NONNEGATIVE WEAK SOLUTIONS OF THIN FILM EQUATIONS RELATED TO VISCOUS FLOWS IN CYLINDRICAL GEOMETRIES

JEREMY L. MARZUOLA, STERLING SWYGERT, AND ROMAN TARANETS

Abstract. Motivated by models for thin films coating cylinders in two physical cases proposed in [Ker94] and [KF94], we analyze the dynamics of corresponding thin film models. The models are governed by nonlinear, fourth-order, degenerate, parabolic PDEs. We prove, given positive and suitably regular initial data, the existence of weak solutions in all length scales of the cylinder, where all solutions are only local in time. We also prove that given a length constraint on the cylinder, long-time and global in time weak solutions exist. This analytical result is motivated by numerical work on related models of Reed Ogrosky [Ogr13] in conjunction with the works [CFL+12, COO14, COO17, CMOV16].

1. Introduction

The analysis of liquid films is an area of mathematical research that has many applications, ranging from biological systems to engineering and has been a rich area of research over the last three decades. Generically, the films have one free boundary whose evolution is determined by the relationship between external forces and the surface tension of the free surface itself. Many modeling and numerical studies have been done in order to understand these flows in different parameters and geometrical setups. In particular [Ben66] and [Lin74] study films along an inclined plane and [Fre92],[LL75],[COO14] consider films in the exterior or interior of vertically oriented tubes. The most significant physical difference between these two geometries is the free surface's azimuthal curvature dictating the surface stress in the cylindrical setting. The interior case of the cylindrical geometry is studied extensively in [CMOV16]. A specific class of the films, called thin films, exploit the ratio between the thicknesses of the film and the cylinder. In [Fre92], an evolution equation is derived for a thin film coating either the outside or the inside of a cylinder. This model was further studied in [KF94] and is explained in greater detail below. Thin films equations have also been studied in the frameworks of the generalized Kuramoto-Sivashinsky equation and the Cahn-Hilliard equation [BJN+13],[HR93].

Much work in the area has drawn on the machinery developed in [BF90], where the equation

$$h_t + (f(h)h_{xxx})_x = 0, \quad f(h) = f_0(h)|h|^n, \quad 0 < f_0 \in C^{1+\alpha}(\mathbb{R}^1), \alpha \in (0,1), \text{ and } n \geq 1 \tag{1.1}$$

is examined. Using the tools developed in [BF90], there has been development from the analytical standpoint, including well-posedness, existence of weak solutions, and finite-time blow-up. In [Gia99] an equation modeling the flow of a thin film over an inclined plane is analyzed and global in time existence of weak solutions is given. There has also been a fine collection of work in proving finite-time blow-up in some of the models. In [CT16], an equation modeling the spreading of a thin film over a flat solid surface and studied and a blow-up result is proved. Finite-time blow-up can also be seen in [BP00]. A comprehensive discussion of the relationship between scaling properties and singularity formation can be found in [SGKM95].

Though there has been some work done in more general settings [ES95], thin films coating a cylinder have been studied extensively. Eres, Schwartz, and Weidner provided models and numerical work for a stationary, horizontally oriented cylinder in the presence of gravity [ESW97]. Aside from modeling and numerical work, much analytic progress has been made by Chugunova, Pugh, and Taranets. For instance, they have studied the dynamics of a thin film on the exterior surface of a horizontally oriented cylinder rotating about it's axis of symmetry and provided arguments for long-time existence of weak solutions [CT12],[CPT10],[Tar06].

In this paper, we study the dynamics of an incompressible thin fluid film on the exterior of a cylinder. In particular we consider two specific one dimensional models. The first model (Model I), derived in [Ker94], is given by the initial boundary value problem

$$\left\{\begin{array}{l} h_t = -hh_x - S\left[h^3(h_x + h_{xxx})\right]_x \text{ in } Q_T, \\ h(x,0) = h_0(x) \in H^1(\Omega), \\ \partial_x^j h(-a,\cdot) = \partial_x^j h(a,\cdot) \text{ for } t \in (0,T),\ j = 0,1,2,3, \end{array}\right. \tag{1.2}$$

where $\Omega = (-a,a)$ is a bounded interval in $\mathbb{R}$ and $Q_T = \Omega \times (0,T)$. The equation models the situation in which the cylinder is horizontally oriented, a horizontally directed air flow is present without gravity, and the cylinder is fully coated so that the only free boundary is that where the surface of the fluid meets the air. Here, h is the thickness of the film with initial value h_0 and x is the longitudinal position. Model II, derived in [KF94], is given by the initial boundary value problem

$$\left\{\begin{array}{l} h_t = -2h^2 h_x - S\left[h^3(h_x + h_{xxx})\right]_x \text{ in } Q_T, \\ h(x,0) = h_0(x) \in H^1(\Omega), \\ \partial_x^j h(-a,\cdot) = \partial_x^j h(a,\cdot) \text{ for } t \in (0,T),\ j = 0,1,2,3. \end{array}\right. \tag{1.3}$$

This equation models the thickness of a thin film fully coating a vertically oriented cylinder in the presence of gravity. In each model, S is a modified Weber number. The rightmost terms in each equation represent the effects of surface tension in the azimuthal and axial directions, respectively, and the first terms on the right hand sides represent the forces acting on the films, e.g., air flow in Model I and gravity in Model II. In the following sections, we provide local in time existence of weak solutions to both Model I and Model II. When $\Omega \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we prove the existence of long-time weak solutions to Model I and global in time weak solutions to Model II when. Furthermore, in all cases, we prove non-negativity property, i.e. positive initial conditions yields non-negative solutions. Schematic diagrams of each model and respective coordinates can be found in [Ker94] and [KF94].

