|_{H^s} \leq \frac{C}{t^{M+\frac{d}{2}}},$$

and hence, we have to check that for our space X^A,

$$\|f_0\|_{X^A} \leq \frac{C}{t^{\frac{d}{2}(p+1)}}.$$

where p is determined by the supercritical power in the nonlinearity and we have assumed the moments condition above for all $|\beta| \leq 2M$. In particular, note that $\gamma > \frac{4}{d}$, hence these terms have at least quadratic decay in t of the L^{∞} norm.

We also have

$$|a| = O(R^{p-1}|v_0| + O(|v_0|^p)),$$

hence

$$\begin{aligned} |D^{\alpha}a(x,t)| &\leq \frac{C}{t^{\frac{d}{2}p}}, \\ |e^{-c|x|}D^{\alpha}a(x,t)| &\leq \frac{C}{t^{M+\frac{d}{2}}}. \end{aligned}$$

Similarly,

$$\begin{aligned} |D^\alpha b(x,t)| &\leq \frac{C}{t^{\frac{d}{2}p}}, \\ |e^{-c|x|}D^\alpha b(x,t)| &\leq \frac{C}{t^{M+\frac{d}{2}}}. \end{aligned}$$

Now, we look at the integral formulation of the equation for w

$$\begin{aligned} w(t) &= -i\int_t^\infty e^{i(\tau-t)\mathcal{H}}P_d[f_0+aw+b\bar{w}+G(w)](\tau)d\tau \qquad (7.1) \\ &+ -i\int_t^\infty e^{i(\tau-t)\mathcal{H}}P_c[f_0+aw+b\bar{w}+G(w)](\tau)d\tau, \qquad (7.2) \end{aligned}$$

where P_S projects onto the singular part of the spectrum and P_M projects onto the discrete part of the spectrum.

Since we are interested in minimal regularity perturbations, we first want to see that

$$\|w(t)\|_{H^2} \leq \frac{1}{t^{N_1}},$$

for $t \geq \frac{1}{\delta}$ and N_1 to be determined.

To do this, we will discuss Equations (7.1) and (7.2) separately.

From Corollary 6.3.7

$$\begin{aligned} \|(7.1)\|_{H^2} &\leq \int_t^\infty [1+(\tau-t)^3]\left\{\int |f_0+aw+b\bar{w}+G(w)](x,\tau)|e^{-c|x|}dx\right\}dt \\ &\leq \int_t^\infty [1+(\tau-t)^3][\frac{C}{\tau^{M+\frac{d}{2}}}+\frac{C}{\tau^{M+\frac{d}{2}}}\|w(\tau)\|_{H^2}+C\|w(\tau)\|_{H^2}^2]d\tau. \end{aligned}$$

Hence, by assuming M, N_1 large enough and using a bootstrapping argument

$$\|(7.1)\|_{H^2} \leq \int_t^\infty [1+(\tau-t)^3]\left[\frac{C}{\tau^{2N_1}}\right]d\tau < \frac{C}{t^{2N_1-4}} < \frac{C\delta}{t^{N_1}}.$$

For the second part of this argument, we see

$$\begin{aligned} \|(7.2)\|_{H^2} &\leq \int_t^\infty \|f_0+aw+b\bar{w}+G(w)](\tau)\|_{H^2}d\tau \\ &\leq \int_t^\infty \left\{\frac{C}{\tau^{\frac{d}{2}p}}+\frac{C}{\tau^{\frac{d}{2}p}}\|w(\tau)\|_{H^2}+C\|w(\tau)\|_{H^2}^2\right\}d\tau \\ &\leq \int_t^\infty \left\{\frac{C}{\tau^{\frac{d}{2}p}}+\frac{C}{\tau^{N_1+\frac{d}{2}p}}+\frac{C}{\tau^{2N_1}}\right\}d\tau \\ &\leq \frac{C}{t^{N_1+1}} \leq \frac{C\delta}{t^{N_1}}, \end{aligned}$$

provided $\frac{d}{2}p$ is large enough.

We are also be able to show

$$\|w\|_{L^2(|x|^A dx)} \leq \frac{C}{t^{N_2}}$$

for $t > \frac{1}{\delta}$ and N_2 to be determined. Then, we will have the desired contraction argument for the linear perturbation.

From the necessary dispersive estimate given by

$$\||x|^\alpha e^{itH}(P_S\phi)\|_{L^2} \leq C(1+|t|^3)\int |\phi| e^{-c|x|} dx,$$

the estimate for Equation (7.1) follows immediately from the H^s argument.

For Equation (7.2), we need the following estimate

$$\||x|^\alpha e^{itH}(P_M\phi)\|_{L^2} \leq C\||x|^\alpha\phi\|_{L^2} + C(1+|t|^\alpha)\|\phi\|_{H^\alpha}.$$

Then,

$$\begin{aligned}
\|(7.2)\|_{L^2(|x|^A dx)} &\leq C\int_t^\infty \|[f_0 + aw + b\bar{w} + G(w)](\tau)\|_{L^2(|x|^A dx)} d\tau & (7.3)\\
&+ \int_t^\infty (1+|t-\tau|^A)\|[f_0 + aw + b\bar{w} + G(w)](\tau)\|_{H^A} d\tau. & (7.4)
\end{aligned}$$

Again, we look at each integral separately. For Equation (7.3),

$$\begin{aligned}
(7.3) &\leq \int_t^\infty \left\{ \frac{C}{\tau^{2N}} + \frac{C}{\tau^2}\|w(\tau)\|_{L^2(|x|^A dx)} + C\|w(\tau)\|_{L^\infty}\|w(\tau)\|_{L^2(|x|^A dx)} \right\} d\tau \\
&< \left\{ \frac{C}{\tau^{2N}} + \frac{C}{\tau^{2+N1}} + \frac{C}{\tau^{N+N1}} \right\} d\tau \\
&< \frac{C}{t^{N1+1}} < \frac{C\delta}{t^{N1}}.
\end{aligned}$$

For (7.4),

$$\begin{aligned}
(7.4) &\leq C\int_t^\infty [(1+(\tau-t)^A]\left\{ \frac{C}{\tau^{\frac{d}{2}p+A}} + C\|w(\tau)\|_{H^A}^2 \right\} d\tau \\
&\leq C\int_t^\infty \tau^A \left\{ \frac{C}{\tau^{2N_2}} + \frac{C}{\tau^{\frac{d}{2}p}} \right\} d\tau \\
&\leq \frac{C}{t^{2N_2-A-1}} < \frac{C\delta}{t^{N_2}},
\end{aligned}$$

for N_2, p sufficiently large.

Hence, the contraction argument goes through and we have the desired bound on $\|w\|_X$.

7.3 Optimization for LS

We seek optimal values for the spaces and decay in the case where the perturbation solves the linear Schrodinger equation.

To begin, allow A and M to be arbitrary for now and we will select them later. Assume that $\phi \in X_A$ and that the first $2M$ moments of ϕ vanish. By writing the linear solution in integral form, we see that

$$\begin{aligned} \|D^\alpha v_0\|_{L^\infty} &\leq \frac{C}{t^{\frac{d}{2}}}, \\ |e^{-c|x|} D^\alpha v_0(x,t)| &\leq \frac{C}{t^{\frac{d}{2}+M}}, \end{aligned}$$

for $\alpha < M$.

To gain in time decay for the linear Schrödinger equation, we make the assumption that

$$\int x^\alpha \phi(x) dx = 0, \tag{7.5}$$

where α is a multi-index where $|\alpha| \leq M$ for M to be determined below. Note that the function space $\mathcal{P}_1^A$ in Section 1 is determined by functions $\phi \in X^A$ coupled with taking moments conditions for $|\alpha| \leq A$.

Lemma 7.3.1. *Since $R \in \mathcal{S}$,*

$$\|(Rv_0)(t)\|_{H^s} \leq \frac{C}{t^M}.$$

Proof. We have

$$\|(Rv_0)(t)\|_{L^2} \leq \|\langle x \rangle^{-N} v_0\|_{L^\infty} \|\langle x \rangle^N R\|_{L^\infty}.$$

Hence, using the principal of nonstationary phase away from the origin and the moments condition near the origin in the fundamental solution for linear Schrödinger, we gain in time decay. Note that in order to gain in time decay away from the origin, it is essential that we have the weight in order to control all of the terms resulting from integrating by parts. The higher derivative terms follow similarly.

□

Lemma 7.3.2. *For f_0 described above for $v_0 = e^{i\Delta t}\phi$, we have*

$$\|f_0\|_{H^2} \leq \frac{C}{t^{\frac{d}{2}p}}$$

and

$$\|e^{-c|x|}f_0\|_{L^\infty} \leq \frac{C}{t^{(\frac{d}{2}+M)p}}.$$

Proof. The $O(Rv_0)$ term is controlled by similar analysis to that in 7.3.1. Hence, we concern ourselves with the $O(|v_0|^{p+1})$ term. To that end, we have

$$\|v_0^{p+1}\|_{L^2} \leq \|v_0\|_{L^\infty}^p \|v_0\|_{L^2}.$$

Consequently, we have

$$\|v_0^{p+1}\|_{L^2} \lesssim \langle t\rangle^{-\frac{d}{2}p}.$$

□

Once again, since the decay rate is determined by the number of moments for the linear equation,

$$\begin{aligned} |e^{-c|x|}D^\alpha a(x,t)| &\leq \frac{C}{t^{M+\frac{d}{2}}}, \\ |e^{-c|x|}D^\alpha b(x,t)| &\leq \frac{C}{t^{M+\frac{d}{2}}}. \end{aligned}$$

As we desire to work with low regularity perturbations, let us simply assume that $\|w\|_{H^2}^2 \lesssim t^{-N}$. Now, we must choose A and M optimally for the contraction argument to work. From Equation (7.1), we require that

$$\begin{aligned} 2N - 4 &\geq N, \\ \frac{d}{2} + M - 4 &\geq N, \end{aligned}$$

where the moments condition is determined by the $O(v_0R^p)$ term. So, we gather that $N > 4$ and $M > 8 - \frac{d}{2}$. The number of moments necessary will depend upon the the dimension d. In $\mathbb{R}^3$, we have $M > \frac{13}{2}$.

From Equation (7.2), we have only one more requirement

$$\frac{d}{2}p - 1 > N.$$

At this stage, we see that given $N > 4$, we need $p > \frac{10}{3}$. Clearly, the restrictions on p lessen as d gets large. In particular, we cannot show the existence of stable perturbations for minimal mass solitons of NLS equations with nonlinearities of type 2 in $\mathbb{R}^3$. A variation of this argument will be explored later to show long time stability under restricted perturbations.

7.4 Linearization Scheme for $\mathcal{H}$-LS perturbations

Again, let

$$u = Re^{it} + z_\phi e^{it} + we^{it},$$

except now we have

$$iz_t + \mathcal{H}z = 0, \tag{7.6}$$

$$z(0,x) = \phi(x), \tag{7.7}$$

where $\mathcal{H}$ is linear operator resulting from linearizing about the minimal mass soliton. We refer to Equation (7.6) as the $\mathcal{H}$linear Schrödinger equation ($\mathcal{H}$-LS).

Now, let $v_0 = z_\phi e^{-it}$. Again, we have the same equation,

$$iv_t + \Delta v - v + F(|R+v|^2)(R+v) - F(R^2)R = 0,$$

where $u(x,t) = e^{it}(R+v)$.

However, since

$$i(v_0)_t + \Delta v_0 - v_0 + [F(R^2) + F'(R^2)R^2]v_0 + F'(R^2)R^2\overline{v_0} = 0,$$

we have

$$\begin{aligned} iw_t &+ \Delta w - w + [F(|R+v_0+w|^2)(R+v_0+w) - F(R^2)R - (F(R^2) + F'(R^2)R^2)v_0 \\ &- F'(R^2)R^2\overline{v_0}] = 0. \end{aligned}$$

Hence, let

$$\begin{aligned} f_0 &= F(|R+v_0|^2)(R+v_0) - F(R^2)R - (F(R^2)+F'(R^2)R^2)v_0 - F'(R^2)R^2\overline{v_0}, \\ a &= [F(|R+v_0|^2)+F'(|R+v_0|^2)|R+v_0|^2] - [F(R^2)+F'(R^2)R^2], \\ b &= F'(|R+v_0|^2)(R+v_0)^2 - F'(R^2)R^2, \\ G(w) &= F(|R+v_0+w|^2)(R+v_0+w) - F(|R+v_0|^2)(R+v_0) \\ &- [F(|R+v_0|^2)+F'(|R+v_0|^2)|R+v_0|^2]w - F'(|R+v_0|^2)(R+v_0)^2\bar{w}. \end{aligned}$$

Hence, we now have

$$\begin{aligned} |f_0| &\lesssim O(R^p|v_0|^2) + O(|v_0|^{p+1})|, \\ |a| &\lesssim O(R^{p-1}|v_0|) + O(|v_0|^p), \\ |b| &\lesssim O(R^{p-1}|v_0|) + O(|v_0|^p) \\ |G| &\lesssim O(|w|^2). \end{aligned}$$

Notice that since the linear terms in v_0 have been removed from f_0, we expect to require fewer moments conditions for the contraction argument to hold.

7.5 Optimization for $\mathcal{H}$-LS

In the scheme where we solve the linear perturbation using $\mathcal{H}$, we have now introduced gain in the f_0 term. Specifically, we now have

$$|f_0| \lesssim O(R|v_0|^2) + O(|v_0|^{p+1}).$$

In the $\mathcal{H}$-LS case, we use the moment conditions derived in Section 6.4 in order to gain decay in time locally in space. Note that the function space $\mathcal{P}_2^A$ in Theorem 2 is determined by functions $\phi \in X^A$ coupled with taking moments conditions for $|\beta| \leq A$.

Lemma 7.5.1. *Since $R \in \mathcal{S}$, if v_0 satisfies the first M moments conditions from Section 6.4*

$$\|(Rv_0)(t)\|_{H^s} \leq \frac{C}{t^{\frac{d}{2}+M}}.$$

Proof. We have

$$\|(Rv_0)(t)\|_{L^2} \leq \|\langle x\rangle^{-N} v_0\|_{L^\infty} \|\langle x\rangle^N R\|_{L^\infty}.$$

Hence, using the principal of nonstationary phase away from the origin and the moments condition near the origin on Equation (6.14), we gain in time decay. Note that in order to gain in time decay away from the origin, it is essential that we have the weight in order to control all of the terms resulting from integrating by parts. The higher derivative terms follow similarly.

□

Lemma 7.5.2. *For f_0 described above for $v_0 = e^{i\mathcal{H}t}\phi$, we have*

$$\|f_0\|_{X_A} \leq \frac{C}{t^{\frac{d}{2}p}},$$

for $s < M$.

Proof. The $O(Rv_0)$ term is controlled by similar analysis to that in 7.5.1. Hence, we concern ourselves with the $O(|v_0|^{p+1})$ term. To that end, we have

$$\|v_0^{p+1}\|_{L^2} \leq \|v_0\|_{L^\infty}^p \|v_0\|_{L^2}.$$

Using a similar analysis from Lemma 7.3.2 on Equation (6.14), we have

$$\|v_0^{p+1}\|_{L^2} \lesssim \langle t\rangle^{\frac{d}{2}p}$$

using the L^2 boundedness results for $e^{i\mathcal{H}t}$.

□

Now, since we are dealing with nonlinearities with minimal smoothness, we wish to run the contraction argument with minimal assumptions on $w(x,t)$. Then, we assume

$$\|w(x,t)\|_{L^2} < \frac{1}{t^N},$$

for some N to be determined. Then,

$$w(t) = -i\int_t^\infty e^{i(\tau-t)H} P_d[f_0 + aw + b\bar{w} + G(w)](\tau)d\tau \tag{7.8}$$

$$+ \quad -i\int_t^\infty e^{i(\tau-t)H} P_c[f_0 + aw + b\bar{w} + G(w)](\tau)d\tau, \tag{7.9}$$

where P_d projects onto the discrete part of the spectrum and P_c projects onto the continuous part of the spectrum. So, using the dispersive estimates, we have

$$\begin{aligned}
\|(7.8)\|_{H^2} &\leq \|\int_t^\infty e^{i(\tau-t)H}P_S[f_0+aw+b\bar{w}+G(w)](\tau)d\tau\|_{H^2} \\
&\leq \int_t^\infty [1+(\tau-t)^3]\left\{\int |f_0+aw+b\bar{w}G(w)](x,t)e^{-c|x|}dx\right\}d\tau \\
&\leq \int_t^\infty [1+(\tau-t)^3]\left\{\int |f_0+aw+b\bar{w}G(w)](x,t)e^{-c|x|}dx\right\}d\tau \\
&\lesssim \int_t^\infty [1+(\tau-t)^3]\left\{\frac{1}{\tau^{2N}}+\frac{1}{\tau^N}\|w(\tau)\|_{L^2}+\|w(\tau)\|_{L^2}^2\right\}d\tau \\
&\lesssim \frac{1}{t^{(\frac{d}{2}+M)2-4}}+\frac{1}{t^{(\frac{d}{2}+M)p+N-4}}+\frac{1}{t^{2N-4}}, \\
\|(7.9)\|_{L^2} &\leq \|\int_t^\infty e^{i(\tau-t)H}P_M[f_0+aw+b\bar{w}+G(w)](\tau)d\tau\|_{L^2} \\
&\leq \int_t^\infty \|\int |f_0+aw+b\bar{w}G(w)](x,t)e^{-c|x|}dx\|_{L^2}d\tau \\
&\leq \left\{\frac{1}{\tau^{\frac{d}{2}p}}+\frac{1}{\tau^N}\|w(\tau)\|_{H^2}+\|w(\tau)\|_{H^2}^2\right\} \\
&\lesssim \frac{1}{t^{2N-1}}+\frac{1}{t^{M+N-1}}+\frac{1}{t^{2M-1}}.
\end{aligned}$$

Hence, we require once again that that $2N-4\geq N$, but the moments condition is determined now by the $O(v_0^2R^{p-1})$ term, so we have

$$d+2M-4>4.$$

In $\mathbb{R}^3$ that $M>\frac{5}{2}$, or $M>2$. The condition on p however does not change whatsoever, therefore are again only considering nonlinearities of type 1 with $p>\frac{10}{3}$.

7.6 Manifolds of Perturbations

From Theorems 1 and 2, we would like to know that our perturbative solution actually lives on a finite codimension submanifold. Specifically, given spaces X_1, X_2 with $X_1\subset X_2$ and norms $\|\cdot\|_{X_1}$, $\|\cdot\|_{X_2}$ respectively, we require a finite codimension subset $\mathcal{S}\subset X_1$ and a map $\Psi:\mathcal{B}\cap\mathcal{S}\to X_2$ where

$$\mathcal{B}=\{\phi\in X_1|\|\phi\|_{X_1}<\delta\}.$$

For Theorems 1 and 2 above, we have

$$\Psi(\phi) = w(t_0). \tag{7.10}$$

Lemma 7.6.1. *For the map Ψ defined by* (7.10)*, we have*

$$\|\Psi(\phi)\|_{X_2} \lesssim \|\phi\|_{X_1}^2,\ \phi \in \mathcal{B} \cap \mathcal{S}, \tag{7.11}$$

$$\|\Psi(\phi_1) - \Psi(\phi_2)\|_{X_2} \lesssim \delta\|\phi_1 - \phi_2\|_{X_1},\ \phi_1, \phi_2 \in \mathcal{B} \cap \mathcal{S}, \tag{7.12}$$

where

$$X_1 = L^2(|x|^{3+}dx) \cap H^2, \tag{7.13}$$

$$X_2 = H^2, \tag{7.14}$$

and

$$\mathcal{S} = \{\phi \in H^2 | \phi = P_c\phi,\ \phi \in \mathcal{P}_2^A\} \tag{7.15}$$

for some $A > 2$.

Remark 7.1. *In Lemma 7.6.1, Equation* (7.11) *shows that the tangent space at* 0 *of the stable submanifold, $\mathcal{M}$, is the space $\mathcal{S}$, while Equation* (7.12) *shows that $\mathcal{M}$ is given by a Lipschitz parametrization.*

Remark 7.2. *The codimension of $\mathcal{S}$ will be at most $2d + 4$ since $H^1 \times H^1 = N_g(\mathcal{H}) \oplus \{N_g(\mathcal{H}^*)\}^\perp$ and $N_g(\mathcal{H}) = 2d + 4$. It is possible that the size of $\mathcal{S}$ can be improved beyond this codimension, which the author will explore in future work.*

Proof of Lemma 7.6.1. Let us first prove (7.11). Assume that $\|w\|_{X_2} \leq \|\phi\|_{X_1}^2$. Then,

$$\begin{aligned}
\|w\|_{X_2} &\leq \int_{t_0}^{\infty} (1 + (t_0 - \tau)^3) \left[\int v_0^2 e^{-c|x|} dx + \int v_0^2 e^{-c|x|} dx + \int v_0^2 e^{-c|x|} dx \right] d\tau \\
&+ \int_{t_0}^{\infty} [\|f_0\|_{X_2} + \|aw\|_{X_2} + \|b\bar{w}\|_{X_2} + \|G(w)\|_{X_2}]\, d\tau \\
&\lesssim t_0^{-\epsilon} \|\phi\|_{X_1}^2
\end{aligned}$$

using our assumptions as well as the decay of $\|w\|_{H^2}$ from the proof of Theorem 2. Hence, taking t_0 to be large, the result follows.

Now, for (7.12), we have

$$w_1 - w_2 = \int_{t_0}^{\infty} e^{i\mathcal{H}(\tau - t_0)}[(f_0^1 - f_0^2) + a(w_1 - w_2) + b(\bar{w}_1 - \bar{w}_2) + (G(w_1) - G(w_2))]d\tau.$$

Since

$$|G(w_1) - G(w_2)| \sim |w_1 + w_2||w_1 - w_2|$$

and

$$|f_0^1 - f_0^2| \sim R|\phi_1 + \phi_2||\phi_1 - \phi_2| + (|\phi_1|^p + |\phi_2|^p)|\phi_1 - \phi_2|,$$

the result follows from a similar continuity argument to that above using (7.11). □

Remark 7.3. *Note that for p large enough in type 1 nonlinearities, using the dispersive estimates (iv), (v) from Theorem 26 and the fact that*

$$\|\phi\|_{L^1} \leq C(\epsilon)\|\phi\langle x\rangle^{d+\epsilon}\|,$$

we can take $X_1 = X_2 = X^{d+}$ to have a true manifold of perturbations.

7.7 Long Time Analysis for Type 2 Nonlinearities

For NLS with saturated nonlinearities of type 2, we can no longer do the global scattering analysis from above. Instead, we have Theorem 3:

Theorem 29. *Given Equation (1.8), where β is an admissible saturated nonlinearity of type 2, for any $\phi = P_c\phi \in W^{2,1} \cap H^2$ with $\|\phi\|_{W^{2,1}\cap H^2} < \delta < 1$, Equation (1.8) has a solution for $t \in [0, \left(\frac{1}{2\delta}\right)^{\frac{1}{4}}$ of the form*

$$u(x,t) = R_{min} + v(t) = R_{min} + e^{i\mathcal{H}t}\phi + w(x,t),$$

where

$$u(x,0) = R_{min} + \phi.$$

Proof. Instead of the scattering point of view, we look at solving for the perturbation forward in time. Namely, we have

$$\begin{aligned} w(t) &= i\int_0^t e^{i\mathcal{H}(t-\tau)}[f_0 + aw + b\bar{w} + G(w)](\tau)d\tau \\ &= i\int_0^t e^{i\mathcal{H}(t-\tau)}P_d[f_0 + aw + b\bar{w} + G(w)](\tau)d\tau \\ &+ i\int_0^t e^{i\mathcal{H}(t-\tau)}P_c[f_0 + aw + b\bar{w} + G(w)](\tau)d\tau. \end{aligned}$$

Let us assume that

$$\|w\|_{L^\infty[0,T]H^2} < \delta, \tag{7.16}$$

$$\|\phi\|_{W^{1,2}\cap H^2} \leq \delta^{\frac{1}{2}}. \tag{7.17}$$

Then

$$\begin{aligned} \|w\|_{L^\infty[0,T]H^2} &\lesssim \Big\| \int_0^t [1+(t-\tau)^3]\left(\int f_0 e^{-c|x|}dx + \int awe^{-c|x|}dx\right. \\ &+ \left.\int b\bar{w}e^{-c|x|}dx + \int G(w)e^{-c|x|}dx\right)d\tau \\ &+ \int_0^t [\|f_0\|_{H^2} + \|aw\|_{H^2} + \|b\bar{w}\|_{H^2} + \|G(w)\|_{H^2}]d\tau\Big\|_{L_t^\infty[0,T]} \\ &\lesssim \Big\| \int_0^t [1+(t-\tau)^3]\left(\langle\tau\rangle^{-3(M+\frac{d}{2})}\|\phi\|_{L^1}^3\right. \\ &+ \left.\langle\tau\rangle^{-(M+\frac{d}{2})}\|w\|_{H^2}\|\phi\|_{L^1} + \|w\|_{H^2}^2\right)d\tau \\ &+ \int_0^t [\langle\tau\rangle^{-d}\|\phi\|_{L^1}^2\|\phi\|_{H^2} + \langle\tau\rangle^{-\frac{d}{2}}\|\phi\|_{L^1}\|w\|_{H^2} + \|w\|_{H^2}^2]d\tau\Big\|_{L_t^\infty[0,T]} \\ &\lesssim T^4\|w\|_{L^\infty[0,T]H^2} + T^4\|\phi\|_{L^1}^2\|w\|_{L^\infty[0,T]H^2} + T^4\|\phi\|_{L^1}^3 \\ &\lesssim \frac{\delta}{2}. \end{aligned}$$

Hence, by a continuity argument, w exists on $[0,T]$ with $\|w\|_{L^\infty[0,T]H^2} \leq \delta$. □

Appendix A

Existence, Uniqueness, and Continuity of Solitons

A.1 Existence

We prove the existence in dimension $d \geq 3$ of a unique C^2, postive, radial, exponentially decreasing solution to the equation

$$-\Delta u = g(u), \tag{A.1}$$

for $u \in H^1$ and $g(0) = 0$. We follow the fundamental work of (BerLion). Specifically, g satisfies

$$\begin{aligned} -\infty &< \liminf_{s\to 0^+} \frac{g(s)}{s} \leq \limsup_{s\to 0^+} \frac{g(s)}{s} = -m < 0, \\ -\infty &\leq \limsup_{s\to\infty} \frac{g(s)}{s^{\frac{d+2}{d-2}}} \leq 0, \end{aligned}$$

and there exists $\eta_0 > 0$ such that

$$G(\eta_0) = \int_0^{\eta_0} g(s)ds > 0.$$

Note that we have $g(s) = -\lambda^2 s + \beta(s^2)s$ and all of the necessary assumptions are satisfied.