We first prove the existence of weak solutions to Model I. In particular, we prove local in time existence of weak solutions with any bounded spatial domain and long-time existence of weak solutions when the spatial domain is restricted to $\Omega \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The work for both the local in time and long-time solutions is broken up into four steps. First, we provide the definitions for functionals used throughout the paper and derive some straightforward identities in section 2. In section 3, we derive energy estimates for a regularized version of Model I and show control of the regularized solutions in different norms. In section 4, we define weak solutions and demonstrate that the limit of the solution to the regularized problem exists and satisfies this definition assuming non-negativity. Finally, we prove in section 5 that the limit is indeed nonnegative. In section 6, we examine Model II and give a brief description of how to prove the local in time existence of weak solutions. We then provide a proof for the existence of global in time solutions to Model II. Though many of the components of the proof are naturally analogous to those for Model I, the energy estimates are treated with a modified approach and are given in details. Finally, in section 7 we discuss future work in this analysis, including natural extensions of the arguments found here to long-wave models and a mixing of Model I and Model II.

2. Model I Preliminaries

One notices that (1.2) is degenerate if h vanishes at any point in the domain, and in order for the equation to be uniformly parabolic, it must be the case that $h \geqslant \delta$ in Q_T for some

$\delta > 0$. In order to remedy this, one may consider the regularized problem

$$\begin{cases} h_{\varepsilon,t} = -h_\varepsilon h_{\varepsilon,x} - S\left[\left(|h_\varepsilon|^3+\varepsilon\right)\left(h_{\varepsilon,x}+h_{\varepsilon,xxx}\right)\right]_x \text{ in } Q_T, \\ h_\varepsilon(x,0) = h_{0,\varepsilon}(x) \in C^{4+\gamma}(\Omega) \text{ for some } \gamma\in(0,1), \\ \partial_x^j h_\varepsilon(-a,\cdot) = \partial_x^j h_\varepsilon(a,\cdot) \text{ for } t\in(0,T),\ j=0,1,2,3, \end{cases} \tag{2.1}$$

Observe that the right hand sides of both (1.2) and (2.1) have a gradient form. This fact and the periodic boundary conditions tell us that integrating over Q_T yields

$$\int_\Omega h_\varepsilon(x,T)\ dx = \int_\Omega h_{0,\varepsilon}\ dx =: M_\varepsilon < \infty$$

for each $0 \le T \le T_\varepsilon$ and for each $\varepsilon > 0$. In other words, (1.2) and (2.1) are both conservation laws and conserve $\int_\Omega h(x,t)\ dx$ over time. We assume that $h_{0,\varepsilon} \to h_0$ strongly in $H^1(\Omega)$. Then we can bound M_ε uniformly by $M := \int_\Omega h_0\ dx > 0$. Thus for $\varepsilon > 0$ sufficiently small we have $0 < \int_\Omega h_{0,\varepsilon}\ dx \le M < \infty$.

2.1. **Functionals.** Here we define some different energy terms:

$$E_0(h) = \frac{1}{2}\int_\Omega h^2\ dx, \quad E_1(h) = \frac{1}{2}\int_\Omega h_x^2\ dx, \quad \mathcal{E}(h) = \frac{1}{2}\int_\Omega (h_x^2 - h^2)\ dx.$$

We also define the functions g_ε and G_ε by

$$g_\varepsilon(s) = -\int_s^A \frac{dr}{|r|^3+\varepsilon}, \tag{2.2}$$

$$G_\varepsilon(s) = -\int_s^A g_\varepsilon(r)\ dr, \tag{2.3}$$

where $A > 0$ is a finite real number to be specified later.

The use of these functionals naturally draws on their use in [BF90]. There are some useful statements [BF90] makes of g_ε and G_ε:

$$\begin{aligned} &G_\varepsilon'(s) = g_\varepsilon(s), && G_\varepsilon''(s) = g_\varepsilon'(s) = \frac{1}{|s|^3+\varepsilon}, \\ &g_\varepsilon(s) \le 0\ \forall s \le A, && G_\varepsilon(s) \ge 0\ \forall s\in\mathbb{R}, \\ &G_\varepsilon(s) \le G_0(s)\ \forall s \in \mathbb{R}, \end{aligned}$$

where $G_0 = \lim_{\varepsilon\to 0} G_\varepsilon$. Finally,

$$G_0(s) = \frac{1}{2s} - \frac{1}{A} + \frac{s}{2A^2} = \frac{(A-s)^2}{2A^2 s} \ge 0\ \forall s > 0.$$

2.2. **Model I Energy Identities.** We first work with the regularized equation (2.1) to derive a priori estimates. To begin, we draw on general parabolic theory in order to demonstrate that the perturbed equation is well-posed. Consider the operator

$$P_\varepsilon(x,y,t) = -y\partial_x - S\partial_x\left[(|y|^3+\varepsilon)(\partial_x + \partial_{xxx})\right].$$

Then the equation

$$h_t = P_\varepsilon h$$

is uniformly parabolic in a region $Q = [0,R]\times\Omega\times[0,T_\varepsilon]$ in the sense of Petrovsky [Eid69] as the characteristic equation

$$D(\lambda,\sigma) = -S(|y|^3+\varepsilon)\sigma^4 - \lambda = 0$$

has root $\lambda = -S(|y|^3+\varepsilon)\sigma^4$ which can be bounded above by $-\delta(\varepsilon) < 0$ so long as $|y| < R := R(\varepsilon)$ and $S > 0$. Theorem 7.3 in [Eid69] tells us that there exists a unique classical solution $h_\varepsilon \in C_{x,t}^{4+\gamma,1+\frac{\gamma}{4}}(Q_{T_\varepsilon})$ to (2.1), where $\gamma \in (0,1)$. In the rest of this section and the following two sections we will write $h = h_\varepsilon$.

Multiplying (2.1) by h and integrating over Ω, we obtain

$$\int_\Omega h h_t\ dx = -\frac{1}{2}\int_\Omega h^2 h_x\ dx - S\int_\Omega \left[\left(|h|^3+\varepsilon\right)\left(h_x + h_{xxx}\right)\right]_x h\ dx$$

$$= S\int_\Omega \left(|h|^3+\varepsilon\right) h_x^2 \, dx + S\int_\Omega \left(|h|^3+\varepsilon\right) h_x h_{xxx} \, dx,$$

where the last line uses integration by parts and periodic boundary conditions. Similarly, we can multiply (2.1) by $-h_{xx}$ and integrate over Ω to see that

$$\begin{aligned}\int_\Omega h_x h_{x,t} \, dx &= -\int_\Omega h_t h_{xx} \, dx \\ &= \int_\Omega h h_x h_{xx} \, dx + S\int_\Omega \left[\left(|h|^3+\varepsilon\right)(h_x+h_{xxx})\right]_x h_{xx} \, dx \\ &= \frac{1}{2}\int_\Omega \left(h^2\right)_x h_{xx} \, dx - S\int_\Omega \left[\left(|h|^3+\varepsilon\right)(h_x+h_{xxx})\right] h_{xxx} \, dx \\ &= -\frac{1}{2}\int_\Omega h^2 h_{xxx} \, dx - S\int_\Omega \left(|h|^3+\varepsilon\right) h_x h_{xxx} \, dx - S\int_\Omega \left(|h|^3+\varepsilon\right) h_{xxx}^2 \, dx.\end{aligned}$$

Adding the left hand and right hand sides of the chains of equalities, we have

$$\frac{1}{2}\frac{d}{dt}||h||^2_{H^1(\Omega)} + S\int_\Omega \left(|h|^3+\varepsilon\right) h_{xxx}^2 \, dx = -\frac{1}{2}\int_\Omega h^2 h_{xxx} \, dx + S\int_\Omega \left(|h|^3+\varepsilon\right) h_x^2. \tag{2.4}$$

3. Model I Energy Estimates

3.1. **Local in Time Estimates.** We can obtain uniform bounds on $||h_\varepsilon(\cdot,T)||^2_{H^1(\Omega)}$ for $\varepsilon>0$ and $T>0$ sufficiently small.

Lemma 1. *Suppose h_0 as in (1.2) and let $h_{0,\varepsilon}\to h_0$ strongly in $H^1(\Omega)$. Let h_ε be a solution to (2.1) in Q_{T_ε}. Then there is a time $T_{loc}>0$ such that h_ε satisfies a priori estimate*

$$||h_\varepsilon(\cdot,T)||^2_{H^1(\Omega)} \le 2^{2/3}\max\left\{1, ||h_0||^2_{H^1(\Omega)}\right\}$$

for $\varepsilon>0$ and $0\le T\le T_{loc}$.