As stated, this reduces to a constrained minimization problem. To begin, we define $\tilde{g} : \mathbb{R} \to \mathbb{R}$ as follows.

Definition A.1.1. *(i) If $g(s) \geq 0$ for all $s > \eta_0$, set $\tilde{g} = g$, and*

(ii) if there exists $s_0 > \eta_0$ such that $g(s_0) = 0$, then

$$\tilde{g}(s) = \begin{cases} g(s) \text{ on } [0, s_0] \\ 0 \text{ for } s \geq s_0. \end{cases}$$

For $s \leq 0$, $\tilde{g}(s) = -g(-s)$.

Note that $\tilde{g}$ satisfies the requirements for g and by the maximum principle, solutions to Equation (A.1) with $\tilde{g}$ are solutions with g. Indeed, if β grows sufficiently that we are in case (ii), we have that a solution, u, would satisfy $|u| < s_0$. In other words, the value of u would never leave the region where $\tilde{g} = g$. To see this, let $E = \{x||u(x)| > s_0\}$. This set is open and bounded since $u \in H^1$. However, on E, $\Delta u = 0$, so u is harmonic and satisfies that maximum or minimum occurs on the boundary. However, the boundary is exactly where $|u| = s_0$.

For simplicity, henceforward we set $\tilde{g} = g$. Therefore, g satisfies a stronger condition

$$\lim_{s \to \pm\infty} \frac{|g(s)|}{|s|^{\frac{d+2}{d-2}}} = 0. \tag{A.2}$$

Claim A.1.2. *Under the stronger condition* (A.2)*, the functional*

$$V(w) = \int G(w)dx$$

is well-defined and C^1 on H^1.

Proof. First of all, the fact that V is well-defined on H^1 is clear. To prove the functional is C^1, we prove first that for u, $v \in H^1$,

$$(i)\ |\frac{1}{t}\{V(u+tv) - V(u) - t\int g(u)vdx\}| \to 0,$$

as $t \to \infty$, $t > 0$.

In this direction, we have using a Taylor expansion

$$\begin{aligned} |\frac{1}{t}\{V(u+tv) - V(u) - t\int g(u)vdx\}| &\leq \int |[G(u+tv) - G(u) - tg(u)v]\frac{1}{t}|dx \quad (A.3) \\ &\leq \int [\sup_{t\in[0,1]} |g(u+tv)| + |g(u)|]|v| \quad (A.4) \\ &\leq C[|u| + |v| + |u|^{\frac{d+2}{d-2}} + |v|^{\frac{d+2}{d-2}}]|v|, \quad (A.5) \end{aligned}$$

since $|g(s)| \leq C[|s| + |s|^{\frac{d+2}{d-2}}]$ by assumption.

Set $h = C[|u| + |v| + |u|^{\frac{d+2}{d-2}} + |v|^{\frac{d+2}{d-2}}]|v|$. From a Sobolev embedding, we have $h \in L^1$.

Since, u, $v \in H^1$ and G is C^1, we know that

$$|[G(u+tv) - G(u) - tg(u)v]\frac{1}{t}| \to 0$$

almost everywhere. Hence, by the dominated convergence theorem and the bound (A.3), (i) is proved.

Next we must see that if $u_n \to u$ in H^1, then

$$(ii) \quad \sup_{\|v\|_{H^1} \leq 1} |\int [g(u_n) - g(u)]dx| \to 0,$$

for $n \to \infty$.

Let $\epsilon > 0$. We prove there exists an R_0 such that

$$\sup_{\|v\|_{H^1} \leq 1} |\int_{|x| \geq R_0} [g(u_n) - g(u)]dx| \leq \epsilon.$$

Since $u_n \to u$ in H^1, we have $u_n \to u$ in $L^{\frac{d+2}{d-2}+1} = L^{2^*}$. Taking subsequences if need be, we have positive functions $f_1 \in L^{\frac{d+2}{d-2}+1}$ and $f_2 \in L^2$ such that

$$\begin{aligned} |u|, |u_n| &\leq f_1 \\ |u|, |u_n| &\leq f_2, \end{aligned}$$

almost everywhere in $\mathbb{R}^d$. Then, for any R,

$$\begin{aligned} \sup_{\|v\|_{H^1} \leq 1} |\int_{|x| \geq R} [g(u_n) - g(u)]dx| &\leq C\|f_2\|_{L^2|x| \geq R}[\sup_{\|v\|_{H^1} \leq 1} \|v\|_{L^2|x| \geq R} \\ &+ C\|f_1\|_{L^{2^*}|x| \geq R}[\sup_{\|v\|_{H^1} \leq 1} \|v\|_{L^{2^*}|x| \geq R} \\ &\leq C[\|f_2\|_{L^2|x| \geq R} + \|f_1\|_{L^{2^*}|x| \geq R}. \end{aligned}$$

Hence, there exists such an R_0.

Now, let Ω be a compact set in $\mathbb{R}^d$. We prove (ii) over Ω. This coupled with the previous result will give us the claim. From our assumptions, we have

$$|g(u_n) - g(u)|^{\frac{2^*}{2^*-1}} \leq C[1 + |f_1|^{2^*}],$$

using that $g(s) \leq C[1 + |s|^{2^*}$. Hence, $g(u_n) \to g(u)$ in $L^{\frac{2^*}{2^*-1}}(\Omega)$. Using Hölder, we see the result follows. $\square$

By a similar argument, though far simpler, we have

$$T(w) = \int |\nabla w|^2 dx,$$

is a well-defined, C^1 functional as well. Look at the constrained minimization problem

$$\min\{T(w) | w \in H^1,\ V(w) = 1\}.$$

Since both T, V are C^1, there exists a Lagrange multiplier θ such that $T'(u) = \theta V'(u)$. In other words,

$$-\Delta u = \theta g(u)$$

in the sense of distributions. A simple rescaling gives us a solution to (A.1). Hence, we must prove that there exists a solution to this minimization problem with the properties required.

Step 1:

We prove here that class of functions $\{w | w \in H^1, V(w) = 1\}$ is not empty. Let η_0 be selected such that $G(\eta_0) > 0$. For $R > 1$, define

$$w_R(x) = \begin{cases} \eta_0 & \text{for } |x| \leq R \\ \eta_0(R + 1 - |x|) & \text{for } |x| \in [R, R+1] \\ 0 & \text{for } |x| \geq R. \end{cases}$$

Clearly, $w_R \in H^1$ and

$$V(w_R) \geq G(\eta_0)|B(0,R)| - |B(0,R+1) - B(0,R)| \max_{s \in [0,\eta_0]} |G(s)|,$$

where $B(x, M)$ is the ball of radius M about x and $|\cdot|$ represents the Lebesgue measure. Hence, there exists C_1, C_2 such that

$$V(w_R) = C_1 R^d - C_2 R^{d-1}.$$

For R large enough, $V(w_R) > 0$ and scaling gives us a large class of functions for which $V = 1$.

Step 2:

Now, we must select a minimizing sequence. There exists a sequence $u_n \in H^1$ such that $V(u_n) = 1$ and $\lim_{n\to\infty} T(u_n) = A = \inf\{T(w)|w \in H^1, V(w) = 1\} \geq 0$.

Before we continue, we must introduce the Schwarz symmetrization scheme. For $f \in L^1$, the Schwarz symmetrization f^* is a radial, nonincreasing, measurable function such that for any $\alpha > 0$,

$$|\{f^* \geq \alpha\}| = |\{|f| \geq \alpha\}|.$$

For every continuous function F such that $F(f)$ is integrable, we have

$$\int F(f)dx = \int F(f^*)dx.$$

Also, for f, $g \in L^2$,

$$\int f(x)g(x)dx \leq \int f^*(x)g^*(x)dx.$$

This is known as the Riesz inequality and shows

$$\|f^* - g^*\|_{L^2} \leq \|f - g\|_{L^2}.$$

Finally, if $u \in H^1$, then $u^* \in H^1$ and

$$\int |\nabla u^*|^2 dx \leq \int |\nabla u|^2 dx.$$

Hence, let u_n^* be the Schwarz symmetrization of $|u_n|$. Then, $u_n^* \in H^1$, $V(u_n^*) = 1$, and $A \leq T(u_n^*) \leq T(u_n)$ implying that $\{u_n^*\}$ is also a minization sequence. Hence, we replace $\{u_n\}$ with $\{u_n^*\}$, so that we have a positive, nonincreasing, radial minimization sequence.

Step 3:

We now wish to show that $\|u_n\|_{H^1}$ is bounded. For $s \geq 0$, define $g_1(s) = (g(s)+ms)_+ = \max(g(s)+ms, 0)$ and $g_2 = g_1(s) - g(s)$, where m is defined as above. For $s \leq 0$, extend g_1

and g_2 as odd functions. Then, $g = g_1 - g_2$ for g_1, $g_2 \geq 0$ on $\mathbb{R}^+$. Also,

$$\begin{aligned} g_1(s) &= o(s) \text{ as } s \to 0, && \text{(A.6)} \\ \lim_{s\to\infty} \frac{g_1(s)}{s^{2^*-1}} &= 0, && \text{(A.7)} \\ g_2(s) &\geq ms, \ \forall\ s \geq 0. && \text{(A.8)} \end{aligned}$$

It is particularly easy to verify this for the soliton equation resulting from NLS.

Defince G_1, G_2 accordingly by

$$G_i(t) = \int_0^t g_i(s)ds$$

for $i = 1, 2$. From Equation (A.6), have for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$G_1(s) \leq C_\epsilon |s|^{2^*} + \epsilon G_2(s),$$

since near 0, $g_1 << g_2$ and near ∞, $g_1 << s^{2^*-1}$. Hence,

$$g_1(s) \leq C_\epsilon |s|^{2^*-1} + \epsilon g_2(s).$$

Since $T(u_n) \to A$, $\|\nabla u_n\|_{L^2}$ is bounded, hence by Sobolev,

$$\|u_n\|_{L^{2^*}} \leq C$$

for all n. Since $V(u_n) = 1$, we have

$$\int G_1(u_n)dx = \int G_2(u_n)dx + 1. \tag{A.9}$$

Hence, setting $\epsilon = \frac{1}{2}$,

$$C + \frac{1}{2}\int G_2(u_n)dx \geq \int G_2(u_n)dx + 1.$$

Thus,

$$\frac{m}{2}\int u_n^2 dx \leq \int G_2(u_n)dx \leq C,$$

from Equation (A.6).

Hence, $\|u_n\|_{H^1}$ is bounded, as is $\|u_n\|_{L^p}$ for any $2 \leq p \leq 2^*$ by interpolation.

Step 4:

Since u_n is radial, nonincreasing and bounded in L^2, we have

$$|u_n(x)| \leq C|x|^{-\frac{d}{2}}.$$

This is proved simply by writing the L^2 integral in polar coordinates. Since u_n is bounded in H^1, one can extract a subsequence which converges weakly in H^1 and almost everywhere to a function u. Note that u is radial and nonincreasing.

Let $H(s) = s^2 + |s|^{2^*-1}$. From Equation (A.6),

$$\frac{G_1(s)}{H(s)} \to 0,$$

as $s \to 0$ and ∞.

Also,

$$\begin{aligned} \sup_n \int H(u_n)dx &< \infty, \\ G_1(u_n) &\to G_1(u) \text{ almost everywhere,} \\ u_n(x) \to 0 \text{ as } |x| &\to \infty, \text{ uniformly in } n. \end{aligned}$$

Claim A.1.3.

$$\int G_1(u_n)dx \to \int G(u)dx$$

as $n \to \infty$.

Proof. This is a result of a more general compactness lemma of Strauss.

We first prove that for any bounded Borel set, B, we have

$$\int_B |G_1(u_n) - G_1(u)|dx \to 0,$$

as $n \to \infty$.

To prove this, we need G_1 uniformly integrable on B. Since $G_1(s)/H(s) \to 0$ as $s \to \infty$, we have

$$|G_1(u_n(x))| \leq C[1 + H(u_n(x))].$$

Hence $G_1(u_n)$ and $G_1(u)$ are in $L^1(B)$. We have,

$$\int_{B\cap\{|G_1(u_n(x))|\geq K\}} G_1(u_n(x))dx \leq \int_{B\cap\{|u_n(x)|\geq\phi(K)\}} |G_1(u_n(x))|dx,$$

for some function ϕ such that $\phi(K)\to\infty$ as $K\to\infty$. This follows just from the structure of G_1. Thus,

$$\int_{B\cap\{|G_1(u_n(x))|\geq K\}} G_1(u_n(x))dx \leq \epsilon(K)\int_B |H(u_n(x))|dx \leq C\epsilon(K),$$

for $\epsilon(K)\to 0$ as $K\to\infty$ since $G_1(s)/H(s)\to 0$ as $s\to\infty$. Uniform integrability follows, hence the claim holds on bounded sets.

Due to the uniform boundedness near ∞ of the u_n's and the limiting properties of $G_1(s)/H(s)$ near 0, there exists an R_0 such that

$$\int_{|x|\geq R_0} |G_1(u_n(x))| \leq \epsilon\int_{|x|\geq R_0} |H(u_n(x))| \leq C\epsilon,$$

for all $|x|\geq R_0$.

Hence, by Fatou's lemma $G_1(u)\in L^1(|x|\geq R_0)$ and

$$\int_{|x|\geq R_0} |G_1(u)|dx \leq C\epsilon.$$

From the first part of the proof, we have n_0 such that for all $n\geq n_0$,

$$\int_{|x|<R_0} |G_1(u_n)-G_1(u)|dx \leq \epsilon.$$

Thus, we have proven our claim. □

Now, using the claim, Fatou's lemma and Equation (A.9), we have

$$\int G_1(u)dx \geq \int G_2(u_n)dx + 1,$$

or in other words, $V(u)\geq 1$. We also have

$$T(u)\leq \liminf_{n\to\infty} T(u_n) = A.$$

Suppose $V(u)>1$. A scale change gives us $V(u)=1$ with $T(u)<A$, a contradiction. Hence, $V(u)=1$ and u is a solution to the constrained minimization problem.

Step 5:

We have proved that there is a solution u to the constrained minimization problem with V and T are C^1. So, there exists a Lagrange multiplier such that

$$\frac{1}{2}T'(u) = \theta V'(u).$$

If $\theta = 0$, then $u = 0$, a contradiction. We wish to show that $\theta > 0$. Assume not. Let $w \in \mathcal{S}$ be such that

$$\langle V'(u), w\rangle = \int g(u)w dx > 0.$$

This is possible since $V'(u) \neq 0$.

Since

$$V(u + \epsilon w) \approx V(u) + \epsilon\langle V'(u), w\rangle,$$

and

$$T(u + \epsilon w) \approx T(u) + 2\epsilon\theta\langle V'(u), w\rangle,$$

for $\epsilon \to 0$ and $\theta < 0$. Hence, one can find ϵ small enough such that $V(u + \epsilon w) > 1$ and $T(u + \epsilon w) < J$. Again, by a scale change, we get a contradiction, so $\theta > 0$. Therefore, in the sense of distributions, there exists a solution to Equation (A.1) which is positive and radial.

Note, for $d = 1, 2$, the constrained minimization approach fails. However, using ODE theory in $d = 1$ and critical point theorems for the action functional in $d = 2$, similar results can be obtained. See (Caz), Chapter 8 for a description.

A.1.1 Properties

Lemma A.1.4. *The solution u is C^2.*

Proof. We have that u satisfies

$$-\Delta u = q(u)u,$$

for $q(u) = g(u)/u$. From Equation (A.2), we get

$$|q(u)| \leq C[1 + |u|^{2^*-2}].$$

Since $u \in H^1$, $u \in L^{2^*}$. Given that $u \in L^p$ solves the equation, we have that $u \in W^{2,\frac{p}{2^*-1}}$. Using Sobolev embeddings, we have $u \in L^q$ for all $q \geq \frac{p}{2^*-1}$ such that

$$\frac{1}{q} \geq \frac{2^*-1}{p} - \frac{2}{d}.$$

Define a sequence q_j by

$$\frac{1}{q_j} = (2^*-1)^j \left(\frac{1}{2^*} - \frac{2}{d(2^*-2)} + \frac{2}{d(2^*-1)^j(2^*-2)} \right).$$

Since $(d-2)(2^*-2) < 4$, we have

$$\frac{2^*-2}{2^*} - \frac{2}{d} = -\delta,$$

for $\delta > 0$. Then,

$$\frac{1}{q_{j+1}} - \frac{1}{q_j} = -(2^*-1)^j\delta \leq -\delta,$$

so $\frac{1}{q_j}$ is decreasing and $\frac{1}{q_j} \to -\infty$ as $j \to \infty$. Since $q_0 = 2^*$, there exists $k \geq 0$ such that

$$\frac{1}{q_l} > 0 \text{ for } 0 \leq l \leq k, \ \frac{1}{q_{k+1}} \leq 0.$$

Claim A.1.5. $u \in L^{q_k}$.

Proof. Since $u \in L^{2^*}$. Also, from Sobolev, if $u \in L^{q_l}$ for some $l \leq k-1$, then $u \in L^q$ for all

$$q \geq \frac{q_l}{2^*-1}$$

such that

$$\frac{1}{q} \geq \frac{2^*-1}{q_l} - \frac{2}{d} = \frac{1}{q_{l+1}}.$$

Specifically, $u \in L^{q_{l+1}}$, so the claim is proved. □

Applying Sobolev once again, we have $u \in L^q$ for all $q \geq q_k/(2^* - 1)$ such that $1/q \geq 1/q_{k+1}$. This includes $q = \infty$. Thus, $|u|^{2^*-2}u \in L^2 \cap L^\infty$, so $u \in W^{2,p}$ for all $2 \leq p < \infty$. This implies that $u \in W^{1,p}$ for all $2 \leq p < \infty$. However, this allows us to look at the elliptic equation for $\partial_j u$ and show that $\partial_j u \in W^{2,p}$. Thus, $u \in W^{3,p}$ for all $2 \leq p < \infty$. By Sobolev, $u \in C^{2,\delta}$ for all $0 < \delta < 1$ and hence, $|D^\beta u(x)| \to 0$ as $|x| \to \infty$ for all $|\beta| \leq 2$. □

Lemma A.1.6. *There exists some $\delta > 0$ such that*

$$|D^\alpha u(x)| \leq Ce^{-\delta|x|},$$

for some C and $|\alpha| \leq 2$.

Proof. Since $u \in C^2$, it actually satisfies Equation (A.1). Set

$$v = r^{(d-1)/2}u,$$

then v satisfies

$$v_{rr} = [q(r) + \frac{b}{r^2}]v,$$

where $q(r) = -g(u(r))/u(r)$ and $b = (d-1)(d-3)/4$. For r large enough, we have

$$q(r) + \frac{b}{r^2} \geq \frac{m}{2}.$$

Let $w = v^2$, so we have

$$\frac{1}{2}w_{rr} = v_r^2 + [q(r) + \frac{b}{r^2}]w.$$

For $r > r_0$ large, we have $w_{rr} \geq mw$ and $w \geq 0$.

Let $q = e^{-\sqrt{(m)}r}(w_r + \sqrt{m}w)$. Then,

$$z_r = e^{-\sqrt{(m)}r}(w_{rr} - mw) \geq 0.$$

So, z is nondecreasing on (r_0, ∞). If there exists $r_1 > r_0$ such that $z(r_1) > 0$, then $z(r) \geq z(r_1) > 0$ for all $r \geq r_1$. Hence,

$$w_r + \sqrt{m}w \geq (z(r_1))e^{\sqrt{m}r},$$

and $w_r + \sqrt{m}w$ is not integrable. However, v^2 and vv_r are integrable near ∞, so w and w_r are also integrable, a contradiction. Thus, $z(r) \leq 0$ and

$$(e^{\sqrt{m}r}w)_r = e^{2\sqrt{m}r}z,$$

for $r \geq r_1$. As a result, $w(r) \leq Ce^{-\sqrt{m}r}$ and

$$|u(r)| \leq Cr^{-\frac{d-1}{2}}e^{-\frac{\sqrt{m}}{2}r},$$

for $r \geq r_1$.

For the decay of u_r, notice that

$$(r^{d-1}u_r)_r = -r^{d-1}g(u). \tag{A.10}$$

Using the properties of g and the exponential decay of u, integrate Equation (A.10) to see that u_r must have exponential decay.

The decay of u_{rr} comes from the radial form Equation (A.1). □

A.2 Uniqueness

Due to the complexity and breadth of work done on this issue, we present here only the key lemmas involved in proving the most efficient version of the result. See (McCleod) and the references therein for the proofs.

From the work of Gidas-Ni-Nirenberg, from the maximum principle we know that any positive ground state solution is radial. Hence, we show the uniqueness for solutions of the problem (GS):

$$\partial_{rr}u + \frac{d-1}{r}\partial_r u + f(u) = 0, \tag{A.11}$$

$$\partial_r u(0) = 0,\ u(r) \to 0 \text{ as } r \to \infty, \tag{A.12}$$

$$u(r) > 0 \text{ for } r \geq 0. \tag{A.13}$$

In the following, we state a number of lemmas that lead us to the conclusion of uniqueness for (A.11). If excluded, the proofs can be found in (McCleod) and the references

therein. Define

$$I(u,\lambda) = \lambda u f'(u) - (\lambda+2)f(u).$$

Theorem 30. *Let f be as in Section A.1, and suppose that for each $U > \alpha$ there is a $\lambda = \lambda(U) > 0$, continuously dependent upon U, such that*

$$\begin{aligned} I(u,\lambda) &\geq 0 \text{ for } 0 < u < U \text{ and} \\ I(u,\lambda) &\leq 0 \text{ for } u > U. \end{aligned}$$

Then, the solution u is unique.

We study this problem using a shooting method

$$\begin{aligned} \partial_{rr}u &+ \frac{d-1}{r}\partial_r u + f(u) = 0, \\ \partial_r u(0) &= 0, u(0) = a > 0, \end{aligned}$$

where u is extended maximally to the right with $u \geq 0$.

Consider a solution $v \in C^2([0,\sigma))$ for some $\sigma > 0$. Assume there are two such solutions, $v(r)$ and $w(r)$, and set $z = v - w$. Then, we have

$$\partial_{rr}z + \frac{d-1}{r}\partial_r z + \Phi(r) = 0,$$

for $\Phi(r) = f(v(r)) - f(w(r))$. Then, since $z(0) = \partial_r z(0) = 0$, we have from the fundamental solution of the Laplacian that

$$z(r) = -\int_0^r \frac{1}{d-2}\left(1 - \left(\frac{s}{r}\right)^{d-2}\right) s\Phi(s)ds.$$

Since Φ is Lipschitz continuous for small values $0 \leq s \leq s_0$, $|\Phi(s)| \leq K|z(s)|$, so

$$|z(r)| \leq \frac{Kr}{|d-2|}\int_0^r |z(s)|ds, \ r < s_0,$$

therefore $z = 0$ for $0 \leq r < s_0$ using Gronwall's inequality. Since the equation is non-singular for $r \geq s_0$, $z = 0$ wherever both solutions are defined. Hence, the shooting method used gives a unique solution. We denote such a solution by $u(a,r)$. Define

$$\begin{aligned} S^+ &= \{a > 0 : u(r,a) \text{ remains bounded away from } 0\}, \\ S^0 &= \{a > 0 : u(r,a) \text{ solves (GS)}\}, \\ S^- &= \{a > 0 : u(R,a) = 0 \text{ for some } R = R(\alpha) > 0\}. \end{aligned}$$

For $a \in S^0$, set $R(a) = \infty$. The variation $\delta(r,a) = \frac{\partial u}{\partial a}$ satisfies the equation

$$\begin{aligned} \partial_{rr}\delta &+ \frac{d-1}{r}\partial_r\delta + f'(u)\delta = 0, \\ \partial_r\delta(0) &= 0, \delta(0) = 1. \end{aligned}$$

Lemma A.2.1. *(a) The sets S^+, S^0, and S^- are disjoint and cover the interval $(0,\infty)$, with $(0,\alpha] \in S^+$. If u sovles (GS), then $u(0) \geq \alpha$. Also, S^+ and S^- are both open in $(0,\infty)$.*

(b) Any solution u with $u(0) \in S^0 \cup S^-$ is monotone decreasing. In particular, any solution of (GS) is monotone decreasing.