Proof. Let $\varepsilon>0$ and let $h:=h_\varepsilon$ be a solution to (2.1). Recalling (2.4), we can bound using Cauchy's inequality and the compact embedding of H^1 in L^∞:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||h||^2_{H^1(\Omega)} + S\int_\Omega \left(|h|^3+\varepsilon\right) h_{xxx}^2 \, dx \\ &\qquad \le \frac{S}{4}\int_\Omega \left(|h|^3+\varepsilon\right) h_{xxx}^2 \, dx + \frac{1}{4S}\int_\Omega |h| \, dx + S\int_\Omega \left(|h|^3+\varepsilon\right) h_x^2 \, dx.\end{aligned}$$

As

$$\int_\Omega |h| \, dx \le |\Omega|^{1/2}\left(\int_\Omega h^2 \, dx\right),$$

then

$$\frac{1}{2}\frac{d}{dt}||h||^2_{H^1(\Omega)} + \frac{3S}{4}\int_\Omega \left(|h|^3+\varepsilon\right) h_{xxx}^2 \, dx \le C_S||h||^5_{H^1(\Omega)} + S\varepsilon||h||^2_{H^1(\Omega)} + \frac{|\Omega|^{1/2}}{4S}||h||_{H^1(\Omega)}.$$

Setting $V_\varepsilon(t)=\max\left(1, ||h_\varepsilon(\cdot,t)||^2_{H^1(\Omega)}\right)$, it follows that V_ε satisfies

$$V_\varepsilon'(t) \le C_S V_\varepsilon^{5/2}(t).$$

Dividing by $V_\varepsilon^{5/2}(t)$ and integrating yields

$$V_\varepsilon(t) \le \left(V_\varepsilon^{-3/2}(0) - \frac{3}{2}C_S t\right)^{-2/3} \tag{3.1}$$

for $0\le t<T^*=\frac{2}{3C_S}V_\varepsilon^{-3/2}(0)$. As $h_{0,\varepsilon}\to h_0$ strongly in $H^1(\Omega)$, (3.1) implies that $||h_\varepsilon||_{H^1(\Omega)}$ is uniformly bounded for $0\le T\le T_{\text{loc}}=\frac{1}{3C_S}V^{-3/2}(0)$ and independent of $\varepsilon>0$. □

3.2. **Long-time Estimates on $\Omega \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.** Here, we require that for $\Omega = (-a, a)$, we have $a < \frac{\pi}{2}$. Consider (2.1) and fix $\varepsilon > 0$. The existence theory in [Eid69] (Theorem 6.3 on page 302) tells us that there is a classical solution $h_\varepsilon \in C^{4+\gamma,1+\gamma/4}(Q_{\tau_\varepsilon})$ to (2.1) for some small time $\tau_\varepsilon > 0$. It is further demonstrated in [Eid69] (Theorem 9.3 on page 316) that if we have a priori control $||h_\varepsilon||_{L^\infty(Q_{T_\varepsilon})} \leq A$ and control on the Hölder norms in $C^{1/2,1/8}_{x,t}(Q_{T_\varepsilon})$ for some $T > \tau_\varepsilon$, then, in fact, h_ε can be continued in time as a classical solution to (2.1) on Q_T. We use the the functional $\mathcal{E}(h) = \frac{1}{2}\int_\Omega (h_x^2 - h^2)\, dx$ to demonstrate such control.

Before proceeding we require the Grönwall type inequality found in [Gyö71]:

Lemma 2. *Suppose that $y(t)$ satisfies the inequality*

$$y(t) \leq at + b + c\int_{t_0}^{t} g(y(s))\, ds \;\forall t \geq t_0 \geq 0,$$

where y is a non-negative continuous function, g is a positive nondecreasing function, and $a, b, c > 0$. Then

$$y(t) \leq G^{-1}\left\{G(at_0 + b) + \left(\frac{a}{g(at_0+b)} + c\right)(t - t_0)\right\},$$

where

$$G(t) = \int_\eta^t \frac{ds}{g(s)} \text{ for } \eta, t > 0.$$

Proof. Begin by defining

$$w(t) = at + b + c\int_{t_0}^{t} g(y(s))\, ds.$$

Because g is nondecreasing, $g(y(t)) \leq g(w(t))$. Notice that

$$w'(t) = a + cg(y(t)).$$

whence it follows that

$$w'(t) \leq a + cg(w(t)).$$

Using the fact that $g > 0$, we obtain

$$\frac{w'(t)}{g(w(t))} \leq \frac{a}{g(w(t))} + c.$$

Again using that $g > 0$ is nondecreasing and noticing that w is non-decreasing, it follows that

$$\frac{w'(t)}{g(w(t))} \leq \frac{a}{g(at_0 + b)} + c.$$

Integrating yields

$$G(w(t)) \leq G(w(t_0)) + \left(\frac{a}{g(at_0+b)} + c\right)(t - t_0),$$

and applying the inverse of G yields the result so long as $t \in [t_0, T^*]$ where T is chosen such that

$$G(w(t_0)) + \left(\frac{a}{g(at_0+b)} + c\right)(T - t_0) \in \mathrm{Dom}(G^{-1}) \;\forall T \in [t_0, T^*].$$

□

Lemma 3. *Fix $\varepsilon > 0$ and let h_ε be a solution of (2.1) up to time $T > 0$. Then h_ε satisfies the a priori estimate*

$$||h_{\varepsilon,x}(\cdot,t)||^2_{L^2(\Omega)}\, dx \leq K_0 + K_1 t + K_2 t^2. \tag{3.2}$$