(c) If u is a solution of (GS), then for any $\epsilon \in (0,m)$,

$$\limsup_{r\to\infty} u(r)e^{r\sqrt{m-\epsilon}} < \infty, \ \limsup_{r\to\infty} |u'(r)|e^{r\sqrt{m-\epsilon}} < \infty$$

and

$$\frac{|u'(r)|}{u(r)} \to \sqrt{-m}$$

as $r \to \infty$.

(d) If u is a solution of the shooting problem with $u(0) \in S^+$, then u has an infinite number of local maxima and minima. Furthermore, if $u(r_0)$ is a local maximum then $u(r) < u(r_0)$ for $r < r_0$.

Lemma A.2.2. *(a) If $u(0) \in S^0 \cup S^-$, and δ has only a finite number of critical points and a finite number of zeroes in $(0,R)$.*

(b) If $u(0) \in S^0$, then as $r \to \infty$ either

$$\delta(r) \to \pm\infty, \ \delta'(r) \to \pm\infty \text{ with } \delta(r)\delta'(r) > 0 \text{ for large } r,$$

or

$$\delta(r) \to 0, \ \delta'(r) \to 0 \text{ with } \delta(r)\delta'(r) < 0 \text{ for large } r.$$

In the second case, for any $\epsilon \in (0,m)$,

$$\limsup_{r\to\infty} |\delta(r)|e^{r\sqrt{m-\epsilon}} < \infty, \ \limsup_{r\to\infty} |\delta'(r)|e^{r\sqrt{m-\epsilon}} < \infty.$$

Lemma A.2.3. *(a) Let $a_0 \in S^-$ and suppose that $\delta(R(a_0)) < 0$. Then $R(a)$ is a decreasing function of a near a_0.*

(b) Let $a_0 \in S^0$ and suppose $\delta(r) \to -\infty$ as $r \to \infty$. Then for some $\epsilon > 0$, the interval $(a_0, a_0 + \epsilon) \subset S^-$, while $(a_0 - \epsilon, a_0) \subset S^+$. Similarly, if $\delta(r) \to \infty$, there exists $\epsilon > 0$ such that the interval $(a_0, a_0 + \epsilon) \subset S^+$, while $(a_0 - \epsilon, a_0) \subset S^-$.

Definition A.2.4. *Let $a \in S^0 \cup S^-$. Then, a is said to be admissible if $\delta(r, a)$ has exactly one zero in $[0, R)$. It is said that a is strictly admissible if a is admissible and $\delta(r, a) < 0$ (or $\delta(r) \to -\infty$ as $r \to \infty$ if $a \in S^0$).*

Lemma A.2.5. *Suppose that $a \in S^0 \cup S^-$ and $\delta(r) \to 0$ as $r \to R$. Then, if $r_1 \in (0, R)$ is such that $\delta(r_1) = 0$, we have $u(r_1) > \alpha$.*

Lemma A.2.6. *Let Y and Z be nontrivial solutions of*

$$\begin{aligned} Y''(r) + \frac{d-1}{r} Y'(r) + g(r)Y(r) &= 0 \\ Z''(r) + \frac{d-1}{r} Z'(r) + G(r)Z(r) &= 0, \end{aligned}$$

on some interval $(\mu, \nu) \subset (0, \infty)$, where g, G are continuous on (μ, ν) and $G \geq g$ on (μ, ν), $G \neq g$. If either

$$\mu > 0 \text{ and } Y(\mu) = Y(\nu) = 0,$$

or

$$\mu = 0,\ Y, Z \text{continuous at } \mu, \text{ and } Y'(\mu) = Z'(\mu) = Y(\nu) = 0,$$

then Z has at least one zero in (μ, ν). The same conclusions hold for $G = g$ on (μ, ν), provided Y, Z are linearly independent.

Definition A.2.7. *Define*

$$\rho = \inf_r \{ r \in (0, \infty) |\ \exists\ Y \text{ with no zeroes in } (r, \infty) \text{ s.t. } Y'' + \frac{d-1}{r} Y' + gY = 0 \ \}.$$

The interval (ρ, ∞) is called the disconjugacy interval of the equation $Y''(r) + \frac{d-1}{r} Y'(r) + g(r)Y(r) = 0$.

Lemma A.2.8. *Let $g(r)$ be continuous of $(0,\infty)$, and suppose that $g(r) < 0$ for large r. Let the disconjugacy interval of $Y''(r) + \frac{d-1}{r}Y'(r) + g(r)Y(r) = 0$ be (ρ,∞) with $\rho > 0$, and suppost that $Y''(r) + \frac{d-1}{r}Y'(r) + g(r)Y(r) = 0$ has a solution which goes to 0 as $r \to \infty$. If Y is a nontrivial solution of $Y''(r) + \frac{d-1}{r}Y'(r) + g(r)Y(r) = 0$ such that $Y(\rho) = 0$, then Y has no subsequent zeroes and $Y(r) \to 0$ as $r \to \infty$. Conversely, if Y is a nontrivial solution of $Y''(r) + \frac{d-1}{r}Y'(r) + g(r)Y(r) = 0$ with a zero in (ρ,∞), then Y does not approach 0 as $r \to \infty$.*

Lemma A.2.9. *Under the assumptions on the nonlinearity, for any $a \in S^0 \cup S^{-1}$ the variation $\delta(r,a)$ has at least one zero in $(0,R)$.*

Lemma A.2.10. *Under the conditions on f, if $a \in S^0 \cup S^{-1}$ is admissible, then it is strictly admissible.*

Lemma A.2.11. *The set of strictly admissible $a \in S^0 \cup S^{-1}$ is both open and closed in $S^0 \cup S^{-1}$.*

Lemma A.2.12. *The value $a_0 = \inf(S^0 \cup S^{-1})$ is admissible.*

Theorem 31. *Let f satisfy the requirements for existence and suppose that for each $U > \alpha$, there exists $\lambda = \lambda(U) > 0$, continuously depending upon U, such that*

$$\begin{aligned} I(u,\lambda) &\geq 0 \text{ for } 0 < u < U, \\ I(u,\lambda) &\leq 0 \text{ for } u > U. \end{aligned}$$

Then, there is a unique solution to (GS).

Corollary A.2.13. *Let f satisfy the hypotheses in Section A.1. Then for each $R > 0$ the problem*

$$\begin{gathered} u'' + \frac{d-1}{r}u' + f(u) = 0 \\ u > 0,\ u'(0) = 0,\ u(R) = 0 \end{gathered}$$

has at most one solution.

A.2.1 Sturm Oscillation Theory

In $\mathbb{R}^3$, a change of variables shows that eigenvalues satisfy

$$\left(-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+V(r)\right)u_l(r;E)=Eu_l(r;E). \tag{A.14}$$

Let $N_l(E;V)$ be the number of zeros of $u_l(r;E)$ other than $r=0$.

Lemma A.2.14. *Let $E_0 \leq 0$. If $N_l(E_0;V) \geq m_0$, then* (A.14) *has at least m_0 square integrable solutions with $E < E_0$. In particular, since the essential spectrum is $[0,\infty)$, $N_l(E;V) < \infty$ if $E_0 < 0$.*

Lemma A.2.15. *$N_l(E;V)$ is a monotone increasing function of E. If $E_0 < 0$ is an eigenvalue, then $N_l(E;V) \geq N_l(E_0;V)+1$ for $E > E_0$.*

A.3 Continuity in λ.

We present here a collection of results from (Shatah1), (Shatah2) and (ShatStr1).

A solution for the soliton equation (1.4)

$$-\Delta\phi_\omega+\omega\phi_\omega-f(\phi_\omega),$$

minimizes the Lagrangian given by

$$J_\omega(\phi)=\frac{1}{2}\int|\nabla\phi|dx+\frac{\omega}{2}\int|\phi|^2dx-\int F(\phi)dx,$$

for $F'=f$.

Using a rescaling argument, we see that if ϕ_ω minimizes J_ω, then

$$K_\omega(\phi_\omega)=\frac{d-2}{2}\int|\phi_\omega|^2dx+d(\frac{\omega^2}{2}\int|\phi_\omega|^2dx+\int F(\phi_\omega)dx)=0.$$

Define $M_\omega=\{\phi\in H^1_r(\mathbb{R}^d), K_\omega(\phi)=0, \phi\neq 0\}$.

Lemma A.3.1. *M is a C^1 hypersurface in $H^1_r(\mathbb{R}^d)$ bounded away from 0.*

Proof. Since f is continuous, $K = K_\omega$ is a C^1 functional. If we take a function $u_0 \in H_r^1$ such that $\delta K(u_0) = 0$, it satisfies the equation

$$-(d-2)\Delta u_0 + d\omega^2 u_0 - df(u_0) = 0.$$

Once again using a rescaling argument, we have

$$\frac{(d-2)^2}{2}\int |\nabla u_0|^2 dx + d^2\omega^2 \int |u_0|^2 dx - d^2 \int F(|u_0|)dx = 0.$$

Hence, if $K(u_0) = 0$, then

$$\frac{(d-2)^2}{2}\int |\nabla u_0|^2 dx - \frac{d(d-2)}{2}\int |\nabla u_0|^2 dx = 0.$$

Hence, $u_0 = 0$, so for $u_0 \in M$, $\delta K(u_0) \neq 0$ and M is a C^1 hypersurface. By the restrictions on the nonlinearity, we have

$$K(u) \geq \frac{d-2}{2}\int |\nabla u|^2 dx + \frac{1}{4}\int |u|^2 dx - C\int |u|^{l+1} dx,$$

where $l = 1 + 4/(d-2)$. Using Gagliardo-Niremberg for $d > 2$, we have

$$\frac{\|u\|_{H^1}^2}{4} - C\|u\|_{H^1}^\alpha$$

with $\alpha > 2$. For $\|u\|_{H^1} < \epsilon$, ϵ small, we have $K(u) > 0$, so M is bounded away from 0. □

Theorem 32. *For some $v \neq 0$, there exists*

$$d(\omega) = \inf_{v \in M_\omega} J_\omega(v),$$

and

$$d(\omega) = \{\inf \frac{1}{d}\int |\nabla v|^2 dx,\ K_\omega(v) \leq 0,\ v \neq 0\}.$$

Moreover, v satisfies (1.4).

Proof. Consider any function $v \in H_r^1$ such that $K_\omega(v) < 0$. Let $v_\beta(x) = v(x/\beta)$. Then, we have

$$K_\omega(v_\beta) = \beta^{d-2}\frac{d-2}{2}\int |\nabla v|^2 dx + \beta^d d(\frac{\omega^2}{2}\int |v|^2 dx + \int G(|v|)dx).$$

For $\beta = 1$, $K_\omega(v_1) < 0$ and for β small, $K_\omega(v_\beta) > 0$. Hence, there exists $\beta_0 \in (0,1)$ such that $K_\omega(v_{\beta_0}) = 0$ and

$$\frac{1}{d}\int |\nabla v_{\beta_0}|^2 dx = \frac{\beta_0^{d-2}}{d}\int |\nabla v|^2 dx < \frac{1}{d}\int |\nabla v|^2 dx.$$

Since $J_\omega(v) = 1/d(\int |\nabla v|^2 dx + K_\omega(v))$, then

$$\begin{aligned} d(\omega) &= \inf_{v\in M_\omega} J_\omega(v) = \inf\{1/d\int |\nabla v|^2 dx,\ K_\omega(v) = 0, v \neq 0\} \\ &= \inf\{1/d\int |\nabla v|^2 dx,\ K_\omega(v) \leq 0,\ v \neq 0\}. \end{aligned}$$

Hence, we have equivalence of the minimization problems.

Let v_k be a minimizing sequence. Then, $\int |\nabla v_k|^2 dx$ is bounded and by the properties of the nonlinearity, we have $G(\eta) > -\epsilon/2\eta^2 - C(\epsilon)\eta^{l+1}$, for $l < 1+4/(d-2)$. Since $K_\omega(v_k) \leq 0$,

$$0 \geq K_\omega(v_k) = \frac{d-2}{2}\int |\nabla v_k|^2 dx + d(\frac{\omega^2}{2}\int |v_k|^2 dx + \int G(|v_k|^2) dx),$$

or

$$0 \geq \frac{d-2}{2}\int |\nabla v_k|^2 dx + d(\frac{\omega^2-\epsilon}{2}\int |v_k|^2 dx - C(\epsilon)\int |v_k|^{l+1} dx).$$

Since H_r^1 is compactly embedded in L_r^p for $2 < p < 2 + 4/(d-2)$, we see that $\|v_K\|_{H^1} < \infty$ and there exists a subsequence also denoted v_k such that

$$v_k \rightarrow^w v_0 \in H_r^1,\ v_k \rightarrow v_0 \in L^p \text{ for } 2 < p < 2 + 4/(d-2).$$

Since weak limits are lower semicontinuous, we have

$$\begin{aligned} K_\omega(v_0) &= \frac{d-2}{2}\int |\nabla v_0|^2 dx + d(\frac{\omega^2}{2}\int |v_0|^2 dx + \int G(|v_0|^2) dx) \\ &\leq \frac{d-2}{2}\int |\nabla v_k|^2 dx + d(\frac{\omega^2}{2}\int |v_k|^2 dx + \int G(|v_k|^2) dx) = 0. \end{aligned}$$

Hence, the weak limit is actually strong and

$$d(\omega) = \inf_{v\in M_\omega} J_\omega(v) = J_\omega(v_0).$$

By Lagrange multipliers for J_ω and K_ω,

$$-\Delta v_0 + \omega^2 v_0 + f(|v_0|)v_0 = \lambda(-(d-2)\Delta v_0 + d\omega^2 v_0 + df(|v_0|)v_0).$$

Integrating, using a minimizing/scaling argument, the applying that $K_\omega(v_0) = 0$, we have that $\lambda = 0$, proving the result. □

It is clear that $d(\omega) = 1/d \int |\nabla v|^2 dx$ where v is the minimizer from the previous theorem. For each ω, call the minimizer ϕ_ω.

Lemma A.3.2. *Let $\omega_1 < \omega_2$ be such that $[\omega_1, \omega_2] \subset (0, \infty)$, then $d(\omega)$ and $\int |\phi_\omega|^2 dx$ are uniformly bounded in $\omega \in [\omega_1, \omega_2]$.*

Proof. Since K is continuous in in ω, $d(\omega)$ is bounded for $\omega \in [\omega_1, \omega_2]$. We have $K_\omega(\phi_\omega) = 0$. Since $G(\eta) \geq -c\eta^{l+1}$ for η large and $G(0) = G'(0) = G''(0) = 0$, we have

$$\frac{a}{2}\eta^2 + G(\eta) > -C_a \eta^{l_0+1},$$

for and $a > 0$ and $l_0 = 1 + 4/(d-2)$. Hence,

$$\int (\frac{a}{2}|v|^2 + G(|v|))dx > -C_a \int |v|^{l_0+1} dx.$$

As $K_\omega(\phi_\omega) = 0$, using Gagliardo-Niremberg, we have

$$\frac{d-2}{2} \int |\nabla \phi_\omega|^2 dx + d\frac{(\omega^2 - a^2)}{2} \int |\phi_\omega|^2 dx - C_a (\int |\nabla \phi_\omega|^2 dx)^\alpha \leq 0.$$

As a result, $\|\phi_\omega\|_{L^2}$ is bounded in terms of the bound on $\|\nabla \phi_\omega\|_{L^2}$. □

Lemma A.3.3. *We have that $d(\omega)$ is increasing in ω and if $\omega_1 < \omega_2$, then*

$$\begin{aligned} d(\omega_2) &< d(\omega_1) - \frac{(\omega_2^2 - \omega_1^2)}{2} \int |\phi_{\omega_1}|^2 dx + o(\omega_1 - \omega_2), \\ d(\omega_1) &< d(\omega_2) + \frac{(\omega_2^2 - \omega_1^2)}{2} \int |\phi_{\omega_2}|^2 dx + o(\omega_1 - \omega_2). \end{aligned}$$

Consequently, $d(\omega)$ is continuous in $\omega \in (0, \infty)$.

Proof. Let $\omega_1 < \omega_2$. Then,

$$K_{\omega_1}(\phi_{\omega_2}) = K_{\omega_2}(\phi_{\omega_2}) - d\frac{(\omega_2^2 - \omega_1^2)}{2} \int |\phi_{\omega_2}|^2 dx.$$

Since $K_{\omega_2}(\phi_{\omega_2}) = 0$, we have

$$K_{\omega_1}(\phi_{\omega_2}) < 0,$$

and

$$d(\omega_2) = \frac{1}{d} \int |\nabla \phi_{\omega_2}|^2 dx > \inf\{\frac{1}{d} \int |\nabla v|^2 dx,\ K_{\omega_1(v)} \leq 0,\ v \neq 0\}.$$

Hence, we have $d(\omega_2) > d(\omega_1)$.

Set $\psi_\beta(x) = \phi_{\omega_2}(x/\beta)$. Then,

$$\begin{aligned} K_{\omega_1}(\psi_\beta) &= \frac{(d-2)\beta^{d-2}}{2}\int |\nabla\phi_{\omega_2}|^2 dx \\ &+ d\beta^d(\frac{\omega_1^2}{2}\int |\phi_{\omega_2}|^2 dx + \int G(|\phi_{\omega_2}|)dx) \\ &= \frac{(d-2)\beta^{d-2}}{2}\int |\nabla\phi_{\omega_2}|^2 dx - \beta^d(\frac{(d-2)}{2}\int |\nabla\phi_{\omega_2}|^2 dx \\ &+ d\frac{(\omega_2^2-\omega_1^2)}{2}\int |\phi_{\omega_2}|^2 dx. \end{aligned}$$

Define $\Delta_{12} = (\omega_2^2 - \omega_1^2)/2\int |\phi_{\omega_2}|^2 dx$, then

$$K_{\omega_1}(\psi_\beta) = \frac{d(d-2)}{2}d(\omega_2)\beta^{d-2} - d(\frac{d-2}{2}d(\omega_2) + \Delta_{12})\beta^d,$$

and setting

$$\beta_2 = \frac{1}{(1+\frac{2\Delta_{12}}{(d-2)d(\omega_2)})}^{\frac{1}{2}},$$

we have $K_{\omega_1}(\psi_{\beta_2}) = 0$ and hence

$$d(\omega_1) \le \frac{1}{d}\int |\psi_{\beta_2}|^2 dx = \beta_2^{d-2} d(\omega_2).$$

For $\omega_2 - \omega_1$ small, we have $|\Delta_{12}| < C(\omega_2^2 - \omega_1^2$ since $\|\phi_\omega\|_{L^2}$ is bounded. So,

$$\beta_2^{d-2} = 1 - \frac{\Delta_{12}}{d(\omega_2)} + o(\Delta_{12}),$$

and

$$d(\omega_1) \le d(\omega_2) - \frac{(\omega_2^2-\omega_1^2)}{2}\|\phi_{\omega_2}\|_{L^2}^2 + o(\omega_2-\omega_1), \tag{A.15}$$

which gives the first inequality in the lemma. A similar procedure using a rescaling of ϕ_{ω_1} in K_{ω_2} gives

$$d(\omega_2) \le d(\omega_1) + \frac{(\omega_2^2-\omega_1^2)}{2}\|\phi_{\omega_1}\|_{L^2}^2 + o(\omega_2-\omega_1). \tag{A.16}$$

Continuity is a direct consequence of (A.15) and (A.16). □

Theorem 33. *Select $\omega_0 \in (0, \infty)$. For ω in a neighborhood of ω_0, let ϕ_ω be the unique ground state solution satisfying the equation*

$$-\Delta\phi_\omega + \omega^2\phi_\omega - f(\phi_\omega) = 0.$$

Assume also that $L = -\Delta + \omega_0^2 - f'(\phi_{\omega_0})$ does not have an eigenvalue at 0. If $f \in C^2$ satisfying conditions in Section A.1, then the map $\omega \to \phi_\omega$ is a C^2 map from $\mathbb{R} \to H_r^1$.

Proof.

Claim A.3.4. *We have that the map $\omega \to \phi_\omega$ is continuous in a neighborhood of ω_0 with values taken in in H_r^1.*

Proof. From Lemma A.3.3, we know that $d(\omega)$ is continuous with respect to ω and that $\|\phi_\omega\|_{L^2}$ is bounded locally in terms of ω. Take (ω_j) to be a sequence converging to ω_0. Then (ϕ_{ω_j}) is bounded in H_r^1 and a subsequence can be chosen to converge weakly in H_r^1 to some v. Then,

$$0 = K_\omega(\phi_\omega) = \frac{d-2}{d}\int |\nabla\phi_\omega|^2 dx + \frac{\omega^2}{2}\int |\phi_\omega|^2 + \int F(\phi_\omega)dx.$$

Letting $\omega_j \to \omega_0$, we see that from the lower semicontinuity of the weak limit and assumptions on F that $K_{\omega_0}(v) \leq 0$. Hence,

$$\int |\nabla v|^2 dx \geq \int |\nabla\phi_{\omega_0}|^2 dx.$$

By lower semicontinuity of the weak limit again, we have $\|\nabla v\|_{L^2} = \|\nabla\phi_{\omega_0}\|_{L^2}$, which by uniqueness implies $v = \phi_{\omega_0}$. Hence $\phi_\omega \to^w \phi_{\omega_0}$ in H_r^1. By continuity of d, we can take a strong limit in the expression for K_ω to see that $\phi_\omega \to \phi_{\omega_0}$ in L^2. Hence, $\phi_\omega \to \phi_{\omega_0}$ in H^1. □

Claim A.3.5. *In a neighborhood of $\phi_{\omega_0} \in H_r^1$, all solutions lie on a C^1 curve.*

Proof. We have $-\Delta\phi + \omega^2\phi + f(\phi) = 0$. Define a function

$$G(\omega, u) = u - (\omega^2 - \Delta)^{-1} f(u),\ u \in H_r^1.$$

The function $G(\omega, u) \in H_r^1$ since $u \in H_r^1$, $f(u) \in L^{2d/(d+2)}$ using the assumptions on the nonlinearity and $(\omega^2 - \Delta)^{-1} f(u) \in H^1$ by Sobolev embeddings. The function G is actually a C^1 operator from $(0, \infty) \times H_r^1$ into H_r^1. Clearly, $G(\omega_0, \phi_{\omega_0}) = 0$.

Look at the operator given by $L = -\Delta + \omega_0^2 + f'(\phi_{\omega_0})$. Using Weyl's criteria, we see the continuous spectrum is precisely (ω_0^2, ∞) and L has only discrete spectrum to the left of ω_0^2. Via our uniqueness assumption, we have assumed that L is invertible. Therefore, the compact operator

$$(\lambda_0 - \Delta)^{-\frac{1}{2}} f'(\phi_{\omega_0})(\lambda_0 - \Delta)^{-\frac{1}{2}}$$

on L_r^2 does not have -1 in its spectrum. So,

$$\frac{\partial G}{\partial u}(\omega_0^2, \phi_{\omega_0}) = I + (\omega_0^2 - \Delta)^{-1} f'(\phi_{\omega_0}),$$

acting from H_r^1 to H_r^1 is also invertible. Using the implicit function theorem, the solutions of $G(\omega, u) = 0$ in a neighborhood of $\omega_0, \phi_{\omega_0}$ form a C^1 curve in $(0, \infty) \times H_r^1$. □

Using the claims, we have that $\omega \to \phi_\omega$ is a C^1 curve near $\omega = \omega_0$, $\phi_\omega = \phi_{\omega_0} \in H_r^1$. Since ϕ_ω is bounded in H_r^1, by the aymptotic analysis at ∞, on a neighborhood of ω_0 the functions

$$|x|^{\frac{d-1}{2}} \phi_\omega(x)$$

are bounded independently of ω and x. Bootstrapping using the soliton equation, we have that ϕ_ω is uniformly bounded. Since

$$[-\Delta + \omega^2 + f'(\phi_\omega)] \frac{\partial \phi_\omega}{\partial \omega} = -2\omega\phi_\omega. \tag{A.17}$$

Since L is invertible, $\frac{\partial \phi_\omega}{\partial \omega}$ is uniformly bounded. Hence $[\frac{\partial \phi_\omega}{\partial \omega}]^2$ is bounded in H^1. Taking difference quotients in (A.17), we have that ϕ_ω is C^2 with values in H^1. Then, $\frac{\partial^2 \phi_\omega}{\partial \omega^2}$ exists and satisfies

$$[-\Delta + \omega^2 + f'(\phi_\omega)] \frac{\partial^2 \phi_\omega}{\partial \omega^2} = -4\omega\phi_\omega - 2\phi_\omega - f''(\phi_\omega)[\frac{\partial \phi_\omega}{\partial \omega}]^2.$$

□

Appendix B

Functional Analysis

B.1 Spherical Harmonics

We present here the discussion from (Taylor2).