Proof. Multiplying (2.1) by $-h_\varepsilon - h_{\varepsilon,xx}$, integrating over Q_T, integrating by parts, and using the periodic boundary conditions, one obtains
(3.3)

$$\mathcal{E}(h_\varepsilon(\cdot,T)) + S\iint_{Q_T}\left(|h_\varepsilon|^3+\varepsilon\right)\left(h_{\varepsilon,x}+h_{\varepsilon,xxx}\right)^2\,dx\,dt = \mathcal{E}(h_{0,\varepsilon}) - \frac{1}{2}\iint_{Q_T} h_\varepsilon^2 h_{\varepsilon,xxx}\,dx\,dt.$$

This implies

$$\begin{aligned}||h_x(\cdot,T)||^2_{L^2(\Omega)} &+ 2S\iint_{Q_T}\left(|h|^3+\varepsilon\right)\left(h_x+h_{xxx}\right)^2\,dx\;dt\\ &= ||h(\cdot,T)||^2_{L^2(\Omega)} + 2\mathcal{E}(h_{0,\varepsilon}) - \iint_{Q_T} h^2 h_{xxx}\,dx\;dt.\end{aligned}$$

Applying the Poincaré inequality to $h(x,T)$, we obtain

$$\begin{aligned}\left(1-\left(\frac{|\Omega|}{\pi}\right)^2\right)||h_x(\cdot,T)||^2_{L^2(\Omega)} &+ 2S\iint_{Q_T}\left(|h|^3+\varepsilon\right)\left(h_x+h_{xxx}\right)^2\,dx\;dt\\ &\le 2\mathcal{E}(h_{0,\varepsilon}) + \frac{M_\varepsilon^2}{|\Omega|} - \iint_{Q_T} h^2 h_{xxx}\,dx\;dt.\end{aligned} \tag{3.4}$$

Now, observe that we can bound the integral on the right hand side:

$$\begin{aligned}-\iint_{Q_T} h^2 h_{xxx}\,dx\;dt &\overset{\substack{\text{periodic}\\ \text{boundary}\\ \text{conditions}}}{=} -\iint_{Q_T} h^2\left(h_x+h_{xxx}\right)\\ &\overset{\substack{\text{Cauchy's}\\ \text{inequality}}}{\le} \frac{S}{2}\iint_{Q_T}\left(|h|^3+\varepsilon\right)\left(h_x+h_{xxx}\right)^2\,dx\;dt + \frac{1}{2S}\iint_{Q_T}|h|\,dx\;dt.\end{aligned}$$

Therefore, it follows from (3.4) that

$$\begin{aligned}\left(1-\left(\frac{|\Omega|}{\pi}\right)^2\right)||h_x(\cdot,T)||^2_{L^2(\Omega)} &+ \frac{3S}{2}\iint_{Q_T}\left(|h|^3+\varepsilon\right)\left(h_x+h_{xxx}\right)^2\,dx\;dt\\ &\le 2\mathcal{E}_\varepsilon(0) + \frac{M_\varepsilon^2}{|\Omega|} + \frac{1}{2S}\iint_{Q_T}|h|\,dx\;dt.\end{aligned} \tag{3.5}$$

Again, we can use the Cauchy-Schwarz and Poincaré inequalities to see that

$$\begin{aligned}\iint_{Q_T}|h|\,dx\;dt &\le |\Omega|^{1/2}\int_0^T\left(\int_\Omega h^2\,dx\right)^{1/2}\,dt\\ &\le |\Omega|^{1/2}\int_0^T\left(\left(\frac{|\Omega|}{\pi}\right)^2\int_\Omega h_x^2\,dx + \frac{M_\varepsilon^2}{|\Omega|}\right)^{1/2}\,dt\\ &\le \frac{|\Omega|^{3/2}}{\pi}\int_0^T ||h_x(\cdot,t)||_{L^2(\Omega)}\,dt + M_\varepsilon T.\end{aligned}$$

Applying this bound to (3.5) we have

$$||h_x(\cdot,T)||^2_{L^2(\Omega)} \le \alpha(T) + K\int_0^T ||h_x(\cdot,t)||_{L^2(\Omega)}\,dt,$$

where

$$\alpha(T) = \left(1 - \left(\frac{|\Omega|}{\pi}\right)^2\right)^{-1} \left(2\mathcal{E}(h_{0,\varepsilon}) + \frac{M_\varepsilon^2}{|\Omega|} + \frac{M_\varepsilon}{2S}T\right),$$

$$K = \left(1 - \left(\frac{|\Omega|}{\pi}\right)^2\right)^{-1} \frac{|\Omega|^{3/2}}{2S\pi}.$$

An application of Lemma 2 completes the proof of (3.2), with

$$K_0 = \alpha(0), \quad K_1 = \alpha'(0) + K\sqrt{\alpha(0)}, \quad K_2 = \frac{1}{4}\left(\frac{\alpha'(0)}{\sqrt{\alpha(0)}} + K\right)^2.$$

□

Application of Poincaré and Sobolev inequalities immediately implies that for any finite time T, we have a priori bound for $||h_\varepsilon||_{L^\infty(Q_T)}$.

3.3. **Hölder Continuity of $\{h_\varepsilon\}_{\varepsilon>0}$.** Let $T < \infty$ be a uniform time of existence for a family of solutions $\{h_\varepsilon\}_{\varepsilon>0}$. Using the uniform boundedness of $||h_\varepsilon||_{H^1(Q_T)}$, an application of Morrey's inequality ([Eva98] page 282) implies that $h_\varepsilon(\cdot, t)$ are uniformly bounded in $C^{1/2}(\bar{\Omega})$ for $0 \le \varepsilon \le \varepsilon_0$, $0 \le t \le T$, i.e. there is a constant K_3 such that

(3.6) $$|h_\varepsilon(x_1, t) - h_\varepsilon(x_2, t)| \le K_3|x_1 - x_2|^{1/2},$$

where the constant K_3 is independent of ε.

Lemma 4. *There is a constant $M < \infty$ so that for every $0 \le \varepsilon \le \varepsilon_0$ and $0 \le t_1 < t_2 \le T$, h_ε satisfies*

(3.7) $$|h_\varepsilon(x_0, t_1) - h_\varepsilon(x_0, t_2)| \le M|t_1 - t_2|^{1/8}$$

for each $x_0 \in \Omega$.

Proof. Suppose that

(3.8) $$|h_\varepsilon(x_0, t_1) - h_\varepsilon(x_0, t_2)| > M|t_2 - t_1|^{1/8}$$

for some $x_0 \in \Omega$ and some $0 \le t_1 < t_2 \le T$. We will derive an upper bound for M independent of ε. Without loss of generality, assume that $h_\varepsilon(x_0, t_2) > h_\varepsilon(x_0, t_1)$.

Following the work in [BF90], we define $\xi_0 \in C_0^\infty$ so that ξ_0 is even, $\xi_0(x) = 1$ if $0 \le x \le \frac{1}{2}$, $\xi_0(x) = 0$ if $x \ge 1$, and $\xi_0'(x) \le 0$ for $x \ge 0$. Setting

$$\xi(x) = \xi_0\left(\frac{x - x_0}{(M^2/16K_3^2)(t_2 - t_1)^{2\beta}}\right),$$

where $\beta = \frac{1}{8}$. It follows that

(3.9) $$\xi(x) = \begin{cases} 0 & \text{if } |x - x_0| \ge \frac{M^2}{16K_3^2}(t_2 - t_1)^{2\beta} \\ 1 & \text{if } |x - x_0| \le \frac{1}{2}\frac{M^2}{16K_3^2}(t_2 - t_1)^{2\beta}. \end{cases}$$

We next define θ_δ by

$$\theta_\delta(t) = \int_{-\infty}^t \theta_\delta'(s)\, ds,$$

where θ_δ' is given by

$$\theta_\delta'(t) = \begin{cases} \frac{1}{2\delta} & \text{if } |t - t_2| < \delta \\ -\frac{1}{2\delta} & \text{if } |t - t_1| < \delta \\ 0 & \text{otherwise} \end{cases}$$

and $0 < \delta < \frac{t_2 - t_1}{2}$. It is easy to see that θ_δ is Lipschitz continuous and that $|\theta_\delta| \le 1$. Furthermore, $\theta_\delta = 0$ near $t = 0$ and $t = T$ provided δ is small enough.

Setting $\phi(x,t) = \xi(x)\theta_\delta(t)$, it is clear that integration by parts yields

$$\iint_{Q_T} h_\varepsilon \phi_t \; dx\, dt = - \iint_{Q_T} f_\varepsilon \phi_x \; dx\, dt,$$

where $f_\varepsilon = \frac{h_\varepsilon^2}{2} + S\left(|h|^3 + \varepsilon\right)(h_{\varepsilon,x} + h_{\varepsilon,xxx})$. Using the definition of ϕ, we see that

$$\iint_{Q_T} h_\varepsilon(x,t)\xi(x)\theta_\delta'(t) \; dx\, dt = - \iint_{Q_T} f_\varepsilon(x,t)\xi'(x)\theta_\delta(t) \; dx\, dt. \tag{3.10}$$

We first work with the left hand side of (3.10). Taking the limit as δ tends to 0, it is clear that

$$\lim_{\delta\to 0} \iint_{Q_T} h_\varepsilon(x,t)\xi(x)\theta_\delta'(t) \; dx\, dt = \int_\Omega \xi(x)\left(h_\varepsilon(x,t_2) - h_\varepsilon(x,t_1)\right) \; dx. \tag{3.11}$$