To begin, we write the Laplacian in $\mathbb{R}^d$ in spherical coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{d-1}}.$$

To understand Δ_{S^d}, we consider the Dirichlet problem for the unit ball $B = \{x \in \mathbb{R}^{d+1} | |x| < 1\}$:

$$\Delta u = 0 \text{ in } B,\ u = f \text{ on } S^d = \partial B, \tag{B.1}$$

given $f \in \mathcal{D}'(S^d)$. There exists the Poisson integral formula for the solution to (B.1):

$$u(x) = \frac{1-|x|^2}{A_d}\int_{S^d} \frac{f(y)}{|x-y|^{d+1}} dS(y),$$

where A_d is the volume of S^d. Set $x = r\omega$ for $r = |x|$ and $\omega \in S^d$, then we have

$$u(r\omega) = \frac{1-r^2}{A_d}\int_{S^d} \frac{f(\omega')}{(1-2r\omega\cdot\omega'+r^2)^{d+1}} dS(\omega'). \tag{B.2}$$

We can define an alternative formula for the solution to (B.1) using an operator-valued ODE, namely

$$u(r\omega) = r^{A-\frac{d-1}{2}} f(\omega),\ r \leq 1, \tag{B.3}$$

where A is an operator on $\mathcal{D}'(S^d)$, given by

$$A = \left(-\Delta_S + \frac{(d-1)^2}{4}\right)^{\frac{1}{2}}.$$

Set $r = e^{-t}$, then we can obtain a formula for the semigroup e^{-tA} from the formulae (B.3) and (B.3). Let $\theta(\omega, \omega')$ be the geodesic distance on S^d from ω to ω', so $\cos\theta(\omega, \omega') = \omega \cdot \omega'$, then we can rewrite (B.2) as

$$\begin{aligned} u(r\omega) &= \frac{2}{A_d} \sinh(\log r^{-1}) r^{-\frac{(d-1)}{2}} \\ &\times \int_{S^d} \frac{f(\omega')}{[2\cosh(\log r^{-1}) - 2\cos(\theta(\omega, \omega'))]^{\frac{d+1}{2}}} dS(\omega'). \end{aligned}$$

Then, by (B.3)

$$e^{-tA} f(\omega) = \frac{2}{A_d} \sinh(t) \int_{S^d} \frac{f(\omega')}{[2\cosh t - 2\cos\theta(\omega, \omega')]^{\frac{d+1}{2}}} dS(\omega').$$

Identifying an operator on $\mathcal{D}'(S^d)$ with its Schwartz kernel in $\mathcal{D}'(S^d \times S^d)$, we write

$$e^{-tA} = \frac{2}{A_d} \frac{\sinh(t)}{[2\cosh t - 2\cos\theta(\omega, \omega')]^{\frac{d+1}{2}}},\ t > 0. \tag{B.4}$$

Integrating (B.4) from t to ∞, we have

$$A^{-1} e^{-tA} = 2C_d (2\cosh t - 2\cos\theta(\omega, \omega'))^{-\frac{d-1}{2}},\ t > 0 \tag{B.5}$$

where $n \geq 2$ and

$$C_d = \frac{1}{(d-1)A_d} = \frac{1}{4}\pi^{-\frac{d-1}{2}} \Gamma\left(\frac{d-1}{2}\right).$$

Proposition B.1.1. *The spectrum of the self-adjoint operator A on $L^2(S^d)$ is*

$$spec(A) = \left\{ \frac{1}{2}(d-1) + k | k = 0, 1, 2, \dots \right\}. \tag{B.6}$$

Proof. The spectrum of A must be contained in the set (B.6) since $u(x)$ represented by (B.3) must be smooth at $x = 0$, hence the exponent of r can only take nonnegative integer values.

Let V_k be the eigenspace of A for the eigenvalue $v_k = \frac{d-1}{2} + k$. We would like to show that $V_k \neq 0$ as well as identify V_k exactly. For $f \in V_k$, we have

$$u(x) = u(r\omega) = r^{A - \frac{d-1}{2}} f(\omega) = r^k f(\omega)$$

is a harmonic function on $\mathbb{R}^{d+1}$, which is homogeneous and smooth at $x = 0$. Hence, this must be a harmonic polynomial, homogeneous of degree k in x by standard complex analysis results. Let $\mathcal{H}_k$ be the space of harmonic polynomials, homogeneous of degree k in x. Restricting $\mathcal{H}_k$ to $S^d \subset \mathbb{R}^{d+1}$, we have an isomorphism

$$\rho : \mathcal{H}_k \to V_k.$$

Hence, we need to show that $\mathcal{H}_k \neq 0$ for each $k = 0, 1, 2, \ldots$.

To this end, let $c = (c_1, c_2, \ldots, c_{d+1}) \in \mathbb{C}^{d+1}$. Define

$$p_c(x) = (c_1 x_1 + \cdots + c_{d+1} x_{d+1})^k.$$

Then,

$$\Delta p_c(x) = k(k-1)\langle c, c\rangle (c_1 x_1 + \cdots + c_k x_k)^{k-2},$$

where $\langle c, c\rangle = c_1^2 + \cdots + c_k^2$. Hence, the set $\mathcal{H}_k$ is not empty. □

Now, we want to understand the orthogonal projections E_k of $L^2(S^d)$ on V_k. From (B.5), we have

$$\sum_{k=0}^{\infty} \nu_k^{-1} e^{-t\nu_k} E_k(x, y) = 2C_d (2\cosh t - 2\cos\theta)^{-\frac{(d-1)}{2}},$$

where $\theta(x, y)$ is the geodesic distance from x to y in S^d. Setting $r = e^{-t}$ and using $\nu_k = \frac{d-1}{2} + k$, we get

$$\begin{aligned} \sum_{k=0}^{\infty} r^k \nu_k^{-1} E_k(x, y) &= 2C_d (1 - 2r\cos\theta + r^2)^{-\frac{(d-1)}{2}} \\ &= \sum_{k=0}^{\infty} r^k p_k(\cos\theta), \end{aligned}$$

so

$$E_k(x, y) = \nu_k p_k(\cos\theta).$$

To see that p_K is a polynomial in $\cos\theta$, set $\cos\theta = t$. Then,

$$(1 - 2tr + r^2)^{-\alpha} = \sum_{k=0}^{\infty} C_k^{\alpha}(t) r^k, \tag{B.7}$$

where

$$C_k^\alpha(t) = \sum_{l=0}^{[k/2]} (-1)^l \begin{pmatrix} k-l+\alpha-1 \\ k-l \end{pmatrix} \begin{pmatrix} k-l \\ l \end{pmatrix} (2t)^{k-2l}.$$

Hence, we have the following:

Proposition B.1.2. *The orthogonal projection of $L^2(S^d)$ on V_k has kernel*

$$E_k(x,y) = 2C_d \nu_k C_k^\alpha(\cos\theta),\ \alpha = \frac{d-1}{2}.$$

Note that for $d=2$, $E_k(x,y) = \frac{2k+1}{4\pi} P_k(\cos\theta)$, where P_k is the kth Legendre polynomial.

Integrating E_k over the diagonal, we have

$$Tr(E_k) = 2C_d A_d \nu_k C_k^{\frac{d-1}{2}}(1) = \frac{2\nu_k}{d-1} C_k^{\frac{d-1}{2}}(1).$$

For $t=1$ in (B.7), we have

$$C_k^\alpha(1) = \begin{pmatrix} k+2\alpha-1 \\ k \end{pmatrix}.$$

Corollary B.1.3. *The eigenspace V_k of $-\Delta_S$ on S^d, with eigenvalue*

$$\lambda_k = \nu_k^2 - \frac{1}{4}(d-1)^2 = k^2 + (d-1)k,$$

satisfies

$$dim(V_k) = Tr(E_k) = \frac{2k+d-1}{d-1} \begin{pmatrix} k+d-2 \\ k \end{pmatrix} = \begin{pmatrix} k+d-2 \\ k-1 \end{pmatrix} + \begin{pmatrix} k+d-1 \\ k \end{pmatrix}.$$

In particular, on S^2, $dim(V_k) = 2k+1$.

B.2 Complex Interpolation

We present here varioius arguments from (Stein) and (BeLo).

Suppose we are given a family of operators $\{U^s\}$ on a strip $a \le \mathrm{Re}\,(s) \le b$, given by

$$(U^s f)(x) = \int_{\mathbb{R}^d} K_s(x,y) f(y) dy.$$

We assume that for each (x, y), the function $K_s(x, y)$ is analytic in $a < \mathrm{Re}\,(s) < b$ and is continuous in the closure $a \leq \mathrm{Re}\,(s) \leq b$. Also, we assume

$$\|U^s f\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}},$$

for $\mathrm{Re}\,(s) = a$ and

$$\|U^s f\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}},$$

for $\mathrm{Re}\,(s) = b$. Let us call this set of assumption (UB).

Lemma B.2.1. *Under the assumptions (UB), we have*

$$\|U^{a(1-\theta)+b\theta} f\|_{L^q} \leq M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p},$$

where $0 \leq \theta \leq 1$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Before we begin, we must state and prove an important result from comlex analysis. We present here the version found in (BeLo).

Theorem 34 (Three Lines Theorem)**.** *Assume that $F(z)$ is analytic on the open strip $0 < \mathrm{Re}\, z < 1$ and bounded and continuous on the closed strip $0 \leq \mathrm{Re}\, z \leq 1$. If*

$$|F(it)| \leq M_0,\ |F(1+it)| \leq M_1,\ -\infty < t < \infty,$$

then we have

$$|F(\theta + it)| \leq M_0^{1-\theta} M_1^{\theta},\ -\infty < t < \infty.$$

Proof of Three Lines Theorem. Let $\epsilon > 0$ and $\lambda \in \mathbb{R}$. Set

$$F_\epsilon(z) = e^{\epsilon z^2 + \lambda z} F(z).$$

Then, $F_\epsilon(z) \to 0$ as $\mathrm{Im}\, z \to \pm\infty$, and

$$|F_\epsilon(it)| \leq M_0,\ |F_\epsilon(1+it)| \leq M_1 e^{\epsilon+\lambda}.$$

Hence, for all z in the strip,

$$|F_\epsilon(z)| \leq \max(M_0, M_1 e^{\epsilon+\lambda}),$$

by the Phragmén-Lindelöf principle. In other words,

$$|F(\theta + it)| \leq e^{-\epsilon(\theta^2 - t^2)} \max(M_0 e^{-\theta\lambda}, M_1 e^{(1-\theta)\lambda + \epsilon}).$$

Then, taking $\epsilon \to 0$, we have

$$|F(\theta + it)| \leq \max(M_0 \rho^{-\theta}, M_1 \rho^{1-\theta}).$$

Setting $\rho = M_0/M_1$, we get

$$|F(\theta + it)| \leq M_0^{1-\theta} M_1^{\theta}.$$

□

Proof of Interpolation. Let q' be the dual of q. It suffices to show that

$$\left| \int U^{a(1-\theta)+b\theta}(f) g dx \right| \leq M_0^{1-\theta} M_1^{\theta},$$

for f, g simple functions with $\|f\|_{L^p} \leq 1$ and $\|g\|_{L^{q'}} \leq 1$. Set

$$f_s = |f|^{\alpha(s)} \frac{f}{|f|}, \ \alpha(s) = p\left[\frac{1-s}{p_o} + \frac{s}{p_1}\right],$$

$$g_s = |g|^{\beta(s)} \frac{g}{|g|}, \ \beta(s) = q'\left[\frac{1-s}{q_0'} + \frac{s}{q_1'}\right]$$

and

$$I(s) = M_0^{s-1} M_1^{-s} \int U^{a(1-s)+bs}(f_s) f_s dx.$$

Then, $I(s)$ is analytic and bounded is the strip $0 < \operatorname{Re} s < 1$, continuous at the closure and satisfies $|I(s)| \leq 1$ for $\operatorname{Re} s = 0$ or $\operatorname{Re} s = 1$ by the assumptions on U. Hence, the three lines lemma gives us $|I(\theta)| \leq 1$, which proves the lemma. □

B.3 Hardy-Littlewood-Sobolev Inequality

The crucial result for proving the Strichartz estimates for the Schrödinger equation is the following lemma. This is also utilized heavily to get bounds for the distorted Fourier bases. The discussion we present here come from (Stein).

Lemma B.3.1. *Given $0 < \gamma < d$, $1 < p < q < \infty$, and $\frac{1}{q} = \frac{1}{p} - \frac{d-\gamma}{d}$, then the following holds*

$$\|f * (|y|^{-\gamma})\|_{L^q(\mathbb{R}^d)} \leq A_{p,q}\|f\|_{L^p(\mathbb{R}^d)}.$$

Proof. We have

$$\begin{aligned}[f * (|y|^{-\gamma})](x) &= \int f(x-y)|y|^{-\gamma}dy \\ &= \int_{|y|<R} f(x-y)|y|^{-\gamma}dy + \int_{|y|\geq R} f(x-y)|y|^{-\gamma}dy.\end{aligned}$$

The first integral is convolution of f with the function $|y|^{-\gamma}\chi_{B(R)}(y)$, for χ a characteristic function. This function is radial, decreasing and integrable.

Claim B.3.2. *For ϕ radial and decreasing,*

$$\sup_{t>0} |f * \phi_t(x)| \leq Mf(x) \int \phi(y)dy,$$

where $\phi_t(x) = t^{-d}\phi(x/t)$ and M is the standard maximal operator such that

$$(Mf)(x) = \sup_{r>0} c_d r^{-d} \int_{|y|<r} |f(x-y)|dy.$$

Proof. It suffices to prove

$$|f * \phi(x)| \leq Mf(x),$$

where $\int \phi dy = 1$ since the result follows from scaling. To begin, let $\phi = \sum_{j=1}^N a_j \chi_{B_j}$, where each a_j is positive and each B_j is a ball centered at the origin. Since $\sum_j a_j|B_j| = 1$ and $(f * \chi_{B_j})(x) \leq |B_j|Mf(x)$, the result follows immediately. Since any radial, decreasing ϕ can be approximated by finite sums of this type, the result follows in the limit. □

Using the claim, we have

$$\int_{|y|<R} f(x-y)|y|^{-\gamma}dy \leq (Mf)(x) \int_{|y|<R} |y|^{-\gamma}dy = cR^{d-\gamma}(Mf)(x).$$

Using Hölder's Inequality, we have

$$\int_{|y|\geq R} f(x-y)|y|^{-\gamma}dy \leq \|f\|_{L^p(\mathbb{R}^d)} \||y|^{-\gamma}\chi_{B(R)^c}\|_{L^{p'}(\mathbb{R}^d)},$$

where $\||y|^{-\gamma}\chi_{B(R)^c}\|_{L^{p'}} = cR^{-\frac{n}{q}}$ by the requirement that $\frac{1}{q} = \frac{1}{p} - \frac{n-\gamma}{n}$.

Summing the two integrals, we have

$$|(f * |y|^{-\gamma})(x)| \leq C[Mf(x)R^{n-\gamma} + \|f\|_{L^p}R^{-\frac{n}{q}}.$$

Let R be chosen such that

$$\frac{Mf(x)}{\|f\|_{L^p}} = R^{-n+\gamma-\frac{n}{q}} = R^{-\frac{n}{p}}.$$

Then,

$$|(f * |y|^{-\gamma})(x)| \leq [(Mf)(x)]^{\frac{p}{q}}\|f\|_{L^p}^{1-\frac{p}{q}}.$$

and the result follows from standard L^p bounds on maximal operators, as proved below.

□

B.3.1 Maximal Functions

Definition B.3.3. *We define the centered maximal function of f:*

$$(Mf)(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy.$$

Similarly, we define the uncentered maximal function:

$$(\widetilde{M}f)(x) = \sup_{x\in B} \frac{1}{|B|}\int_{B} |f(y)|dy.$$

Observe $\widetilde{M}f(x) \geq Mf(x)$.

Lemma B.3.4 (Properties of Maximal Functions)**.** *(i) For $f \in L^p$,*

$$\|\widetilde{M}f\|_{L^p} \leq C\|f\|_{L^p}, \quad 1 < p \leq \infty. \tag{B.8}$$

(ii) For $f \in L^1$,

$$\left|\{|\widetilde{M}f| > \alpha\}\right| \leq \frac{C}{\alpha}\|f\|_{L^1}. \tag{B.9}$$

In order to prove these results, we need the following lemma.

Lemma B.3.5 (Vitali Covering Lemma)**.** *Suppose $\{B_k\}$ is a finite family of balls, then there exists a disjoint subset of the balls $\{B_\alpha\}$ such that*

$$\left|\bigcup B_k\right| \leq C \sum_\alpha |B_\alpha|. \tag{B.10}$$

Proof. Begin by choosing the largest ball, say B_{α_1}. Then, choose B_{α_2} the next largest ball such that $B_{\alpha_1} \cap B_{\alpha_2} = \emptyset$. Continue this process.

We have

$$\bigcup B_k \subset \bigcup_i 4B_{\alpha_i}.$$

Suppose that B_j is in the set $\{B_k\}$, but that $B_j \neq B_{\alpha_i}$ for any i. Then, B_j must intersect a larger ball B_α. By the triangle inequality, we have $B_j \subset 4B_\alpha$ since for $x \in B_\alpha$, $y \in B_j$, we have:

$$d(x,y) \leq d(x,x_\alpha) + d(x_\alpha, y) \leq R + d(y,y_j) + d(x_\alpha, y_j) \leq 4R.$$

Thus, Equation (B.10) follows from properties of measures. □

Proof. **(ii)** $x \in \left\{|\widetilde{M}f| \geq \alpha\right\}$ implies there exists B_x such that $x \in B_x$ and

$$\frac{1}{|B_x|}\int_{B_x} |f| dy \geq \alpha.$$

Then

$$\left\{|\widetilde{M}f| \geq \alpha\right\} \subset \bigcup_{x \in \left\{|\widetilde{M}f| \geq \alpha\right\}} B_x.$$

By the Vitali Covering Lemma, locally there exists $\{B_{x_k}\}$ such that $B_{x_i} \cap B_{x_j} = \emptyset$ and

$$\begin{aligned} |\{\, |\widetilde{M}f| \geq \alpha \,\}| &\leq 4 \sum_k |B_{x_k}| \\ &\leq \frac{4}{\alpha} \sum_k \int_{B_{x_k}} |f| dy \\ &\leq \frac{4}{\alpha} \|f\|_{L^1}. \end{aligned}$$

This result can then be extended by looking at $\left\{|\widetilde{M}f| \geq \alpha\right\}$ as the limit of an increasing sequence of compact sets.

(i) Let $f \in L^p$. Write $f = f_1 + f_2$ where:

$$\begin{aligned} (a) \quad f_1 &= f \cdot \chi_{\{f \leq \frac{\alpha}{2}\}} \in L^p, \text{ and} \\ (b) \quad f_2 &= f \cdot \chi_{\{f \geq \frac{\alpha}{2}\}} \in L^1. \end{aligned}$$

Then $\widetilde{M}f(x) > \alpha \Rightarrow \widetilde{M}(f_1 + f_2)(x) > \alpha$, so $\widetilde{M}f_2(x) > \frac{\alpha}{2}$. Thus,

$$\begin{aligned} \left|\left\{|\widetilde{M}f| > \alpha\right\}\right| &\leq \left|\{|\widetilde{M}f_2| > \frac{\alpha}{2}\}\right| \\ &\leq \frac{C}{\alpha}\|f_2\|_{L^1} \\ &\leq \frac{C}{\alpha}\int_{\frac{\alpha}{2}}^{\infty} |\{|f| > h\}|\, dh + |\{|f| > \alpha\}| \end{aligned}$$

which implies

$$\begin{aligned} \left\|\widetilde{M}f\right\|_{L^p}^p &= \int_0^\infty p\alpha^{p-1} \left|\{\widetilde{M}f(x) > \alpha\}\right| d\alpha \\ &\leq \int_0^\infty p\alpha^{p-2} \int_{\frac{\alpha}{2}}^\infty |\{|f| > h\}|\, dh + \int_0^\infty p\alpha^{p-1} |\{|f| > \alpha\}| \\ &\leq \|f\|_{L^p} + \int_0^\infty \int_0^h p\alpha^{p-2} d\alpha\, |\{|f| > h\}|\, dh \\ &= C\|f\|_{L^p}^p. \end{aligned}$$

□

B.4 Spectral Theory, Resolvents and Resolvent Estimates

We present here several results from (RSv4), (HS) and (Taylor2).

Definition B.4.1. *Let A be a linear operator on a Banach space X. The spectrum of A, denoted $\sigma(A)$ is the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda$ is not invertible. This can happen in the following ways:*

(1) $Ker(A - \lambda) \neq 0$,

(2) $Ker(A - \lambda) = 0$, and $Ran(A - \lambda)$ is dense,

(3) $Ker(A - \lambda) = 0$, but $Ran(A - \lambda)$ is not dense.

The resolvent set of an operator A, denoted $\rho(A)$ is the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda$ is invertible. If $\lambda \in \rho(A)$, the resolvent of A is given by $R_A(\lambda) = (A - \lambda)^{-1}$. The discrete spectrum of A, $\sigma_d(A)$ is given by the set of finite multiplicity, isolated points of $\sigma(A)$. The residual spectrum of an operator A, denoted $\sigma_{res}(A)$ is given by the set $\{\lambda \in \mathbb{C} | Ker(A - \lambda) = \{0\}$ and $Ran(A - \lambda)$ is not dense$\}$. The essential spectrum is given by $\sigma_{ess} = \sigma(A) - \sigma_d(A)$.

Theorem 35. *The resolvent set $\rho(A)$ is an open subset of $\mathbb{C}$ ($\sigma(A)$ is then closed) and $R_A(\lambda)$ is an analytic operator-valued funtion of λ on $\rho(A)$.*

Proof. To show that $\rho(A)$ is open, assume $\lambda_0 \in \rho(A)$ and let $\lambda \in \mathbb{C}$ such that

$$A - \lambda = \|(A - \lambda_0)^{-1}\|^{-1}.$$

Then, we have

$$A - \lambda = (A - \lambda)(1 - (A - \lambda_0)^{-1}(\lambda - \lambda_0)). \tag{B.11}$$

By construction, $\|(A-\lambda_0)^{-1}(\lambda-\lambda_0)\| < 1$, hence using Neumann series and operator norms in Banach spaces, we have $1-(A-\lambda_0)^{-1}(\lambda-\lambda_0)$ is invertible, so $A-\lambda$ is invertible. Thus, for any $\lambda_0 \in \rho(A)$, there exists an open, symmetric ϵ-neighborhood, I, such that for any $\lambda \in I$, $A-\lambda$ is invertible. Note, ϵ can be taken to be $\|(A-\lambda_0)^{-1}\|^{-1}$. Hence, the resolvent set $\rho(A)$ is open.

To show that $R_A(\lambda)$ is analytic at λ_0, use (B.11) to say that

$$(A - \lambda)^{-1} = (A - \lambda_0)^{-1}(1 - (\lambda - \lambda_0)(A - \lambda_0)^{-1})^{-1}.$$

Taking

$$|\lambda - \lambda_0|\|(A - \lambda_0)^{-1}\|^{-1} < 1,$$

we again use Neumann series to say

$$\begin{aligned}(A - \lambda)^{-1} &= (A - \lambda_0)^{-1} \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k (A - \lambda_0)^{-k} \\ &= \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k (A - \lambda_0)^{-k-1}.\end{aligned}$$

Since this is norm convergent, we have a power series representation and hence analyticity.

□

Definition B.4.2. *We define the adjoint of an operator A with domain $D(A)$, denoted A^*, as a map $A^* : D(A*) \to \mathcal{H}$ satisfying the following relation*

$$\langle Ax, y\rangle = \langle y, A^*x\rangle,$$

for all $y \in D(A)$, $x \in D(A^*)$, where

$$D(A^*) = \{x \in \mathcal{H} | |\langle Ay, x\rangle| \leq C_x\|y\| \text{ for } C_x \text{ indepent of } y \text{ and all } y \in D(A)\}.$$

Theorem 36. *Let $\mathcal{H}$ be a Hilbert space. For any densely defined linear operator A,*

$$\overline{Ran(A)} \oplus Ker(A^*) = \mathcal{H}.$$

Proof. It suffices to prove that $Ker(A^*)$ is the orthogonal complement of $Ran(A)$, then use the following lemma.

Lemma B.4.3. *Let M be any closed subspace of $\mathcal{H}$, and let $M^\perp$ be its orthogonal complement. Then, any $x \in \mathcal{H}$ can be written uniquely as $x = y + z$ with $y \in M$ and $z \in M^\perp$.*

Proof of lemma. Let y be the projection of x onto M, set $z = x - y$. Let $t \in \mathbb{R}$ and $m \in M$. If $d = \|x - y\| = \inf_{n\in M} \|x - n\|$, then

$$d^2 \leq \|x - (y + tm)\|^2 = \|x - y\|^2 + t^2\|m\|^2 - 2t\,\mathrm{Re}\,\langle x - y, m\rangle.$$

Hence, for all t,

$$t^2\|m\|^2 - 2t\,\mathrm{Re}\,\langle x - y, m\rangle \geq 0.$$

However, this implies that $\langle z, m\rangle = 0$. Uniquesness follows from standard arguments. □

Now, let $u \in Ran(A)$ and $v \in Ker(A^*)$. Then, there exists $f \in D(A)$ such that $u = Af$ and

$$\langle u, v\rangle = \langle Af, v\rangle = \langle f, A^*v\rangle = 0.$$

Hence, $Ker(A^*) \subset (Ran(A))^\perp$. Now, let $w \in (Ran(A))^\perp$. For any $u = Af \in Ran(A)$, we have

$$0 = \langle u, w \rangle = \langle Af, w \rangle = \langle f, A^* w \rangle.$$

As $D(A)$ is dense, $A^* w = 0$, so $(Ran(A))^\perp \subset Ker(A^*)$. □

Proposition B.4.4. *Let A be a densely defined operator. Then,*

(1) A^* is always closed,

(2) if A is closed, then $D(A^*)$ is dense,

(3) if A is closed, then $A^{**} = A$.

Theorem 37. *Let A be self-adjoint. If for some $M > 0$ and for all $u \in D(A)$,*

$$\|(\lambda - A)u\| \geq M\|u\|,$$

then $\lambda \in \rho(A)$. Moreover,

$$\{zin\mathbb{C} || z - \lambda| < M\} \subset \rho(A).$$

Theorem 38. *Let A be self-adjoint. Then,*

(1) $\sigma(A) \subset \mathbb{R}$,

(2) $\sigma_{res}(A) = \emptyset$,

(3) eigenvectors corresponding to distinct eigenvalues are orthogonal.

Corollary B.4.5. *Let A be self-adjoint and $z \in \mathbb{C}$, $\operatorname{Im}(z) \neq 0$. Then,*

$$\|R_A(z)\| \leq |\operatorname{Im} z|^{-1}.$$

Theorem 39. *Let A be self-adjoint, $\lambda \in \rho(A)$. Then,*

$$\|R_A(\lambda)\| \leq [dist(\lambda, \sigma(A))]^{-1}.$$

Definition B.4.6. *The Riesz projection for an operator A is given by the contour integral*

$$P_{\lambda_0} = \frac{1}{2\pi i} \oint_{\Gamma_{\lambda_0}} R_A(\lambda) d\lambda,$$

where $\lambda_0 \in \sigma_d(A)$ and Γ_{λ_0} is a contour which does not intersect $\sigma(A)$ anywhere except λ_0. Note, P_{λ_0} is a projection onto the eigenspace at λ_0. Hence, if we allow Γ to encircle all of $\sigma(A)$, we can say

$$f(A) = \frac{1}{2\pi i} \oint_\Gamma f(\lambda) R_A(\lambda) d\lambda,$$

for sufficiently regular f.

Theorem 40. *Let A be a self-adjoint operator and B be a closed operator such that:*
(a) For some (and hence all) $z \in \rho(B) \cap \rho(A)$, $(A-z)^{-1} - (B-z)^{-1}$ is compact,
and either
(b1) $\sigma(A) \neq \mathbb{R}$ and $\rho(B) \neq \emptyset$
or
(b2) There are points of $\rho(B)$ in both the upper and lower half-planes.
Then, $\sigma_{ess}(A) = \sigma_{ess}(B)$.

Theorem 41 (Weyl's Theorem)**.** *Let A be a self-adjoint operator and let C be a relatively compact perturbation of A. Then $\sigma_{ess}(A) = \sigma_{ess}(B)$.*

Note that from here, for V with sufficient decay, it is clear that $\sigma_{ess}(-\Delta + V) = \sigma_{ess}(-\Delta) = [0, \infty)$.

B.5 Analytic Fredholm Theory

We present results here from (Taylor2). Let $\mathcal{L}(V, W)$ be the space of linear operators from Banach space V to Banach space W. An operator $G \in \mathcal{L}(V, W)$ is said to be compact if G maps any bounded set of V to a relatively compact subset of W. Namely, for any bounded sequence $\{v_n\} \in V$, $\{Tv_n\} \in W$ has a convergent subsequence.

Let K be a conpact operator from a Banach space V to itself. Then, we have the following result.

Theorem 42. *For $\lambda \neq 0$, the operator $T = \lambda I - K$ is surjective if and only if it is injective.*

Proof. Assume that T is injective. Then, $T : V \to R(T)$ is injective.

To begin, we establish the following claim.

Claim B.5.1. *Let $T : V \to W$ be a linear operator where V, W are both Banach spaces. If $K : V \to Y$ is a compact operator where Y is another Banach space and*

$$\|u\|_V \leq C\|Tu\|_W + C\|Ku\|_Y, \tag{B.12}$$

for all $u \in V$, the T has closed range.

Proof of claim. Let $Tu_n \to f \in W$. We would like to find a $v \in V$ such that $Tv = f$. Let $L = Ker(T)$ and divide argument into to cases.

Case I:

If $dist(u_n, L) \leq a < \infty$, let $v_n = u_n \; modL$, $\|v_n\| \leq 2a$. Then, $Tv_n = Tu_n \to f$. Also, by passing to a subsequence, we have $Kv_n \to g \in Y$ by the compactness of K. Then, using (B.12) on $u = v_n - v_m$, we have that $\{v_n\}$ is a Cauchy sequence in V, so $v_n \to v$ and $Tv = f$.

Case II:

If $dist(u_n, L) \to \infty$, assume that $dist(u_n, L) \geq 2$ for all n. Set $v_n = u_n \; modL$ such that $dist(u_n, L) \leq \|v_n\| \leq dist(u_n, L) + 1$. Choose $w_n = \frac{v_n}{\|v_n\|}$. Then, $dist(w_n, L) \geq \frac{1}{2}$. Since $\|w_n\| = 1$, we take a subsequence so that $Kw_n \to g \in Y$. Since $Tw_n \to 0$, (B.12) applied to $w_n - w_m$ implies that $\{w_n\}$ is a Cauchy sequence since $\|Tw_n\| \to 0$ as $\|w_n\|$ is bounded and $\|v_n\| \to \infty$. Hence, $w_n \to w \in V$, so $Tw_n \to 0$, but $dist(w, L) \geq \frac{1}{2}$. We have therefore arrived at a contradiction, so Case II is impossible. □

By the claim, $R(T)$ is closed and hence a Banach space. Then, the open mapping theorem implies that T is an isomorphism. If $R(T) = V_1 \neq V$, then we have $V_n = T^n(V_1)$ is a strictly decreasing family of closed subspaces. Hence, we have $v_n \in V_n$ such that $\|v_n\|_V = 1$ and $dist(v_n, V_{n+1}) \geq \frac{1}{2}$ by the following lemma from functional analysis.

Lemma B.5.2. *If W is a proper closed, linear subspace of a Banach space V, there exists $v_n \in V$ such that $\|v_n\|_V = 1$ and $dist(v_n, W) \nearrow 1$.*

Proof of lemma. If W is a proper closed, linear subspace of V, given $v \in V \setminus W$, pick $w_n \in W$ such that $\|v - w_n\| \to dist(v, W)$. Normalizing $v - w_n$ gives the result. □

Thus, for $n > m$,

$$\begin{aligned} Kv_m - Kv_n &= \lambda v_m + [-\lambda v_n - (Tv_m - Tv_n)] \\ &= \lambda v_m + w_{mn} \end{aligned}$$

with $w_{mn} \in V_{m+1}$. Hence, $\|Kv_m - Kv_n\| \geq \frac{|\lambda|}{2}$, contradicting the compactness of K. Hence, T is surjective if it is injective.

Now, assume that T is surjective. Again, we need a lemma from functional analysis.

Lemma B.5.3. *Let $'$ represent taking the dual. If V and W are Banach spaces and $T : V \to W$ is a linear operator, then*

$$Ker(T') = T(V^{\perp}).$$

If $T(V)$ is closed in W, then $T'(W')$ is closed in V' and

$$T'(W') = (KerT)^{\perp}.$$

Proof of lemma. □

Using the first part of Lemma B.5.3, we have $T' = \lambda I - K'$ is injective on V'. Since K' is compact through a simple duality argument, T' is surjective. Hence, by the second part of Lemma B.5.3, T is injective. □

Theorem 43. *The discrete spectrum of K has only 0 as an accumulation point.*

Proof. Suppose to the contrary that we have linearly independent $v_n \in V$, $\|v_n\| = 1$, with $Kv_n = \lambda_n v_n$, with $\lambda_n \to \lambda \neq 0$. Let

$$V_n = span\{v_1, \ldots, v_2\},$$

then by Lemma B.5.2, there exists $y_n \in V_n$, $\|y_n\| = 1$ such that $dist(y_n, V_{n-1}) \geq \frac{1}{2}$. Let $T_\lambda = \lambda I - K$. For $n > m$,

$$\begin{aligned} \lambda_n^{-1} K y_n - \lambda_m^{-1} K y_m &= y_n + [-y_m - \lambda_n^{-1} T_{\lambda_n} y_n + \lambda_m^{-1} T_{\lambda_m} y_m] \\ &= y_n + z_{nm}, \end{aligned}$$

where $z_{nm} \in V_{n-1}$ since $T_{\lambda_n} y_n \in V_{n-1}$ by construction. Thus,

$$\|\lambda_n^{-1} K y_n - \lambda_m^{-1} K y_m\| \geq \frac{1}{2},$$

which contradicts compactness since $\lambda_n \to \lambda \neq 0$. Hence, 0 is the only possible accumulation point. □

An alternative and rather efficient approach to analytic Fredholm theory is given in (SjZw). Before stating their result, we need to state the following:

Definition B.5.4. *A bounded operator $P : H_1 \to H_2$ between two Banach spaces is called Fredholm if the kernel of P,*

$$ker(P) = \{u \in H^1 | Pu = 0\},$$

and the cokernel of P,

$$coker(P) = H_2 / \{Pu \in H^2 | u = H^1\},$$

are finite dimensional.

Then, the authors show:

Proposition B.5.5. *Suppose that for $z \in \Omega \subset \mathbb{C}$, a connected open set, $A(z)$ is family of Fredholm operators depending holomorphically on z. If $A(z_0)^{-1}$ exists at a point $z_0 \in \Omega$, then $z \to A(z)^{-1}$ is a meromorphic family of operators for all $z \in \Omega$.*

The proof relies on relating Fredholm operators to the Grushin problem,

$$\begin{cases} Pu + R_- u_- = v \\ R_+ r = v_+, \end{cases}$$

where

$$P : H_1 \to H_2,\ R_- : H_- \to H_2,\ R_+ : H_1 \to H_+,$$

If the problem is invertible and well-posed, we have

$$\begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix},$$

where E_{-+} is referred to as the effective Hamiltonian.

In particular, they prove:

Proposition B.5.6. *Suppose that for some choice of $R_\pm$, the Grushin problem is well-posed. Then, $P : H_1 \to H_2$ is a Fredholm operator if and only if $E_{-+} : H_+ \to H_-$ is a Fredholm operator, and*

$$\textit{dim ker}(P) - \textit{dim coker}(P) = \textit{dim ker}(E_{-+}) - \textit{dim coker}(E_{-+}).$$

B.6 Agmon's Bootstrapping Lemma

We include here the discussion of improved regularity from (Agmon). To begin, we have the following theorem regarding estimates resulting from null manifolds for a function.

Definition B.6.1. *We define the spaces*

$$\begin{aligned} L^{2,s}(\mathbb{R}^d) &= \{u(x) | (1+|x|^2)^{\frac{s}{2}} u(x) \in L^2(\mathbb{R}^d)\}, \\ \mathcal{H}_{m,s}(\mathbb{R}^d) &= \{u(x) | D^\alpha u \in L^{2,s}(\mathbb{R}^d),\ 0 \le |\alpha| \le m\}. \end{aligned}$$

Note that the spaces $\mathcal{H}_{m,0}$ are the standard L^2 Sobolev spaces.

Theorem 44. *Let $F(x)$ be a real C^∞ function on $\mathbb{R}^d$ such that $F(x) \neq 0$ for $|x| \ge R_0$, and which satisfies*

$$\left| D^\alpha \left(\frac{1}{F(x)} \right) \right| \le C_\alpha$$

for $|x| \ge R_0$ and all multi-indices α. Set $\Gamma = \{x | F(x) = 0\}$. Assume $\Gamma \neq \emptyset$ and that $\nabla F(x) \neq 0$ for $x \in \Gamma$, hence Γ is a smooth, compact $d-1$ dimensional manifold in $\mathbb{R}^d$.

Let $u(x)$ be a function in $\mathcal{H}_{m,0}$ with $m > \frac{1}{2}$, such that $u(x) = 0$ of Γ in the trace sense, meaning that it is simply the restriction to Γ if $u \in C(\mathbb{R}^d)$ and is otherwise defined via a density argument. Then,

$$\frac{u}{F} \in \mathcal{H}_{m-1,0}(\mathbb{R}^d) \cap L^1_{loc}(\mathbb{R}^d),$$

and

$$\left\| \frac{u}{F} \right\|_{H^{m-1}} \leq C \|u\|_{H^m},$$

where C depends only upon F and m.

In the end, the portion of this theorem we will need most to bootstrap through regularity will be the L^1_{loc} bound.

Lemma B.6.2. *The following indentity holds:*

$$\operatorname{Im} \langle R_0^{\pm}(\lambda) f, f \rangle = \pm \frac{\pi}{2\sqrt{\lambda}} \int_{|\xi|=\sqrt{\lambda}} |(\tau \hat{f})(\xi)|^2 d\sigma,$$

where $\tau \hat{f}$ is the trace of $\hat{f}$ on the frequency sphere $|\xi| = \sqrt{\lambda}$.

Proof. The identity follows from the formal calculation using Plancherel and the frequency space representation of R_0. □

Theorem 45 (Agmon's Bootstrapping Theorem)**.** *Let $V(x)$ be a real function of $\mathbb{R}^d$ such that $(1 + |x|)^{\delta} V(x)$ has sufficient decay properties (as discussed in Section 6.1) for some $\delta > 0$. Let $u \in \mathcal{H}^{loc}_{2,0}$ be a solution of*

$$-\Delta u + V(x)u = \lambda u$$

for $\lambda > 0$. Assume $u \in \mathcal{H}_{2,s_0}$ for some $s_0 > -\frac{1}{2} - \delta$. If $\hat{u} \in L^1_{loc}$, then $u \in \mathcal{H}_{2,s}$ for any real s.

Proof. It suffices to show that $u \in \mathcal{H}_{2,s_0+\delta}$. Then, by repeating the argument, we have that $u \in \mathcal{H}_{2,s_0+j\delta}$ for $j = 1, 2, \ldots$. However, this gives the result through interpolation. □

B.7 Derivation of Convolution Operators

In this note, we derive the integral kernel in $\mathbb{R}^3$ for the inverse of the differential operator

$$P_\mu = -\Delta - \mu^2.$$

Specifically, given $u, f : \mathbb{R}^3 \to \mathbb{R}$, we find $K_\mu(x, y)$ such that if

$$P_\mu u = f,$$

then

$$u = \int_{\mathbb{R}^3} K_\mu(x, y) f(y) dy.$$

To begin, we Fourier transform the equation to see

$$(\xi^2 - \mu^2)\hat{u} = \hat{f},$$

hence

$$u = \mathcal{F}^{-1}[(\xi^2 - \mu^2)^{-1}] * f.$$

So,

$$K_\mu(x, y) = \mathcal{F}^{-1}[(\xi^2 - \mu^2)^{-1}](x - y),$$

if we can define

$$G(x) = \mathcal{F}^{-1}[(\xi^2 - \mu^2)^{-1}](x)$$

in a meaningful sense.

Without loss of generality, set $\mu > 0$. Initially, asssume that $x \neq 0$, though this will be easily seen as a limiting case in the end. We have

$$\begin{aligned} G(x) &= \int_{\mathbb{R}^3} \frac{e^{ix\cdot\xi}}{(\xi^2 - \mu^2)} d\xi \\ &= \int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{e^{i|x|rcos(\theta)}}{(r - \mu)(r + \mu)} r^2 \sin(\theta) d\theta d\phi dr \\ &= \frac{4\pi}{|x|} \int_0^\infty \frac{r \sin(|x|r)}{(r - \mu)(r + \mu)} dr \end{aligned}$$

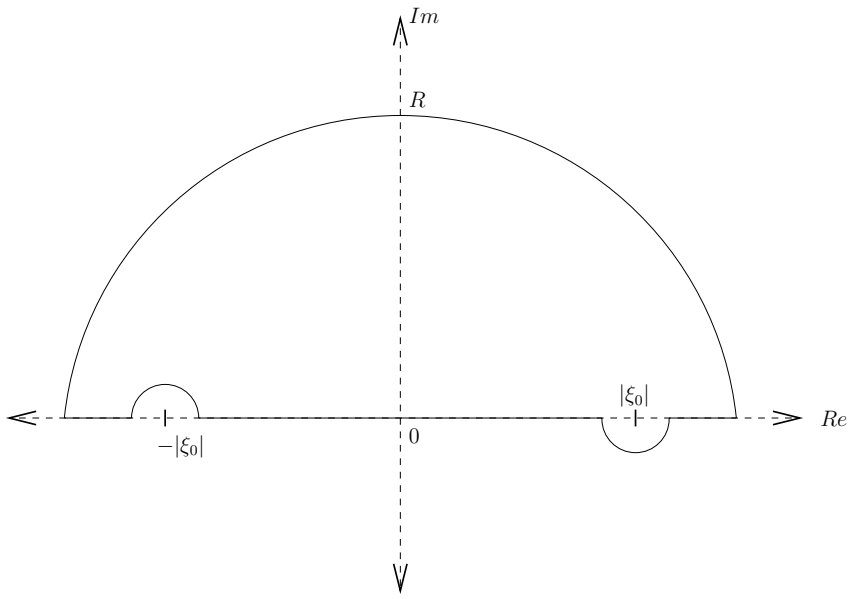

Figure B.1. The contour for computing the behavior of K_μ

by first making a rotational change of variables where $\xi_3 \to \frac{x}{|x|}$, then using polar coordinates.

Now, we are set up to use contour integration to find $G(x)$. See Figure B.1 for the contour over which we integrate. We call this contour $\Gamma_{R,\epsilon}$.

Then, we have from residue theory

$$\begin{aligned}\int_{\Gamma_{R,\epsilon}} \frac{ze^{iz|x|}}{(z-\mu)(z+\mu)}dz &= 2\pi i\left[\frac{\mu e^{i|x|\mu}}{2\mu}\right] \\ &= \pi i e^{i|x|\mu}.\end{aligned}$$

However, breaking Γ down, we also have

$$\int_{\Gamma_{R,\epsilon}} \frac{ze^{iz|x|}}{(z-\mu)(z+\mu)}dz = 2i\int_0^R \frac{r\sin(|x|r)}{(r-\mu)(r+\mu)}dr + \frac{\pi e^{i|x|\mu}}{2} - \frac{\pi e^{-i|x|\mu}}{2}.$$

Combining terms and taking $R \to \infty$, we have

$$G(x) = \frac{4\pi^2\cos(\mu|x|)}{|x|}.$$

This is valid for all x since the integral diverges as $x \to 0$.

Notice that in this analysis we have used the distributional convention

$$f(\lambda) = \frac{1}{2}\left[f(\lambda + i0) + f(\lambda - i0)\right].$$

However, it is not hard to see that taking the convention

$$f(\lambda) = f(\lambda \pm i0)$$

results in the

$$G(x) = \frac{4\pi^2 e^{\pm i|x|\mu}}{|x|}.$$

Appendix C

Microlocal Analysis and Oscilatory Integrals

C.1 Symbol Calculi and Microlocal Analysis

We begin with some preliminaries from the notes of Tataru (Tataru) and (Taylor2).

C.1.1 Pseudodifferential Operators

In this section, we discuss properties of Pseudodifferential Operators (PDO's). In $\mathbb{R}^n$, we define the Weyl quantization of an operator $a(x, hD)$ where $a \in \mathcal{S}(\mathbb{R}^{2n}), a = a(x, \xi)$ by:

$$a(x, D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a(x, \xi)u(y)dyd\xi,$$

for $u \in \mathcal{S}$ and a symbol class by:

$$S^k_\delta(m) = \{a \in C^\infty(\mathbb{R}^{2n}) | |\partial^\alpha a| \leq C_\alpha |\xi|^{-\delta|\alpha|-k} m \text{ for all multi-indices } \alpha\},$$

where $m : \mathbb{R}^{2n} \to (0, \infty)$ is is an order function, i.e. there exist constants C, N such that

$$m(z) \leq C\langle z - w\rangle^N m(w).$$

We also define

$$S^{-\infty}_\delta(m) = \bigcap_{k=-\infty}^{\infty} S^k_\delta(m).$$

For $k, \delta = 0$ we write simply $S(m)$.

Look at $P(x, D)u = f$. One possible approach to such problems involves approximate solutions or parametrices:

$$K \Rightarrow P(x, D)K = I + \text{error}.$$

If the error is small, iterative methods give an exact solution. If the error is smooth, the problem is solved modulo smooth functions.

As seen in (Strichartz), let $R =$ error from above. Then, to solve the equation locally, we need to find g such that $(PK) * g = g + (K * g)$ in some set U. Thus, we have changed the problem to solving the integral equation $g + K * g = f$ on U. Choose a cut-off function ψ which is one on U and zero off V where $U \subset V$. Choose ϕ a cut-off function on V. Note that $\phi\psi = \psi$. Hence, we need only solve the system

$$g + \psi(K * (\phi g)) = \psi f.$$

We define

$$\tilde{K}g = \int \psi(x)K(x - y)\phi(y)\ dy,$$

where $\tilde{K}g$ is supported on V. Then, we have

$$g = \sum_{k=0}^{\infty} (-1)^k (\tilde{K})^k (\psi f),$$

which converges for for U, V small enough to make $\|\tilde{K}\| < 1$.

Let us approach the problem more formally. Consider $u \to P(x, D)u$, and suppose that $P(x, D)u = c(x)D_x^\alpha u$ which implies

$$\begin{aligned} c(x)D^\alpha u &= c(x)\mathcal{F}^{-1}\mathcal{F}D^\alpha u \\ &= c(x)\mathcal{F}^{-1}\xi^\alpha \hat{u} \\ &= \int c(x)\xi^\alpha e^{ix\xi}\hat{u}d\xi. \end{aligned}$$

Thus for $P(x, D) = \sum c_\alpha(x)D^\alpha$, we have $p(x, \xi) := \sum c_\alpha \xi^\alpha$, which we call the symbol of the operator. That is,

$$P(x, D)u = \int p(x, \xi)e^{ix\xi}\hat{u}(\xi)d\xi.$$

Suppose now that $p(x,\xi)$ is a smooth function on $\mathbb{R}^{2n}$. Define $P(x,D)u$ by

$$P(x,D)u = \int p(x,\xi)e^{ix\xi}\hat{u}(\xi)d\xi. \tag{C.1}$$

Since $u \in \mathcal{S} \Rightarrow \hat{u} \in \mathcal{S}$, if

$$|\partial_x^\beta \partial_\xi^\alpha p(x,\xi)| \leq C_\alpha(1+|\xi|+|x|)^{N_{\alpha,\beta}},$$

then

$$P(x,D) : \mathcal{S} \to \mathcal{S}.$$

We want to use the Fourier transform to get an idea of what the adjoints will look like. Let $Pu(x) = \int p(x,\xi)e^{i\xi(x-y)}u(y)dyd\xi$.

$$\begin{aligned}
\langle P(x,D)u, v\rangle &= \int\int p(x,\xi)\hat{u}(\xi)e^{ix\xi}\bar{v}(x)d\xi dx \\
&= \int \hat{u}\int \overline{\bar{p}(x,\xi)e^{-ix\xi}v(x)}dxd\xi \\
&= \int u(y)\overline{\mathcal{F}^{-1}\int(\bar{p}(x,\xi)e^{-ix\xi}v(x))dxdy} \\
\Rightarrow P^*v(y) &= \int\int \bar{p}(x,\xi)e^{-i(x-y)\xi}v(x)dxd\xi.
\end{aligned}$$

Thus if $P(x,D) = \sum c_\alpha(x)D^\alpha$, $(P(x,D))^* = \sum D^\alpha\overline{c_\alpha(x)}$.

Remark C.1. *We say* $(P(x,D))^* = \bar{P}(D,x)$.