We will estimate (3.11) from below. Because of (3.9), it is clear that we must only consider values x such that

$$|x - x_0| \le \frac{M^2}{16K_3^2}(t_2 - t_1)^{2\beta}. \tag{3.12}$$

Note that for such values of x, we have

$$\begin{aligned}
&h_\varepsilon(x,t_2) - h_\varepsilon(x,t_1) \\
&\quad = (h_\varepsilon(x,t_2) - h_\varepsilon(x_0,t_2)) + (h_\varepsilon(x_0,t_2) - h_\varepsilon(x_0,t_1)) + (h_\varepsilon(x_0,t_1) - h_\varepsilon(x,t_1)) \\
&\quad\ge -2K_3|x - x_0|^{1/2} + M(t_2 - t_1)^\beta, \text{ by (3.6) and (3.8).} \\
&\quad\ge \frac{M}{2}(t_2 - t_1)^\beta, \text{ by (3.12).}
\end{aligned}$$

If we assume that $\{\xi = 1\} \subset \Omega$, then we have

$$\int_\Omega \xi(x)\left(h_\varepsilon(x,t_2) - h_\varepsilon(x,t_2)\right) \; dx \ge \frac{M}{2}(t_2 - t_1)^\beta \frac{M^2}{16K_3^2}(t_2 - t_1)^{2\beta}.$$

We now work to bound the right hand side of (3.10). Observe that by Cauchy-Schwarz

$$\begin{aligned}
&\left|\iint_{Q_T} f_\varepsilon(x,t)\xi'(x)\theta_\delta(t) \; dx\, dt\right| \le ||f_\varepsilon||_{L^2(Q_T)} \left(\int_\Omega \xi'(x)^2 \; dx \int_0^T \theta_\delta(t)^2 \; dt\right)^{1/2} \\
&= ||f_\varepsilon||_{L^2(Q_T)} \left(\int_\Omega \left(\frac{d}{dx}\xi_0\left(\frac{x - x_0}{(M^2/16K_3^2)(t_2 - t_1)^{2\beta}}\right)\right)^2 \; dx \int_0^T \theta_\delta(t)^2 \; dt\right)^{1/2} \\
&= \frac{1}{(M^2/16K_3^2)(t_2 - t_1)^{2\beta}} ||f_\varepsilon||_{L^2(Q_T)} \\
&\quad\times \left(\int_\Omega \xi_0'\left(\frac{x - x_0}{(M^2/16K_3^2)(t_2 - t_1)^{2\beta}}\right)^2 \; dx \int_0^T \theta_\delta(t)^2 \; dt\right)^{1/2} \\
&= \frac{1}{(M^2/16K_3^2)(t_2 - t_1)^{2\beta}} ||f_\varepsilon||_{L^2(Q_{T_{\text{loc}}})} \\
&\quad\times \left(\int_\Omega \xi_0'\left(\frac{x - x_0}{(M^2/16K_3^2)(t_2 - t_1)^{2\beta}}\right)^2 (t_2 - t_1 + 2\delta)\right)^{1/2} \\
&\le \frac{1}{(M^2/16K_3^2)(t_2 - t_1)^{2\beta}} ||f_\varepsilon||_{L^2(Q_{T_{\text{loc}}})} \frac{C\sqrt{2}M}{4K_3}(t_2 - t_1 + 2\delta)^{1/2},
\end{aligned}$$

where we obtain the last inequality from the support of $\xi'(x)$ and taking

$$C = \sup_{x\in\Omega} \xi_0'\left(\frac{x - x_0}{(M^2/16K_3^2)(t_2 - t_1)^{2\beta}}\right).$$

It is easy to see that $||f_\varepsilon||_{L^2(Q_T)}$ is uniformly bounded for $0 < \varepsilon \le \varepsilon_0$.

Hence, taking $\delta \to 0$ and using the statements we have derived regarding the left and right hand sides of (3.10), we see that

$$\frac{M}{2}(t_2-t_1)^{\beta}\frac{M^2}{16K_3^2}(t_2-t_1)^{2\beta} \leq \frac{1}{(M^2/16K_3^2)(t_2-t_1)^{2\beta}}||f_\varepsilon||_{L^2(Q_T)}\frac{C\sqrt{2}M}{4K_3}(t_2-t_1)^{\beta}(t_2-t_1)^{1/2}.$$

This implies that

$$M \leq \tilde{C}^{1/4},$$

where $\tilde{C}$ is a constant independent of M and ε. This proves the lemma. □

Because $h_\varepsilon(\cdot,T) \in C_x^{1/2}(\bar{\Omega})$ and $h_\varepsilon(x,\cdot) \in C_t^{1/8}[0,T_{\text{loc}}]$ for $x \in \Omega$ and $0 \leq T \leq T_{\text{loc}}$, [Eid69] (Theorem 9.3 on page 316) implies that h_ε can be extended as a solution to (2.1) on $Q_{T_{\text{loc}}}$. Lemmas 1, 4, and (3.6) imply that $\{h_\varepsilon\}_{\varepsilon>0}$ is a uniformly bounded, equicontinuous family of functions on $Q_{T_{\text{loc}}}$. Due to the Arzelà-Ascoli lemma, this will allow us to find weak solutions to (4.1) in the sense of Definition 1. Similarly, in the setting where $|\Omega| < \pi$, Lemmas 3 and 4 and statement (3.6) imply that $\{h_\varepsilon\}_{\varepsilon>0}$ is a uniformly bounded and equicontinuous family of functions on Q_T for any finite time T.