C.1.2 Weyl Calculus

We make a slight variation in the definition of $P(x,D)$ in (C.1). Set

$$p^w(x,D)(u) = \int p\left(\frac{x+y}{2},\xi\right)e^{i\xi(x-y)}u(y)dyd\xi. \tag{C.2}$$

The Weyl calculus is extremely useful since $(p^w(x,D))^* = \bar{p}^w(x,D)$. We will see more on functional calculi in section 9.

Remark C.2. *Note that Pseudodifferential Operators do not necessarily commute.*

Example C.1.1. $[x,D] = xD - Dx = i$.

Thus, we need a functional calculus. That is, we need to determine how pseudodifferential operators act under compositions. First, we prove the uncertainty principle.

C.1.3 The Uncertainty Principle

Recall that functions of the form $e^{ix\xi_0}$ are purely oscillatory, whereas δ_{x_0} are purely spatial. Suppose we see a concentration around $(0,0)$ in phase space. Note that $x\delta_0 = 0$. Given a function u, $\|xu\|_{L^2}$ measures the concentration around $x = 0$, and $\|\xi\hat{u}\|_{L^2}$ measures the concentration around $\xi = 0$.

Lemma C.1.2 (Uncertainty Principle)**.**

$$\|xu\|_{L^2}\|Du\|_{L^2} \geq \frac{n}{2}\|u\|_{L^2}^2$$

in $\mathbb{R}^n$*.*

Proof.

$$\begin{aligned}\left|\int_{\mathbb{R}^n} n|u|^2dx\right| &= \left|\int_{\mathbb{R}^n} x\cdot\partial_x|u|^2dx\right| \\ &= \left|\int_{\mathbb{R}^n} x(u\cdot\partial_x\bar{u}+\bar{u}\cdot\partial u)dx\right| \\ &\leq 2\|xu\|_{L^2}\|Du\|_{L^2}.\end{aligned}$$

□

Note that equality holds if u is a Gaussian.

C.1.4 L^2 boundedness of Pseudodifferential Operators

Theorem 46. *Suppose* $m(x,\xi)$ *is supported on* $B(0,1)\times\mathbb{R}^n$ *and that* $|\partial_x^\alpha m(x,\xi)| \leq C_\alpha$*, then* $m(x,D) : L^2 \to L^2$*.*

Proof. We know $B(0,1) \subset Q(0,1)$ where $Q(a,r) = \{x \ : \ |x_i - a_i| \leq r, \ \forall\ 1 \leq i \leq n\}$, so we use Fourier series in $L^2(Q(0,1))$. Let ϕ_j be a basis which has the property that each entry is bounded in L^2 and L^∞. Then, $m(x,\xi) = \sum_j \phi_j(x)m_j(\xi)$, where $m_j(\xi) = \int m(x,\xi)\bar{\phi}_j(x)dx$. Using integration by parts and our assumptions on m, $|m_j(\xi)| \leq Cj^{-N}, \forall N$. Therefore,

$$m(x,D) = \sum_{j=1}^{\infty}\phi_j(x)m_j(D)$$

and

$$\|m(x,D)\|_{L^2\to L^2} \le \sum_{j=1}^{\infty} \|\phi_j\|_{L^\infty}\|m_j\|_{L^\infty} < \infty.$$

□

Theorem 47. *Suppose $m(x,\xi)$ is:*

- *bounded*
- *smooth in ξ*
- *homogeneous of order 0 in ξ.*

Then, $m(x,D) : L^2 \to L^2$.

Proof. Since m is homogeneous in ξ, we have $|\partial_\xi^\alpha m| \le c_\alpha(\frac{1}{|\xi|})^\alpha$, $m(x,\xi) = m(x,\frac{\xi}{|\xi|})$. We choose ϕ_j to be an L^2 basis of eigenfunctions for $-\Delta_{S^{n-1}}$. The ϕ_j are restrictions of homogeneous harmonic polynomials of degree j.

$$\begin{aligned} \|\phi_j\|_{L^\infty} &\le j^{-cN}; \\ m(x,\frac{\xi}{|\xi|}) &= \sum \phi_j(\frac{\xi}{|\xi|})m_j(x), \end{aligned}$$

where $m_j(x) = \int m(x,\frac{\xi}{|\xi|})\bar{\phi_j}(\frac{\xi}{|\xi|})dS^{n-1}$. Thus we have rapid decay $|m_j(x)| \le j^{-N}$.

$$\begin{aligned} m(x,D) &= \sum m_j(x)\phi_j(\frac{D}{|D|}) \Rightarrow \\ \|m(x,D)\|_{L^2\to L^2} &\le \sum \|m_j\|_{L^\infty}\|\phi_j\|_{L^\infty} < \infty. \end{aligned}$$

□

C.1.5 Symbol Classes

Definition C.1.3 (Symbol Classes)**.** *We say a symbol $a(x,\xi)$ is the the symbol class $S^m_{\rho,\delta}$ if and only if $a \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ and*

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le c_{\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|},$$

where m is called the order of a, and $0 \le \delta \le \rho \le 1$.

Example C.1.4. *If $|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \le c_{\alpha,\beta}$, then $p \in S^0_{0,0}$.*

Theorem 48. *If $a \in S^0_{\rho,\rho}$, $0 \le \rho \le 1$, then $a(x,D)$ is L^2 bounded.*

Proof. Let us use a partition of unity: $\mathbb{R}^n = \cup_j A_j$, where

$$A_j = \{2^{j-1} \le |\xi| \le 2^{j+1}\},\ A_0 = B(0,1).$$

Then,

$$1 = \sum_{j=0}^{\infty} \phi_j(\xi),\ \text{supp}\ (\phi_j) = A_j.$$

$|\partial^\alpha \phi_j| \le c_\alpha 2^{-|\alpha|j} \approx c_\alpha (1+|\xi|)^{-|\alpha|}$ since ϕ_j moves from 0 to 1 in a region the size of 2^j. Write $a(x,\xi) = \sum_j a_j(x,\xi)$, where $a_j(x,\xi) = a(x,\xi)\phi_j(x,\xi)$. Then

$$\left|\partial_\xi^\alpha \partial_x^\beta a_j(x,\xi)\right| \le c_{\alpha,\beta} 2^{j\rho(|\beta|-|\alpha|)}.$$

Step 1: Boundedness of $a_j(x,D)$.

Let us introduce a change of variables $x = \lambda y$, $\partial_x = \frac{1}{\lambda}\partial_y$ so that $a_j(x,D) \to a_j(\lambda y, \frac{D}{\lambda})$. In other words, we dilate space by λ and contract the frequency by $\frac{1}{\lambda}$. Then we have

$$\partial_y^\alpha \partial_\nu^\beta a_j((\lambda y, \frac{\nu}{\lambda})| \le (\lambda 2^{j\rho})^{(|\beta|-|\alpha|)}.$$

If $\lambda = 2^{-j\rho}$, then $a_j \in S^0_{0,0} \Rightarrow a_j$ is L^2 bounded.

Step 2: Orthogonality of the a_j's (Summing the a_j's).

We will need the Cotlar-Stein Lemma.

Lemma C.1.5 (Cotlar-Stein)**.** *Let $A = \int_M A_x dx$, where $A_x : H \to H$. If*

$$\int \|A_x^* A_y\|^{\frac{1}{2}}_{H\to H} dx \le C$$

and

$$\int \|A_x A_y^*\|^{\frac{1}{2}}_{H\to H} dx \le C,$$

then $\|A\|_{H\to H} \le C$.

Proof of Lemma C.1.5. We will use the following facts from functional analysis:

Fact 1: $\|A\|^2 = \|AA^*\|$

Fact 2: If B is a self-adjoint operator, then $\|B\|^n = \|B^n\|$.

Thus, we have $\|A\|^{2n} = \|A^*A\|^n = \|(A^*A)^n\|$, where

$$(A^*A)^n = \int_{M^{2n}} A^*_{x_1} A_{x_2} A^*_{x_3} A_{x_4} \cdots A^*_{x_{2n-1}} A_{x_{2n}} dx.$$

But we also have

$$\|A^*_{x_1} A_{x_2} A^*_{x_3} A_{x_4} \cdots A^*_{x_{2n-1}} A_{x_{2n}}\| \leq \|A^*_{x_1} A_{x_2}\| \cdots \|A^*_{x_{2n-1}} A_{x_{2n}}\|$$

and

$$\begin{aligned} &\|A^*_{x_1} A_{x_2} A^*_{x_3} A_{x_4} \cdots A^*_{x_{2n-1}} A_{x_{2n}}\| \leq \\ &\leq \quad \|A^*_{x_1}\| \|A_{x_2} A^*_{x_3}\| \cdots \|A_{x_{2n-2}} A^*_{x_{2n-1}}\| \|A_{x_{2n}}\|. \end{aligned}$$

Thus, we can say

$$\|A^*_{x_1} A_{x_2} A^*_{x_3} A_{x_4} \cdots A^*_{x_{2n-1}} A_{x_{2n}}\| \leq \|A^*_{x_1}\|^{\frac{1}{2}} \|A^*_{x_1} A_{x_2}\|^{\frac{1}{2}} \cdots \| \|A_{x_{2n}}\|^{\frac{1}{2}}.$$

Now, let us make some simplifying assumptions:

1. M has a finite measure m.
2. $\|A_x\|$ is uniformly bounded by N.

Then,

$$\begin{aligned} \|(A^*A)^n\| \quad &\leq \quad \int_{M^{2n}} \|A^*_{x_1}\|^{\frac{1}{2}} \|A^*_{x_1} A_{x_2}\|^{\frac{1}{2}} \cdots \| \|A_{x_{2n}}\|^{\frac{1}{2}} dx \leq NC^{2n-1} m \\ &\Rightarrow \|A\| \leq (mN)^{\frac{1}{2n}} C^{1-\frac{1}{2n}} \overset{n\to\infty}{\to} C. \end{aligned}$$

For complete details, see (Stein). □

Now, continuing with our proof of the theorem, let $A = a(x, D)$, $a = \sum A_j$, $A_j = a_j(x, D)$. Thus, $A^*_j = a_j(\bar{D}, x)$, and

$$\begin{aligned} (A_j A^*_k u)(x) \quad &= \quad a_j(x, D) a_k(\bar{D}, x) u(x) \\ &= \quad \int a_j(x, \xi) e^{i\xi x} e^{-i\xi y} a_k(\bar{y}, \xi) u(y) dy d\xi \\ &= \quad 0 \end{aligned}$$

when $|j-k| > 1$ as a_j, a_k have different regions of support. Thus, we get $\sum_{j=k-1}^{k+1} \|A_j A^*_k\|^{\frac{1}{2}} \leq C$.

Lemma C.1.6 (Rapid Decay of Mixed Terms)**.** $|j-k|>1 \Rightarrow \|A_j^* A_k\| \leq (2^j+2^k)^{-N}$.

Proof. Suppose $j < k-1$. We have $A_j^* A_k = A_j^* S_k(D) A_k + A_j^*(1-S_k(D))A_k$ where S_k selects a ball inside the support of A_{k-2}. Now, A_j^* is L^2 bounded, while $S_k(D)$ and A_k have disjoint supports. Similarly, A_k is L^2 bounded, while A_j^* and $(1-S_k(D))$ have disjoint supports. Thus,

$$\begin{aligned}\sum_{j=0}^{\infty} \|A_j^* A_k\|^{\frac{1}{2}} &\leq \sum_{|j-k|>1} (2^j+2^k)^{-N} \\ &\leq C,\end{aligned}$$

if we can properly bound $S_k(D)a_k(x,D)$ and $a_j(\bar{D},x)(1-S_k(D))$. To do this, let us use the Bargmann Transform and scaling:

$$S_k\left(\frac{D}{\lambda}\right) a_k\left(\lambda x, \frac{D}{\lambda}\right) = T^*TS_k\left(\frac{D}{\lambda}\right) T^*Ta_k\left(\lambda x, \frac{D}{\lambda}\right) T^*T.$$

We have the following two kernels to work with,

$$\begin{aligned}K_1(\xi_1,x_1,\xi_2,x_2) &= (Ta_k(\lambda x, D/\lambda)T^*) \\ K_2(\xi_2,x_2,\xi_3,x_3) &= (TS_k(D/\lambda)T^*).\end{aligned}$$

$$\begin{aligned}|K_1(\xi_1,x_1,\xi_2,x_2)| &\leq [1+|\xi_1-\xi_2|+|x_1-x_2|+\text{dist }((x_2,\xi_2),\text{supp } a_k)]^{-N}. \\ |K_2(\xi_2,x_2,\xi_3,x_3)| &\leq [1+|\xi_2-\xi_3|+|x_2-x_3|+\text{dist }((x_2,\xi_2),\text{supp } S_k)]^{-N}.\end{aligned}$$

Thus,

$$\begin{aligned}&|K(\xi_1,x_1,\xi_3,x_3)| \leq \\ &\leq \int [1+|\xi_1-\xi_2|+|x_1-x_2|+\text{dist }((x_2,\xi_2),\text{supp } a_k)]^{-N} \\ &\qquad \cdot[1+|\xi_2-\xi_3|+|x_2-x_3|+\text{dist }((x_2,\xi_2),\text{supp } S_k)]^{-N} d\xi_2 dx_2 \\ &\leq r^{-\frac{N}{2}}(1+|(\xi_1,x_1)-(\xi_3,x_3)|)^{-N},\end{aligned}$$

where $r = 2^{k(1-\rho)}$. □

□

C.1.6 Adjoints and Compositions of Pseudodifferential Operators

Adjoints.

Formally, using Taylor series, we see

$$\begin{aligned}
[A(x,D)]^* u(x) &= \int a(\bar{y},\xi) e^{i\xi(x-y)} u(y) dy d\xi \\
&= \int \sum_\alpha \frac{\partial_x^\alpha a(\bar{x},\xi)(y-x)^\alpha}{\alpha!} e^{i\xi(x-y)} u(y) dy d\xi \\
&= \int \sum_\alpha \frac{(-i)^\alpha \partial_x^\alpha \partial_\xi^\alpha a(\bar{x},\xi)}{\alpha!} e^{i\xi(x-y)} u(y) dy d\xi.
\end{aligned}$$

So, we would like to show $A(x,D)^* = B(x,D)$ where

$$b(x,\xi) \sim \frac{(-1)^\alpha \partial_x^\alpha \partial_\xi^\alpha a(\bar{x},\xi)}{\alpha!}.$$

But does this converge? We need $a(x,\xi)$ nice as $|\alpha| \to \infty$. Let $a(x,\xi) \in S^m_{\rho,\delta}$, then $b(x,\xi) \in \sum S^{m-|\alpha|(\rho-\delta)}_{\rho,\delta}$. Note that this works fine if $\rho > \delta$ but is meaningless if $\rho \leq \delta$.

Products.

Again, formally using Taylor series, we have

$$\begin{aligned}
(A(x,D)B(x,D)u)(x) &= \int a(x,\xi) e^{i\xi(x-y)} b(y,\nu) e^{i\nu(y-z)} u(z) dz d\nu dy d\xi \\
&= \int \sum_\alpha \frac{\partial_\xi^\alpha a(x,\xi) \partial_x^\alpha b(x,\nu)}{i^{|\alpha|}\alpha!} e^{i\xi(x-y)} u(y) dz d\nu dy d\xi.
\end{aligned}$$

Then, by taking the Fourier transform with respect to y, then integrating over ν, we have

$$\int \sum_\alpha \frac{\partial_\xi^\alpha a(x,\xi) \partial_x^\alpha b(x,\xi)}{i^{|\alpha|}\alpha!} e^{i\xi(x-z)} u(y) dz d\xi.$$

So, formally,

$$c(x,\xi) \sim \frac{\partial_\xi^\alpha a(x,\xi) \partial_x^\alpha b(x,\xi)}{i^{|\alpha|}\alpha!},$$

and again we need $\rho > \delta$ for this to make sense.

Definition C.1.7. *Let $a_j \in S^{m_j}_{\rho,\delta}$, $m_j \searrow -\infty$, then we say $a \sim \sum_{j=1}^\infty a_j$ if*

$$a - \sum_{j=1}^N a_j \in S^{m_{N-1}}_{\rho,\delta}.$$

Theorem 49 (Formal Series)**.** *(i) Let $a(x,\xi) \in S^m_{\rho,\delta}$ for $0 \le \delta < \rho \le 1$. Then,*

$$A(x,D)^* = B(x,D) \tag{C.3}$$

where $b(x,\xi) \in S^m_{\rho,\delta}$, and

$$b(x,\xi) \sim \frac{(-i)^\alpha \partial_x^\alpha \partial_\xi^\alpha \overline{a(x,\xi)}}{\alpha!}. \tag{C.4}$$

(ii) Let $a_j \in S^{m_j}_{\rho,\delta}$, $j = 1,2$ for $0 \le \delta < \rho \le 1$. The

$$A_1(x,D)A_2(x,D) = C(x,D) \tag{C.5}$$

for $c(x,\xi) \in S^{m_1+m_2}_{\rho,\delta}$ where

$$c(x,\xi) \sim \frac{\partial_\xi^\alpha a_1(x,\xi) \partial_x^\alpha a_2(x,\xi)}{i^{|\alpha|}\alpha!}. \tag{C.6}$$

We need to prove a few propositions before proving Theorem 49.

Proposition C.1.8. *Let $a_j \in S^{m_j}_{\rho,\delta}$, $m_j \searrow -\infty$, then there exist $a \in S^{m_1}_{\rho,\delta}$ such that*

$$a \sim \sum_{j=1}^\infty a_j.$$

Proposition C.1.9. *Let $f_j \in \mathbb{R}$, then there exists a function $f \in C^\infty$ such that $f^{(j)}(0) = f_j$.*

Proposition C.1.10. *Suppose $a \in S^m_{\rho,\rho}$, $a_j \in S^{m_j}_{\rho,\rho}$, $m_j \searrow -\infty$, $(a - \sum_{j=1}^m a_j) \in S^{n_j}_{\rho,\rho}$, $n_j \searrow -\infty$, then $a \sim \sum_{j=1}^\infty a_j$.*

We will first prove Proposition C.1.9.

Proof of Proposition C.1.9. Define the function

$$\chi = \{1 \text{ in } [-1,1],\ 0 \text{ in } [-2,2]^c\},$$

where χ is smooth. Let

$$f(x) = \sum_{j=0}^\infty \frac{f_j}{j!} x^j \chi(\frac{x}{\epsilon_j})$$

for $\epsilon_j \to 0$ as $j \to \infty$ to be chosen. Then

$$|f| \le \sum_{j=0}^\infty \frac{f_j}{j!}(2\epsilon_j)^j.$$

We wish to choose our ϵ_j's such that the sum converges.

$$|\partial_x f| \leq C_1 \sum_{j=0}^{\infty} \frac{f_j}{j!}(2\epsilon_j)^{j-i} + c_1 \frac{f_0 \epsilon_0^{-1}}{0!}.$$

Similarly,

$$|\partial^k f| \leq C_k \sum_{j=k}^{\infty} \frac{f_j}{(j-k)!}(2\epsilon_j)^{j-k} + c_k \sum_{j=0}^{k-1} \frac{f_j \epsilon_j^{j-k}}{j!}.$$

So, we need $|\frac{f_j}{(j-k)!}(2\epsilon_j)^{j-k}| \leq \frac{1}{2^j}$ for $k \leq j$. Let $h = j-k$. Then, $(2\epsilon_j)^h \frac{f_j}{h!} \leq 2^{-j}$. Choose ϵ_j such that this occurs for $h = 0, 1, ..., j$. □

Proof of Proposition C.1.8. Let $\chi = \{1 \in B_2^c,\ 0 \in B_1\}$,

$$a(x,\xi) = \sum_{j=0}^{\infty} a_j(x,\xi)\chi\left(\frac{\xi}{R_j}\right),$$

where $R_j \to \infty$. Choose the R_j's such that this sum converges, or

$$|\partial_\xi^\alpha \partial_x^\beta a_j(x,\xi)\chi\left(\frac{\xi}{R_j}\right)| \leq \frac{1}{2^j}(1+|\xi|)^{m_{j-1}-|\alpha|\rho+|\beta|\delta}$$

for $|\alpha|+|\beta| \leq j$, $m_{j-1} > m_j$. We have different cases for the different derivatives:

Case 1. All derivatives fall on $a_j \Rightarrow$

(1) A bad constant,

(2) An extra $(1+|\xi|)^{m_j - m_{j-1}}$.

We must choose R_j large enough such that 2 controls 1.

Case 2. Some derivatives fall on $\chi(\frac{\xi}{R_j}) \Rightarrow$ a factor of $\frac{1}{R_j}$ where $\xi \approx R_j$.

Show convergence:

$$\begin{aligned}\sum_{j=0}^{\infty} |\partial_\xi^\alpha \partial_x^\beta a_j(x,\xi)\chi\left(\frac{\xi}{R_j}\right)| &\leq C \sum_{j=|\alpha|+|\beta|}^{\infty} \frac{1}{2^j}(1+|\xi|)^{m_j-|\alpha|\rho+|\beta|\delta} + c \sum_{j=i}^{|\alpha|+|\beta|} ... \\ &< \infty.\end{aligned}$$

□

Remark C.3. *A note on remainder estimates:*

$$a - \sum_{j=1}^{m} = -\sum_{j=1}^{m} a_j(x,\xi)\left(1-\chi\left(\frac{\xi}{R_j}\right)\right) + \sum_{j=m+1}^{\infty} a_j \chi\left(\frac{\xi}{R_j}\right).$$

We have:

$$\sum_{j=m+1}^{\infty} \left| \partial_\xi^\alpha \partial_x^\beta \left[a_j)(x,\xi) \chi\left(\frac{\xi}{R_j}\right)\right]\right| \leq$$
$$\leq \sum_{j=m+1}^{\max\{m+1,|\alpha|+|\beta|\}} \cdots + \sum_{j=|\alpha|+|\beta|}^{\infty} \frac{1}{2^j}(1+|\xi|)^{m_{j-1}-|\alpha|\rho+|\beta|\delta}.$$

Proof of Proposition C.1.10. We want to show

$$\left| \partial_\xi^\alpha \partial_x^\beta (a(x,\xi) - \sum_{j=1}^{m} a_j(x,\xi)) \right| \leq c_{\alpha,\beta}(1+|\xi|)^{m+1-\rho|\alpha|+\delta|\beta|}.$$

But,

$$\begin{aligned} |\partial_\xi^\alpha \partial_x^\beta (a(x,\xi) - \sum_{j=1}^{m} a_j(x,\xi))| \leq \\ \leq \ |\partial_\xi^\alpha \partial_x^\beta [a(x,\xi) - \sum_{j=1}^{\tilde{m}} a_j(x,\xi)]| + |\partial_\xi^\alpha \partial_x^\beta \sum_{j=m+1}^{\tilde{m}} a_j(x,\xi)| \\ \leq \ (1+|\xi|)^{\tilde{m}-\rho|\alpha|+\delta|\beta|} + C. \end{aligned}$$

So,

$$a \sim \sum_{j=1}^{\infty} a_j.$$

□

Definition C.1.11.

$$S^{-\infty} = \bigcap_{m=0}^{-\infty} S^m_{1,0}.$$

This is the symbol class with all derivatives rapidly decaying.

We are now in a position ro prove Theorem 49

Proof of Theorem 49. Use a Littlewood-Paley Decomposition to get

$$a(x,D) = \sum_{j=0}^{\infty} a_j(x,D)$$

where $\text{supp}(a_j) \subset \{|\xi| \sim 2^j\}$.

Given a_j, we want b_j s.t.

$$b_j \approx \sum \frac{\partial_\xi^\alpha \partial_x^\beta \overline{a_j(x,\xi)}}{i^{|\alpha|}\alpha!}.$$

We know that:

$$|\partial_\xi^\alpha \partial_x^\beta a_j(x,\xi)| \leq c_{\alpha,\beta} 2^{j(m-\rho|\alpha|+\delta|\beta|)}.$$

Rescale using $x \to \lambda x$ and $\xi \to \lambda^{-1}\xi$. Then, we have

$$|\partial_\xi^\alpha \partial_x^\beta a_j(\lambda x, \lambda^{-1}\xi)| \leq c_{\alpha,\beta} 2^{j(m-\rho|\alpha|+\delta|\beta|)} \lambda^{-|\alpha|+|\beta|}.$$

We want to choose λ so that $2^{-j\rho}\lambda^{-1} = 2^{j\delta}\lambda$, or $\lambda^2 = 2^{j(\delta-\rho)}$.

Let $\tilde{a}_j = a_j(\lambda x, \lambda^{-1}\xi)$, then

$$|\partial_\xi^\alpha \partial_x^\beta \tilde{a}_j| \leq c_{\alpha,\beta} 2^{jm} \mu^{|\alpha|+|\beta|},$$

where $\mu = \lambda^{-1}2^{-j\rho} = 2^{j\delta}\lambda \Rightarrow \mu = 2^{\frac{j}{2}(\delta-\rho)} \leq 1$. Thus,

$$\begin{aligned} a_j(y,\xi) &= \\ &= \sum_{|\alpha|=0}^{N} \frac{\partial_x^\alpha a_j(x,\xi)}{\alpha!}(x-y)^\alpha + (x-y)^{N+1} \int_0^1 \frac{h^N \partial_x^{N+1} a_j((1-h)x+hy,\xi)}{(N+1)!} dh \end{aligned}$$

Then,

$$\begin{aligned} a_j(x,D)^* u &= \\ &= \int \overline{a_j(y,\xi)} e^{i\xi(x-y)} u(y) dy d\xi \\ &= \int \sum_{|\alpha|}^{N} \frac{\partial_x^\alpha \overline{a_j(x,\xi)}(y-x)^\alpha}{\alpha!} e^{i\xi(x-y)} u(y) dy d\xi \\ &\quad + \int \frac{(x+y)^{N+1}}{(N+1)!} h^N \partial_x^{N+1} \overline{a_j((1-h)x+hy,\xi)} dh dy d\xi, \end{aligned}$$

where $|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq c_{\alpha,\beta}\mu^{|\alpha|+|\beta|}$. We need to prove $a(x,D)^* - b_{N-1}(x,\xi) \in \mu^{-2N} \cdot S^0_{0,0}$. To do this, let us restate the problem in a stronger form.

Assuming $|\partial_x^{\alpha+\alpha_1}\partial_\xi^{\alpha+\beta_1} a(x,\xi)| \leq c_{\alpha,\alpha_1,\beta_1}$, $|\alpha| = N$, then

$$a(x,D)^* - b_{N-1}(x,\xi) \in S^0_{0,0}.$$

This is the same as above, we have simply moved the factors of μ. The proof of this statement goes as follows:

$$\begin{aligned}
a(x,D)^*u = & \\
&= \int \overline{a(y,\xi)} e^{i\xi(x-y)} u(y) dy d\xi \\
&= \int \sum_{|\alpha|}^{N} \frac{\partial_x^\alpha \overline{a(x,\xi)} (y-x)^\alpha}{\alpha!} e^{i\xi(x-y)} u(y) dy d\xi \\
&\qquad + \int \left((y-x)\partial_x\right)^{N+1} h^N \overline{a((1-h)x+hy,\xi)} dh dy d\xi \\
&= \int b_N e^{i\xi(x-y)} u(y) dy \\
&\qquad + \int \int_0^1 \partial_x^N \partial_\xi^N h^N e^{i\xi(x-y)} a((1-h)x+hy,\xi) u(y) dy d\xi dh,
\end{aligned}$$

where

$$b_N = \sum_{|\alpha|\le N} \frac{\partial_x^\alpha \partial_\xi \overline{a(x,\xi)}}{i^{|\alpha|}\alpha!}.$$

Is the remainder in $S^0_{0,0}$?

Claim C.1.12. *The function $q(x,y,\xi) = \partial_x^N \partial_\xi^N a((1-h)x+hy,\xi)$ is bounded with bounded derivatives.*

Proof of Claim. Let

$$Qu(x) = \int q(x,y,\xi) e^{i\xi(x-y)} u(y) dy d\xi.$$

We need the kernel of TQT^*. So, we have:

$$\begin{aligned}
K(\xi_2,x_2,\xi_1,x_1) = & \\
&= e^{-\frac{1}{2}(y-x_1)^2} e^{i\xi_1(y-x_1)} q(x,y,\xi) e^{i\xi(x-y)} e^{-\frac{1}{2}(x-x_2)^2} e^{i\xi_2(x_2-x)} dx dy d\xi \\
&= \int e^{-\frac{1}{2}(y-x_1)^2} e^{i\xi_1(y-x_1)} q(x-x_2,y,\xi) e^{i\xi(x-y)+i\xi_2(x_2-x)} dx dy d\xi \\
&= \int e^{-\frac{1}{2}(x_1-y)^2} e^{i\xi_1(y-x_1)} q(\xi-\xi_2,y,\xi) e^{i\xi(x_2-y)} dy d\xi,
\end{aligned}$$

as $(x-y) = (x-x_2)+(x_2-y)$.

Repeat this argument to get proper bounds on $K(\xi_2,x_2,\xi_1,x_1)$ and thus $q \in S^0_{0,0}$. This will prove (C.3-C.4). □

Now we proceed to prove (C.5-C.6).

$$|\partial_\xi^{\alpha+\alpha_1}\partial_x^\beta a_1(x,\xi)| \le c_{\alpha,\alpha_1,\beta},\ |\alpha| = N$$
$$|\partial_\xi^{\alpha+\alpha_1}\partial_x^\beta a_2(x,\xi)| \le c_{\alpha,\alpha_1,\beta},\ |\alpha| = N.$$

We need to prove $a_1(x,D)a_2(x,D) - b_{N-1}(x,D) \in S^0_{0,0}$.

$$\begin{aligned}
&(a_1(x,D)a_2(x,D)u)(x) = \\
&= \int a_1(x,\xi)e^{i\xi(x-z)}a_2(z,\nu)e^{i\nu(z-y)}u(y)dydzd\xi d\nu \\
&= \int \sum_{|\alpha|<N} a_1(x,\xi)e^{i\xi(x-z)}a_2(z,\nu)e^{i\nu(z-y)}u(y)dydzd\xi d\nu \\
&= \int \sum_{|\alpha|<N} a_1(x,\xi)e^{i\xi(x-z)}\frac{\partial_x^\alpha a_2(x,\nu)}{\alpha!}(z-x)^\alpha e^{i\nu(z-y)}u(y)dydzd\xi d\nu \\
&\quad + \int\int_0^1 a_1(x,\xi)e^{i\xi(x-z)}\partial_x^N a_2((1-h)x+hy,\nu) \\
&\qquad \cdot(z-x)^N e^{i\nu(z-y)}u(y)dydzd\xi d\nu dh \\
&= \int b_{N-1}e^{i\xi(x-z)}e^{i\nu(z-y)}u(y)dydzd\xi d\nu.
\end{aligned}$$

Note that $q \in \mathcal{C}^\infty$ with bounded derivatives. Let

$$Qu(x) = \int q(x,\xi,y,\nu)e^{i\xi(x-z)}e^{i\nu(z-y)}u(y)dyd\xi dzd\nu.$$

We need derivatives on the Bargman Kernel of Q and TQT^*.

$$\begin{aligned}
&K(x_2,\xi_2,x_1,\xi_1) = \\
&= \int e^{-\frac{1}{2}(x-x_2)^2}e^{i\xi_2(x_2-x)}q(x,\xi,z,\nu,y)e^{i\xi(x-z)}e^{i\nu(z-y)} \\
&\qquad \cdot e^{-\frac{1}{2}(x_1-y)^2}e^{i\xi_1(y-x_1)}dxdydzd\xi d\nu \\
&= \int q(x-x_2,\xi,z,\nu,y-x_1)e^{i\xi_2(x_2-x)+i\xi(x-z)}e^{i\nu(z-y)+i\xi_1(y-x_1)}dxdydzd\xi d\nu \\
&= \int q(x-x_2,z,\nu,y-x_1)e^{i\phi}dxdydzd\xi d\nu,
\end{aligned}$$

where $\phi = (\xi_2-\xi)(x_2-x) + \xi(x_2-x) + \nu(z-y) + \xi_1(y-x_1)$.

Integrate in x to get:

$$= \int q(\xi-\xi_2,z,\nu,y-x_1)e^{i(\xi(x_2-z)+\nu(z-y)+\xi_1(y-x_1))}dyd\xi dzd\nu.$$

Repeat this process to get the desired bounds on $K(\xi_2,x_2,\xi_1,x_1)$. □

C.1.7 Properties of the Calculus of P.D.O.'s

Left Calculus: $a(x, D)$

Right Calculus: $a(D, y)$

Weyl Calculus: $a^w(\frac{x+y}{2}, D)$.

$$a\left(\frac{x+y}{2}, D\right) b\left(\frac{y+z}{2}, D\right) = c\left(\frac{x+z}{2}, D\right), \text{ where}$$
$$c(x,\xi) = \sum \frac{(-1)^\alpha \partial_x^\alpha \partial_\xi^\beta a(x,\xi) \partial_x^\beta \partial_\xi^\alpha b(y,\xi)}{(2i)^{|\alpha|+|\beta|}\alpha!\beta!}$$

Using Taylor Series, take y into z from a, y into x from b. In this series, we get

$$c = ab + \frac{1}{2i}\{a, b\} + ...$$

What is the symbol of a P.D.O.?

Normally, we must talk about a Calculus and a symbol. We have:

$$a(x, D) \to a(x, \xi)$$
$$a(D, x) \to b(x, D) \to b(x, \xi) = a(x, \xi) + \frac{1}{i} a_{x\xi}(x, \xi) + ...$$

The Principal Symbol of an Operator.

If $a \in \mathrm{O}PS^m_{\rho,\delta}$ with error in $OPS^{m+\delta-\rho}_{\rho,\delta}$, then the principle symbol is contained in

$$S^m_{\rho,\delta} \Big/ S^{m+\delta-\rho}_{\rho,\delta},$$

and it is independent of the Calculus.

Example C.1.13. *The principal symbol of* $a(x, D)^*$ *is* $\overline{a(x,\xi)}$.

Example C.1.14. *The principal symbol for the product* $a_1(x, D)a_2(x, D)$ *is*

$$(princ.\ symb.(a_1))(princ.\ symb.(a_2)).$$

An important fact is that the the principal symbol of a commutator is well-defined. A commutator of a P.D.O. is given by:

$$[a_1(x, D), a_2(x, D)] = a_1(x, D)a_2(x, D) - a_2(x, D)a_1(x, D),$$

where $a_1 \in S^{m_1}_{\rho,\delta}$, $a_2 \in S^{m_2}_{\rho,\delta}$. In the expansion of the commutator, we see the first term is simply 0, while the second term is $\frac{1}{i}\{a_1, a_2\}$. Hence, the principal symbol of a commutator is the essentially the Poisson Bracket of the two elements.

C.1.8 L^p bounds for P.D.O.'s

For multipliers, $|\partial^\alpha m(\xi)| \leq (1+|\xi|)^{-|\alpha|} \Longrightarrow m(D) : L^p \to L^p$.

Theorem 50. *If $a \in S^0_{1,\delta}$, $\delta < 1$, then $a(x,D) : L^p \to L^p$.*

Proof. For multipliers, we have

$$m(D)u(x) = \int K(x-y)u(y)dy,$$

where $K = \check{m}$, $|K(x)| \leq |x|^{-n}$, and $|\partial_x K(x)| \leq |x|^{-n-1}$. In this case, we have

$$K(x,y) = \int a(x,\xi)e^{i\xi(x-y)}d\xi.$$

Using a Littlewood-Paley decomposition, we can set $a = \sum a_j$ and similarly $K = \sum K_j$. Then,

$$K_j(x,y) = K_j(x, x-y)$$

Let $z = x - y$. We have:

$$|K_j(x,z)| \leq 2^{jm}(1+2^j|z|)^{-N}$$

and

$$|\partial_z K_j(x,z)| \leq 2^{j(n+1)}(1+2^j|z|)^{-N}.$$

Thus, by summing over j, we see:

$$|K(x, x-y)| \leq |x-y|^{-n}$$

and

$$|\partial_y K(x, x-y)| \leq |x-y|^{-n-1}.$$

As $\partial_x a(x,\xi) \in S^{\delta}_{1,\delta}$ and $\partial_x a_j(x,\xi) \approx 2^{j\delta}$,

$$|\partial_x K_j(x, x-y)| \leq 2^{j\delta} 2^{jn}(1+2^j|z|)^{-N} + 2^{j(n+1))}(1+2^j|z|)^{-N}$$
$$\implies |\partial_x K(x,y)| \leq |x-y|^{-n-1}.$$

□

Remark C.4. *We only need $\delta < 1$ for L^2 boundedness. The kernel bounds are not integrable. In general, we have $a(x,D) : L^1 \not\to L^1$. However, we do have:*

$$a(x,D) : H_1 \to H_1$$
$$a(x,D) : BMO \to BMO.$$

C.1.9 Sobolev Spaces

For P.D.O's, we have $OPS^0_{\rho,\delta} : L^2 \to L^2$.

Proposition C.1.15. $OPS^m_{\rho,\delta} : H^s \to H^{s-m}$, $s \in \mathbb{R}$.

Proof. We know that $(1+|D|^2)^{\frac{s}{2}} : H^s \to L^2$ is an isometry simply by Fourier transform. Hence,

$$a(x,D) : H^s \to H^{s-m} \Leftrightarrow (1+|D|^2)^{\frac{s-m}{2}} a(x,D)(1+|D|^2)^{\frac{-s}{2}} : L^2 \to L^2$$

□

Proposition C.1.16. $OPS^m_{1,\delta} : W^{s,p} \to W^{s-m,p}$ *for* $1 < p < \infty$.

Proof. $W^{s,p} = (1+|D|^2)^{\frac{-s}{2}} L^p$, for $1 < p < \infty$. □

Note that derivatives of L^∞ functions are Lipschitz functions. Half derivatives are Holder functions. Roughly, $L^\infty \to C^\alpha \to Lip$ where α is not an integer. We have that

$$C^\alpha \neq (1+|D|^2)^{\frac{-\alpha}{2}} C^\infty$$

but that

$$C^\alpha = (1+|D|^2)^{\frac{-\alpha}{2}} BMO.$$

C.1.10 Invertibility of P.D.O.'s

Let $a^{ij}(x)D_iD_ju = f$, $a^{ij} \in C^\infty$, positive definite.

Proposition C.1.17. *$f \in H^2$, $u \in H^r$ implies that $u \in H^{s+2}$ where u, f are as above.*

We have here an expression of the form $p(x,D)u = f$ where $p(x,D) = a^{ij}\xi_i\xi_j$, so $p(x,\xi) \in S^2_{1,0}$ and $|p(x,\xi)| \geq |\xi|^2$ for large $|\xi|$.

Proof. Let $q(x,\xi) \in S^{-2}_{1,0}$. Act $q(x,D)$ on both sides of $p(x,D)u = f$. Thus,

$$\begin{aligned} p(x,D)u = f \quad &\Rightarrow \quad q(x,D)p(x,D) - I = r(x,D) \\ &\Rightarrow \quad u + r(x,D)u = q(x,D)f, \end{aligned}$$

where $r \in S^{-\infty}$, $q \in S^{-2}_{1,0}$. □

Theorem 51. *Suppose $a \in S^m_{\rho,\delta}$, $\rho > \delta$, $a(x,\xi) \geq |\xi|^m$ for large $|\xi|$, then there is a $b \in S^{-m}_{\rho,\delta}$ s.t. $a(x,D)b(x,D) - I \in S^{-\infty}$.*

Remark C.5. *$S^{-\infty} : \mathcal{S}' \to C^\infty$. The Kernel of an operator in $S^{-\infty}$ satisfies:*

$$|\partial_x^\alpha \partial_\xi^\beta K(x,y)| \leq c_{\alpha,\beta}(1 + |x-y|)^{-N}.$$

Note that inverses are unique up to $S^{-\infty}$.

$$b_l(x,D) = b_l(x,D)a(x,D)b_r(x,D) + OP\mathcal{S}^{-\infty} = b_r(x,D) + OP\mathcal{S}^{-\infty}.$$

Proof. We seek

$$b(x,\xi) \sim \sum_{j=0}^{\infty} b_j(x,\xi),$$

where $b_j \in OP\mathcal{S}^{m-j(\rho-\delta)}_{\rho,\delta}$ s.t. $b(x,D)a(x,D) - I \in S^{-\infty}$. We will choose the b_j's inductively such that:

$$(b_0 + ... + b_N)(x,D)a(x,D) - I \in OPS^{-(N+1)(\rho-\delta)}_{\rho,\delta}.$$

For $N = 0$, seek $b_0 \in OPS^{-m}_{\rho,\delta}$ s.t. $I - b_0(x,D)a(x,D) \in OPS^{-(\rho-\delta)}_{\rho,\delta}$. Take $b_0(x,\xi) = a(x,\xi)^{-1}$ for large $|\xi|$ and use symbol calculus.

For the induction step, assume that $I - (b_0 + ... + b_{N-1})(x,D)a(x,D) = r_N(x,D) \in S^{-N(\rho-\delta)}_{\rho,\delta}$. Choose

$$b_N(x,\xi) = \frac{r_N(x,\xi)}{a(x,\xi)}$$

for large $|\xi|$. Note that we do not worry about small $|\xi|$ values as we can control them by adding $S^{-\infty}$ symbols. Then,

$$b_N(x,\xi) \in OPS^{-N(\rho-\delta)}_{\rho,\delta}.$$

Thus, by the symbol calculus, we get

$$\begin{aligned} b_N(x,D)a(x,D) &= r_N(x,D) + r_{N+1}(x,D) \\ &\Rightarrow r_{N+1} \in S^{-(N+1)(\rho-\delta)}_{\rho,\delta}. \end{aligned}$$

□

Idea: Look at the differential operator $-\triangle$. We have

$$\langle -\triangle u, u\rangle = \|\nabla u\|^2_{L^2}.$$

Definition C.1.18. *We define $a(x,\xi) \in S^m_{1,0}$ to be positive definite if*

$$\langle a(x,D)u, u\rangle \geq c\|u\|^2_{H^{\frac{m}{2}}} - D\|u\|^2_{H^{\frac{m-1}{2}}}.$$

That is, the operator $a(x,D)$ is positive definite in the usual sense, modulo an error of the next lowest order.

Example C.1.19. *$(1+|D|^2)^{\frac{m}{2}}$ is positive definite.*

We also have the notion of a non-negative definite symbol:

$$\begin{aligned} 0 \quad &\leq \quad \|a(x,D)u\|^2 = \langle a(x,D)u, a(x,D)u\rangle = \langle a(x,D)^* a(x,D)u, u\rangle \\ &\Rightarrow \quad a(x,D)^* a(x,D) \text{ is non-negative definite.} \end{aligned}$$

Note that $(1+|D|^2)^{\frac{m}{2}} + a(x,D)^* a(x,D)$ is positive definite. $a \in S^{\frac{m}{2}}$ and the principal symbol is $(1+|\xi|^2)^{\frac{m}{2}} + |a(x,\xi)|^2$.

C.2 Beals' Lemma

We present here the proof that a general bounded operator is in fact a psuedodifferential operator for some symbol b as present in (EvZw). In particular, this allows us to say that our smooth, decaying error is in fact a PDO in the discussion from Section 6.3. This proof is done here from the semiclassical analysis point of view with $h = 1$, while the original microlocal result is presented in (Beals). However, as the proofs are similar, we present the more concise version from (EvZw).

Theorem 52. *There exist constants C, $M > 0$ such that*

$$\|b\|_{L^\infty} \leq C \sum_{|\alpha| \leq M} \|Op(\partial^\alpha b)\|_{L^2 \to L^2},$$

for all $b \in \mathcal{S}'(\mathbb{R}^{2d})$.

Proof. We have

$$b(x, D)u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} b(x, \xi) e^{i\langle x, \xi \rangle} \hat{u}(\xi) d\xi,$$

where the Fourier transform is taken formally here in $\mathcal{S}'$.

Set $\phi = \phi(x)$, $\psi = \psi(x) \in \mathcal{S}$. Then, we can take $\mathcal{F}(b\phi\hat{\psi}e^{i\langle x, \xi \rangle})$ as a function of dual variables $(x^*, \xi^*) \in \mathbb{R}^{2d}$. We have

$$\begin{aligned} \frac{1}{(2\pi)^d} |\mathcal{F}(b\phi\hat{\psi}e^{i\langle x, \xi \rangle})(0, 0)| &= \frac{1}{(2\pi)^d} \left| \int \int b(x, \xi) \phi(x) \hat{\psi}(\xi) e^{i\langle x, \xi \rangle} dx d\xi \right| \\ &= |\langle b(x, D)\psi, \phi \rangle| \leq \|b\|_{L^2 \to L^2} \|\phi\|_{L^2} \|\psi\|_{L^2}. \end{aligned}$$

Fix $(x^*, \xi^*) \in \mathbb{R}^{2d}$ and rewrite the inequality with $\phi(x)e^{i\langle x^*, x \rangle}$ replacing $\phi(x)$ and $\hat{\psi}e^{-i\langle \xi^*, \xi \rangle}$ replacing $\hat{\psi}(\xi)$, which does not alter the L^2 norm. Then,

$$\frac{1}{(2\pi)^d} |\mathcal{F}(b\phi\hat{\psi}e^{i\langle x, \xi \rangle})(x^*, \xi^*)| \leq \|b\|_{L^2 \to L^2} \|\phi\|_{L^2} \|\psi\|_{L^2}.$$

Take $\chi \in C_c^\infty(\mathbb{R}^{2d})$. Take ϕ, $\psi \in \mathcal{S}$ so that

$$\begin{cases} \phi(x) = 1 \text{ if } (x, \xi) \in \operatorname{supp} \chi \\ \hat{\psi}(\xi) = 1 \text{ if } (x, \xi) \in \operatorname{supp} \chi. \end{cases}$$

Write

$$\phi_1 = \chi e^{-i\langle x,\xi\rangle}.$$

Then,

$$\chi(x,\xi) = \phi_1(x,\xi)\phi(x)\hat{\psi}(\xi)e^{i\langle x,\xi\rangle}.$$

Using standard estimates on the Fourier transform, we have

$$\|\mathcal{F}\chi\|_{L^1} \leq C \sum_{|\alpha|\leq 2d+1} \|\partial^\alpha \chi\|_{L^1},$$

so

$$\|\mathcal{F}\phi\|_{L^1} \leq C \sum_{|\alpha|\leq 2d+1} \|\partial^\alpha \chi\|_{L^1}.$$

Thus, for any $(x^*,\xi^*) \in \mathbb{R}^{2d}$, we have

$$\begin{aligned}
|\mathcal{F}(\chi b)(x^*,\xi^*)| &\leq \|\mathcal{F}(\phi_1 b\phi\hat{\psi}e^{i\langle x,\xi\rangle})\|_{L^\infty} \\
&= \frac{1}{(2\pi)^d}\|\mathcal{F}(\phi_1) * \mathcal{F}(b\phi\hat{\psi}e^{i\langle x,\xi\rangle})\|_{L^\infty} \\
&\leq \frac{1}{(2\pi)^d}\|\mathcal{F}(\phi_1)\|_{L^1}\|\mathcal{F}(b\phi\hat{\psi}e^{i\langle x,\xi\rangle})\|_{L^\infty} \\
&\leq C\|b\|_{L^2\to L^2}
\end{aligned}$$

where C depends only on ϕ, ψ and χ, not (x^*,ξ^*). Hence,

$$\|\mathcal{F}(\chi b)\|_{L^\infty} \leq C\|b\|_{L^2\to L^2}. \tag{C.7}$$

Claim C.2.1.

$$|\mathcal{F}(\chi b)(x^*,\xi^*)| \leq C\langle (x^*,\xi^*)\rangle^{-2d-1} \sum_{|\alpha|\leq 2d+1} \|Op(\partial^\alpha b)\|_{L^2\to L^2}.$$

Proof. We have

$$\begin{aligned}
(x^*)^\alpha(\xi^*)^\beta\mathcal{F}(\chi b)(x^*,\xi^*) &= \int_{\mathbb{R}^{2d}} (x^*)^\alpha(\xi^*)^\beta e^{-i(\langle x^*,x\rangle+\langle \xi^*,\xi\rangle)}\chi b(x,\xi)dxd\xi \\
&= \int\int_{\mathbb{R}^{2d}} e^{-i(\langle x^*,x\rangle+\langle \xi^*,\xi\rangle)} D_x^\alpha D_\xi^\beta(\chi b(x,\xi))dxd\xi.
\end{aligned}$$

Summing over (α, β) with $|\alpha| + |\beta| \leq 2d + 1$ and using (C.7), we have

$$\begin{aligned} \|\langle (x^*, \xi^*)\rangle^{2d+1} \mathcal{F}(\chi b)\|_{L^\infty} &\leq C_1 \sum_{|\alpha|+|\beta|\leq 2d+1} \|\mathcal{F}(D_x^\alpha D_\xi^\beta (\chi b))\|_{L^\infty} \\ &\leq C_2 \sum_{|\alpha|\leq 2d+1} \|Op(\partial^\alpha b)\|_{L^2 \to L^2}. \end{aligned}$$

□

Hence,

$$\|\chi b\|_{L^\infty} \leq C \|\mathcal{F}(\chi b)\|_{L^1} \leq C \sum_{|\alpha|\leq 2d+1} \|Op(\partial^\alpha b)\|_{L^2 \to L^2}.$$

Note that this can be applied in any quantization by standard arguments. □

Definition C.2.2. *Write*

$$ad_B A = [B, A],$$

where ad is called the adjoint action.

Definition C.2.3. *A pair* $(x^*, \xi^*) \in \mathbb{R}^{2d}$ *is identified with a linear operator* $l(x, \xi) = \langle x^*, x\rangle + \langle \xi^*, \xi\rangle$.

Theorem 53 (Beals's Theorem)**.** *Let* $A : \mathcal{S} \to \mathcal{S}'$ *be a continuous linear operator. Then* $A = Op(a)$ *for a symbol* $a \in \mathcal{S}$ *if and only if for all* $N = 0, 1, 2, \ldots$ *and all linear functions* $l_1, \ldots, l_N$*, we have*

$$\|ad_{l_1(x,D)} \circ \cdots \circ ad_{l_N(x,D)} A\|_{L^2 \to L^2} = O(1). \tag{C.8}$$

Proof. If $a \in \mathcal{S}$, then (C.8) holds via standard microlocal techniques. Namely, since $\|A\|_{L^2 \to L^2} = O(1)$ and commuting with $l_j(x, D)$ yields a term of order $O(1)$.

Since A is a continuous linear operator, there is a Schwartz kernel

$$Au(x) = \int_{\mathbb{R}^d} K_A(x, y) u(y) dy$$

where $K_A \in \mathcal{S}'(\mathbb{R}^{2d})$. Denote K_A the kernel of A.

If $a \in \mathcal{S}'(\mathbb{R}^{2d})$ is defined by

$$a(x,\xi) = \int_{\mathbb{R}^d} e^{-i\langle w,\xi\rangle} K_A(a + \frac{w}{2}, x - \frac{w}{2}) dw, \tag{C.9}$$

then

$$K_A(x,y) = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d} a(\frac{x+y}{2}, \xi) e^{i\langle x-y,\xi\rangle} d\xi,$$

where the integrals are shorthand for the formal Fourier transform on $\mathcal{S}'$. Indeed,

$$\begin{aligned} K_A(x,y) &= \frac{1}{(2\pi)^d} \int\int e^{i\langle x-y-w,\xi\rangle} K_A\left(\frac{x+y}{2} + \frac{w}{2}, \frac{x+y}{2} - \frac{w}{2}\right) dw d\xi \\ &= \frac{1}{(2\pi)^d} \int a(\frac{x+y}{2}, \xi) e^{i\langle x-y,\xi\rangle} d\xi \end{aligned}$$

since

$$\frac{1}{(2\pi)^d} \int e^{i\langle x-y-w,\xi\rangle} d\xi = \delta_{x-y}(w)$$

in $\mathcal{S}'$. Hence, $A = Op(a)$ for a as defined in (C.9).

Now, we would like to know that $a \in \mathcal{S}$, or

$$\sup_{\mathbb{R}^{2d}} |\partial^\alpha a| \leq C_\alpha$$

for each multi-index α.

To do so, we use (C.8). Define l_j by the pair (x_j, ξ_j). Compute

$$\begin{aligned} Op(D_{\xi_j} a)u(x) &= \frac{1}{(2\pi)^d} \int D_{\xi_j}\left(a\left(\frac{x+y}{2}, \xi\right)\right) e^{i\langle x-y,\xi\rangle} u(y) d\xi dy \\ &= -\frac{1}{(2\pi)^d} \int a\left(\frac{x+y}{2}, \xi\right) D_{\xi_j}\left(e^{i\langle x-y,\xi\rangle}\right) u(y) d\xi dy \\ &= -\frac{1}{(2\pi)^d} \int a\left(\frac{x+y}{2}, \xi\right) e^{i\langle x-y,\xi\rangle} (x_j - y_j) u(y) d\xi dy \\ &= -[x_j, A]u = -ad_{x_j} A u(x). \end{aligned}$$

Similarly,

$$Op(D_{x_j} a)u(x) = ad_{D_{x_j}} A u(x).$$

In other words, for $j = 1, \ldots, n$,

$$\begin{cases} ad_{x_j} A = -Op(D_{\xi_j} a) \\ ad_{D_{x_j}} A = Op(D_{x_j} a). \end{cases}$$

Hence, from our hypothesis, we have

$$\|Op(\partial^\beta a)\| \lesssim 1$$

for all multi-indices β. From Theorem 52, we have

$$\|\partial^\alpha a\|_{L^\infty} \leq C \sum_{|\beta|\leq d+1} \|Op(\partial^{\alpha+\beta} a)\|_{L^2\to L^2} \leq C_\alpha.$$

Hence, $a \in \mathcal{S}$ and we have the result. □

C.3 Resolvent Identities and Pseudodifferential Operators

We prove here that $L_-^{\pm\frac{1}{2}}$ is a PDO for X a compact manifold as in Hörmander (Ho4).

Theorem 54. *Let X be a compact manifold, Ψ a space of pseudo-differential operators and $\Omega^{\frac{1}{2}}$ be the space of half-densities on X. Let $P \in \Psi^m_{phg}(X;\Omega^{\frac{1}{2}},\Omega^{\frac{1}{2}})$ be a positive, elliptic, symmetric operator. Then, P defines a positive, self-adjoint operator $\mathcal{P}$ in $L^2(X,\Omega^{\frac{1}{2}}$. If $m > 0$ and $a \in \mathbb{R}$, then $\mathcal{P}^a$ is also defined by a pseudodifferential operator in $\Psi^{am}_{phg}(X;\Omega^{\frac{1}{2}},\Omega^{\frac{1}{2}})$, with principal and subprincipal symbols p^a and $ap^{a-1}p^s$ if p and p^s are those for P.*

Proof. $\mathcal{P}$ is bounded if $m \leq 0$, hence self-adjoint. If $m > 0$, then $\mathcal{P}$ is the restriction of P to all $u \in L^2$ with $Pu \in L^2$. This implies $u \in H_{(m)}$ and C^∞ is dense in $H_{(m)}$, hence $\mathcal{P}$ is self-adjoint. The resolvent,

$$R(z) = (\mathcal{P} - z)^{-1},$$

is defined and analytic in z except at eigenvalues of $\mathcal{P}$ on $\mathbb{R}_+$, and the L^2 operator norm can be estimated by the reciprocal of the distance to the eigenvalues. If $a < 0$, it follows from the spectral theorem that with absolute convergence in L^2,

$$\mathcal{P}^a u = -(2\pi i)^{-1} \int_{-i\infty}^{i\infty} z^a R(z) u dz,\ u \in L^2,$$

where z^a is analytic in the right half plane and equal to 1 when $z = 1$. Since $\mathcal{P}^{a+1}u = \mathcal{P}^a\mathcal{P}u$ when u is in the domain of $\mathcal{P}$ and $\operatorname{Re}(a) \leq 0$, it follows that the distribution kernel of $\mathcal{P}^a$

is an entire analytic function of a. By approximating R with a parametrix for $P - z$, we shall determine the singularities for $a < 0$ and by analytic continuation for $a \geq 0$ as well.

Let $Y \subset X$ be a local coordinate patch identified with an open set in $\mathbb{R}^n$. In the local coordinates, the symbol for $P - z$ is uniformly bounded with respect to z in

$$S((1 + |z| + |\xi|^m), g),\ g = |dx|^2 + \frac{|d\xi|^2}{1 + |\xi|^2},$$

where $u \in S(m, g)$ if $u \in C^\infty$ and for every $k \geq 0$, we have

$$\sup_k \frac{|u|_k^g(x)}{m(x)} < \infty,$$

where

$$|u|_k^g = \sup_{t_j \in X} \frac{|u^{(k)}(x; t_1, \ldots, t_k)|}{\Pi_{m=1}^k G(t_m)^{\frac{1}{2}}},$$

and $G = g_x$ is a quadratic form. For fixed k, this is equivalent to the maximum of the derivatives of order k with respect to a G orthogonal coordinate system.

Then, $(P - z)^{-1}$ is uniformly bounded in $S((1 + |z| + |\xi|^m)^{-1}, g)$ if $\mathrm{Re}\,(z) = 0$, which we assume henceforward. Note that we may assume $\mathrm{Re}\,(P(x, \xi)) > 0$ everywhere for $p > 0$ since P is strongly elliptic. Hence,

$$(P(x, D) - z)(P - z)^{-1}(x, D) = I - Q_z(x, D),$$

where Q_z is uniformly bounded in $S((1 + |\xi|)^{m-1}(1 + |z| + |\xi|^m)^{-1}, g)$ by standard bounds on inverses for PDO's. Specifically, we have

$$Q_z(x, \xi) \sim \sum_{\alpha \neq 0} \frac{P^{(\alpha)}(x, \xi) D_x^\alpha (P(x, \xi) - z)^{-1}}{\alpha!}.$$

Let E_z be the asymptotic sum of the symbols of

$$(P - z)^{-1}(x, D)(Q_z(x, D))^N,\ N = 0, 1, \ldots.$$

Then,

$$(P(x, D) - z))E_z(x, D) = I - W_z(x, D),$$

where W_z is uniformly bounded in $S((1 + |z|)^{-1}(1 + |\xi|)^{-N}, g)$ for any N.

Up to now, we have worked only within coordinate patch Y. Choose ψ, $\Psi \in C_0^\infty(Y)$ with $\Psi = 1$ on $\mathrm{spt}(\psi)$. Then, $\Psi E_z(x,D)\psi u$ is defined in X for every u in $\mathcal{D}'(X)$, and

$$(P-z)\Psi E_z(x,D)\psi u = \psi u - W_z(x,D)u$$

for another W_z such that $(1+|z|)W_z$ is uniformly bounded in $\Psi^{-\infty}$. Covering X by coordinate patches Y_j and selecting corresponding $\psi_j, \Psi_j \in C_0^\infty(Y_j)$ such that $\Psi_j = 1$ on $\mathrm{spt}(\psi_j)$ and $\sum_j \psi_j = 1$, we have a parametrix E_z satisfying

$$(P(x,D)-z))E_z(x,D) = I - W_z(x,D),$$

globally in X. Then, we have

$$R(z) = E_z + R(z)W_z,$$

where $(1+|z|)W_z$ is uniformly bounded from $H_{(s)}$ to $H_{(t)}$ for arbitrary s, t and

$$\|R(z)\|_{H_{(t)}\to H_{(t)}} \leq C/(1+|z|$$

since $\|u\|_{(t)} \equiv \|\mathcal{P}^{\frac{t}{m}}u\|_{(0)}$, at least if t is a positive multiple of m. Thus, we have for $a<0$

$$\mathcal{P}^a = -(2\pi i)^{-1}\int_{-i\infty}^{i\infty} z^a E_z dz + T(a)u,$$

where $T(a)$ is an operator with kernel analytic in a when $\mathrm{Re}\,(a) < 1$, with values in $C^\infty(X\times X; \Omega^{\frac{1}{2}}(X\times X))$. Every term in the symbol of E_z is of the form $-(z-P)^{-k-1}qdz$ where $q \in S^{mk-\kappa}$ for some $\kappa \geq 0$. We have

$$-(2\pi i)^{-1}\int_{-i\infty}^{i\infty} z^a(z-P)^{-k-1} = a(a-1)\ldots(a-k+1)P^{a-k}q/k!,$$

which is a symbol in S^{am-k} and depends analytically on a. There are a finite number of terms for fixed κ. Terminating the series for E_z after sufficiently many terms adds an extra error to $T(a)$, yet this error will as many derivatives as we need for $\mathrm{Re}\,(a) < 1$. Hence, $\mathcal{P}^a$ is a pseudo-differential operator for $a<1$. Calculating the terms with $\kappa = 0$ or $\kappa = 1$, we find mod S^{am-2}, the symbol is equal to

$$P^a + \frac{a(a-1)}{2} i \sum p^{(j)}p_{(j)}p^{a-2}.$$

Since P is congruent to $p + p^s - \frac{1}{2} i \sum p^{(j)}_{(j)}$, it follows that P^a is congruent to

$$p^a + a p^{a-1}(p^s - \frac{1}{2} i \sum p^{(j)}_{(j)}).$$

To obtain the sum of the principal and subprincipal symbols for $\mathcal{P}^a$, we add $\frac{1}{2} i \sum \frac{\partial^2 p^a}{\partial x_j \partial \xi_j}$ to the symbol of $\mathcal{P}^a$, which proves the statement on the subprincipal symbol when $a < 1$. It follows immediately from the Weyl calculus that if $\mathcal{P}^a$ is defined by a pseudodifferential operator, this the same holds for $\mathcal{P}^{2a}$, with the principal and subprincipal symbols obtained by multiplication of those of $\mathcal{P}^a$. Hence, the result extends through to all $a \in \mathcal{R}$. □

C.4 Principles of non-stationary phase and stationary phase

In this note we discuss the principles of stationary and nonstationary phase used to prove decay in time for the linearized Hamiltonian. The following come from (Stein) and (EvZw).

Lemma C.4.1 (Non-stationary phase)**.** *Suppose ψ is smooth with sufficient decay and that ϕ is a smooth function with no critical points in the support of ψ. Then*

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\phi(x)} \psi(x) dx = O(\lambda^{-N}),$$

as $\lambda \to \infty$, for all $N \geq 0$. In particular,

$$|I(\lambda)| \lesssim \lambda^{-N} \sum_{|\alpha| \leq N} \sup_{\mathbb{R}^d} |\partial^\alpha \psi|.$$

Proof. Define

$$L = \frac{1}{\lambda i} \frac{1}{|\partial\phi|^2} \langle \partial\phi, \partial \rangle$$

for $x \in \text{supp}(\psi)$. Note that

$$L^N(e^{i\lambda\phi(x)}) = e^{i\lambda\phi(x)},$$

hence

$$|I(\lambda)| = \left| \int_{\mathbb{R}^d} L^N(e^{i\lambda\phi(x)}) \psi(x) dx \right| = \left| \int_{\mathbb{R}^d} e^{i\lambda\phi(x)} (L^*)^N \psi(x) dx \right| \lesssim \lambda^{-N}.$$

□

Definition C.4.2. *A critical point x_0 of ϕ is called nondegenerate if the symmetric matrix*

$$\left[\frac{\partial^2 \phi}{\partial_i \partial_j}\right](x_0)$$

is invertible.

Lemma C.4.3 (Stationary phase)**.** *Suppose $\phi(x_0) = 0$, ϕ has a nondegenerate critical point at x_0. If ψ is supported in a sufficiently small neighborhood of x_0, then*

$$\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}\psi(x)dx \sim \lambda^{-\frac{d}{2}} \sum_{j=0}^{\infty} a_j \lambda^{-j},$$

where the a_j's depend only on finitely many derivatives of ϕ and ψ.

Proof. Let us assume that $\phi(x) = \sum_1^m x_j^2 - \sum_{m+1}^d x_j^2$. Let $l = (l_1, \ldots, l_d)$ be a multi-index. Then, note that

$$\prod_{j=1}^{d} \int_{-\infty}^{\infty} e^{\pm i\lambda x_j^2} e^{-x_j^2} x_j^{l_j} dx_j = \prod_{j=1}^{d} \int_{-\infty}^{\infty} e^{-x^2} x^{l_j} dx (1 \mp i\lambda)^{-\frac{1}{2}-\frac{l_j}{2}}$$

by deforming the contour of integration using the decay of e^{-x^2}. Factor out

$$\prod_{j=1}^{d} \lambda^{-(l_j+1)/2} = \lambda^{-(d+|l|)/2},$$

then expand

$$\prod_{j=1}^{d} (\lambda^{-1} \mp i)^{-\frac{1}{2}-\frac{l_j}{2}}$$

into a power series in λ^{-1} for large λ. Then, we see that

$$\int_{\mathbb{R}^d} e^{i\lambda\phi(x)} e^{-x^2} x^l dx \sim \lambda^{-(d+|l|)/2} \sum_{j=0}^{\infty} c_j(m,l)\lambda^{-j}.$$

Let $\eta(x) \in C_0^\infty(\mathbb{R}^d)$. We need that

$$\left| \int_{\mathbb{R}^d} e^{i\lambda\phi(x)} x^l \eta(x) dx \right| \lesssim \lambda^{-(d+|l|)/2}.$$

To prove this, let

$$\Gamma_j = \{x : |x_j|^2 \geq \frac{x^2}{2d}\},$$

and

$$\Gamma_j = \{x : |x_j|^2 \geq \frac{x^2}{d}\}.$$

Since we have

$$\bigcup_{j=1}^{d} \Gamma_j^0 = \mathbb{R}^d,$$

find functions Ω_j, $j = 1, \ldots, d$ such that $\text{supp}(\Omega_j) \subset \Gamma_j$, Ω_j is homogeneous of degree 0 for all j, smooth away from the origin and

$$\sum_{j=1}^{d} \Omega_j(x) = 1$$

for all $x \neq 0$. Then, we have

$$\int_{\mathbb{R}^d} e^{i\lambda\phi(x)} x^l \eta(x) dx = \sum_j \int_{\mathbb{R}^d} e^{i\lambda\phi(x)} x^l \eta(x) \Omega_j(x) dx.$$

In each cone, use integration by parts with the operator

$$D_j f(x) = (\pm 2i\lambda x_j)^{-1} \frac{\partial f}{\partial x_j},$$

such that

$$D_j e^{i\lambda\phi(x)} = e^{i\lambda\phi(x)}.$$

Using the bound $|x_j| \geq (2d)^{-1/2}|x|$ in Γ_j and

$$|(D_j^*)^N \Omega_j(x)| \leq C_N \lambda^{-N} |x|^{-2N}.$$

By splitting the integral into $\int_{B(0,\epsilon)}$ and $\int_{\mathbb{R}^d \setminus B(0,\epsilon)}$, then choosing $\epsilon = \lambda^{-d/2}$, we are done with the proof. By the principal of nonstationary phase, when $\eta \in \mathcal{S}$ and η vanishes near the origin, then

$$\int e^{i\lambda\phi(x)} \eta(x) dx = O(\lambda^{-N})$$

for every $N \geq 0$. Then, using a Taylor series and the previous estimates, we have the result. □

Bibliography

[Agmon] S. Agmon. Spectral properties for Schrödinger operators and scattering theory, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **2**, 151-218 (1975).

[ADKM1] G.D. Akrivis, V. A. Dougalis, O. A. Karakashian, and W. R. McKinney. On fully discrete Galerkin methods of second-order temporal accuracy for the nonlinear Schrödinger equation, *Numer. Math.*, **59**, 31-53 (1991).

[ADKM2] G.D. Akrivis, V. A. Dougalis, O. A. Karakashian, and W. R. McKinney. Solving the systems of equations arising in the discretization of some nonlinear p.d.e's by implicit Runge-Kutta methods, *Model. Math. Anal. Numer.*, **31**, 251-288 (1997).

[ADKM3] G.D. Akrivis, V. A. Dougalis, O. A. Karakashian, and W. R. McKinney. Numerical approximation of blow-up of radially symmetric solutions of the nonlinear Schrödinger equation, *SIAM J. Sci. Comput.* **25**, no. 1, 186-212 (2003).

[Beals] R. Beals. Characterization of pseudodifferential operators and applications, *Duke Mathematical Journal*, **44**, no. 1, 45-57 (1977).

[BerLion] H. Berestycki and P. L. Lion. Nonlinear scalar field equations, I: Existence of a ground state, *Arch. Rational Mech. Anal.*, **82**, no. 4, 313-345 (1983).

[BeLo] J. Berg and J. Lofstrom. *Interpolation Spaces. An Introduction*, Grundlehren der Mathematischen Wissenschaften, **223**. Springer-Verlag, Berlin-New York (1976).

[1] [Braess] D. Braess, *Finite Elements: Theory, fast solvers, and applications in solid mechanics,* Cambridge University Press, (1997).

[BouWa] J. Bourgain, W. Wang. Construction of Blowup Solutions for the Nonlinear Schrodinger Equation with Critical Nonlinearity, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **25**, 197-215 (1998).

[Caz] T. Cazenave. *Semilinear Schrodinger Equations,* Courant Lecture Notes in Mathematics, **10**. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI (2003).

[ChKi] M. Christ and A. Kiselev. Maximal functions associated with filtrations, *Comm. Pure Appl. Math.*, **56**, no. 11, 1565-1607 (2003).

[ComPel] A. Comech and D. Pelinovsky. Purely nonlinear instability of standing waves with minimal energy, *J. Funct. Anal.*, **179**, 409-425 (2001).

[DS] L. Demanet and W. Schlag. Numerical verification of a gap condition for a linearized nonlinear Schrödinger equation, *Nonlinearity*, **19**, no. 4, 829-852 (2006).

[ES1] B. Erdogan and W. Schlag. Dispersive estimates for Schrödinger operators in the presence of a resonance and/or eigenvalue at zero energy in dimension three. II, *J. Anal. Math*, **99**, 199-248 (2006).

[Evans] L.C. Evans. *Partial Differential Equations,* Graduate Studies in Mathematics, **19**. American Mathematical Society, Providence, RI (1998).

[EvZw] L.C. Evans and M. Zworski. *Lectures on semiclassical analysis.* Unpublished Lecture Notes, 2006.

[FoGe] S. V. Fomin and I. M. Gelfand. *Calculus of Variations*, Prentice-Hall, Inc., Englewood Cliffs, N.J. (1963).

[GrilShaStr] M. Grillakis, J. Shatah and W. Strauss. Stability theory of solitary waves in the presence of symmetry. II, *J. Funct. Anal.*, **94**, no. 2, 308-348 (1990).

[HS] Hislop and Sigal. *Introduction to Spectral Theory. With Applications to Schrödinger Operators*, Applied Mathematical Sciences, **113**. Springer-Verlag, New York (1996).

[2] [HMZ1] J. Holmer, J. Marzuola, and M. Zworski. Fast soliton scattering by delta impurities, to appear in Communications in Mathematical Physics.

[3] [HMZ2] J. Holmer, J. Marzuola, and M. Zworski. Soliton splitting by external delta potentials, to appear in Journal of Nonlinear Science.

[Ho1] L. Hörmander. *The Analysis of Linear Partial Differential Operators I*, Classics in Mathematics. Springer-Verlag, Berlin (2003).

[Ho2] L. Hörmander. *The Analysis of Linear Partial Differential Operators II*, Classics in Mathematics. Springer-Verlag, Berlin (2005).

[Ho3] L. Hörmander. *The Analysis of Linear Partial Differential Operators III*, Grundlehren der Mathematischen Wissenschaften, **274** . Springer-Verlag, Berlin (1994).

[Ho4] L. Hörmander. *The Analysis of Linear Partial Differential Operators IV*, Grundlehren der Mathematischen Wissenschaften, **275** . Springer-Verlag, Berlin (1994).

[KriSch1] J. Krieger and W. Schlag. Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension, *J. Amer. Math. Soc.*, **19**, no. 4, 815-920 (2006).

[McCleod] K. McLeod. Uniqueness of Positive Radial Solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$, II, *Transactions of the American Mathematical Society*, **339**, no. 2, 495-505 (1993).

[Mourre] E. Mourre. Absence of Singular Continuous Spectrum for Certain Self-Adjoint Operators, *Communications in Mathematical Physics*, **78**, 391-408 (1981).

[RSv4] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of Operators*, Academic Press, New York-London (1978).

[RodSchSof] I. Rodnianski, W. Schlag, and A. Soffer. Asymptotic stability of N-soliton states of NLS, preprint (2003).

[Schlag1] W. Schlag. *Stable manifolds for an orbitally unstable NLS*, to appear in *Annals of Math*, preprint (2004).

[Shatah1] J. Shatah. Stable Standing Waves of Nonlinear Klein-Gordon Equations, *Communications in Mathematical Physics*, **91**, 313-327 (1983).

[Shatah2] J. Shatah. Unstable Ground State of Nonlinear Klein-Gordon Equations, *Transactions of the American Mathematical Society*, **290**, no. 2, 701-710 (1985).

[ShatStr1] J. Shatah and W. Strauss. Instability of Nonlinear Bound States, *Communications in Mathematical Physics*, **100**, 173-190 (1985).

[SjZw] J. Sjöstrand and M. Zworski. Elementary linear algebra for advanced spectral problems, to appear in *Annales de l'Institut Fourier*.

[Stein] E. Stein. *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, **43**. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ (1993).

[Strain] J. Strain. *Lecture Notes on Applied Mathematics*, unpublished notes from a course taught during the academic year 2004-2005.

[Strichartz] R. Strichartz. *A Guide to Distribution Theory and the Fourier Transform*, World Scientific Publishing Co., Inc., River Edge, NJ (2003).

[SulSul] C. Sulem and P. Sulem. *The Nonlinear Schrodinger Equation. Self-focusing and wave-collapse*, Applied Mathematical Sciences, **39**. Springer-Verlag, New York (1999).

[Taylor1] M. Taylor. *Partial Differential Equations I*, Applied Mathematical Sciences, **115**, Springer-Verlag, New York (1997).

[Taylor2] M. Taylor. *Partial Differential Equations II*, Applied Mathematical Sciences, **116**, Springer-Verlag, New York (1996).

[Taylor3] M. Taylor. *Partial Differential Equations III*, Applied Mathematical Sciences, **117**, Springer-Verlag, New York (1996).

[Tataru] D. Tataru *Microlocal Analysis*, unpublished lecture notes from a course taught Fall 2003.

[Wein1] M. Weinstein. Modulation Stability of Ground States of Nonlinear Schrodinger Equations. *SIAM Journal of Mathematical Analysis*, **16**, no. 3, 472-491 (1985).

[Wein2] M. Weinstein. Lyapunov Stability of Ground States of Nonlinear Dispersive Evolution Equations. *Communications on Pure and Applied Mathematics*, **39**, 51-68 (1986).