4. Weak Solutions to Model I

We now consider the initial boundary value problem

$$\begin{cases} h_t = -hh_x - S\left[|h|^3(h_x + h_{xxx})\right]_x \text{ in } Q_T, \\ h(x,0) = h_0(x) \in H^1(\Omega), \\ \partial_x^j h(-a,\cdot) = \partial_x^j h(a,\cdot) \text{ for } t \in (0,T),\ j = 0,1,2,3. \end{cases} \tag{4.1}$$

We define a weak solution to (4.1) as follows:

Definition 1. *Let h be defined on Q_T such that*

$$h \in C_{x,t}^{1/2,1/8}(\overline{Q_T}) \cap L^\infty(0,T;H^1(\Omega)), \tag{4.2}$$

$$h_t \in L^2(0,T;(H^1(\Omega))^*), \tag{4.3}$$

$$h \in C_{x,t}^{4,1}(P), \tag{4.4}$$

$$|h|^{3/2}(h_{xxx} + h_x) \in L^2(P), \tag{4.5}$$

where $P = \overline{Q_T} \setminus (\{(h=0)\} \cup \{t=0\})$. Suppose that h satisfies (4.1) in the following sense:

$$\iint_{Q_T} h_t\phi \; dx \; dt - \iint_P \left(\frac{h^2}{2} + S|h|^3(h_x + h_{xxx})\right)\phi_x \; dx \; dt = 0 \tag{4.6}$$

for all $\phi \in C^1(Q_T)$ with $\phi(a,\cdot) = \phi(-a,\cdot)$. Further

(4.7) $h(\cdot,t) \to h(\cdot,0)$ *pointwise and strongly in* $L^2(\Omega)$ *as* $t \to 0$

(4.8) $\partial_x^j h(a,t) = \partial_x^j h(-a,t)$ *for* $j = 0,1,2,3$ *on* $(\partial\Omega \times (0,T)) \setminus (\{h=0\} \cup \{t=0\})$.

Then we call h a weak solution *to the problem* (4.1).

Let T be a uniform time of existence for a family of solutions $\{h_\varepsilon\}_{\varepsilon>0}$. Because $\{h_\varepsilon\}_{\varepsilon>0}$ is a uniformly bounded and equicontinuous family of functions, then by the Arzelà-Ascoli lemma there is a subsequence $\varepsilon_k \to 0$ such that

$$h_{\varepsilon_k} \to h \quad \text{uniformly in } \bar{Q}_T. \tag{4.9}$$

Henceforth, we refer to this subsequence as $\varepsilon \to 0$.

Theorem 5. *Any function obtained as in* (4.9) *is a weak solution to* (4.1).

Proof. It is clear that (4.2) follows by the fact that $h_\varepsilon \to h$ uniformly in Q_T. Now take $\phi \in \text{Lip}\,(Q_T)$ such that $\phi = 0$ near $t = 0$ and $t = T$. Then for each $0 < \varepsilon \leq \varepsilon_0$, we have

$$0 = \iint_{Q_T} h_\varepsilon\phi_t \; dx\, dt + \iint_{Q_T} \frac{h_\varepsilon^2}{2}\phi_x \; dx\, dt$$

$$+ S \iint_{Q_T} |h_\varepsilon|^3 (h_{\varepsilon,x} + h_{\varepsilon,xxx})\phi_x \, dx\, dt + S\varepsilon \iint_{Q_T} (h_{\varepsilon,x} + h_{\varepsilon,xxx})\phi_x \, dx\, dt.$$

By (3.2), Cauchy's inequality, and the Sobolev inequality, it follows that the expression $\varepsilon^{1/2} ||h_{\varepsilon,x} + h_{\varepsilon,xxx}||_{L^2(Q_T)}$ is uniformly bounded with respect to ε. Then, we see that

$$\varepsilon \iint_{Q_T} |(h_{\varepsilon,x} + h_{\varepsilon,xxx})\phi_x| \, dx\, dt \le \varepsilon ||h_{\varepsilon,x} + h_{\varepsilon,xxx}||_{L^2(Q_T)} ||\phi_x||_{L^2(Q_T)}$$
$$\le C\varepsilon^{1/2},$$

where C is a constant independent of ε. Therefore

$$\lim_{\varepsilon\to 0} \varepsilon \iint_{Q_T} (h_{\varepsilon,x} + h_{\varepsilon,xxx})\phi_x \, dx\, dt = 0,$$

Note that our a priori estimates imply $H_\varepsilon = (|h_\varepsilon|^3 + \varepsilon)^{1/2}(h_{\varepsilon,x} + h_{\varepsilon,xxx})$ is uniformly bounded in $L^2(Q_T)$, which in turn implies that $H_\varepsilon \rightharpoonup H \in L^2(Q_T)$. Regularity theory of uniformly parabolic equations and the fact that h_ε are uniformly Hölder continuous imply that

(4.10)
$h_{\varepsilon,t}, h_{\varepsilon,x}, h_{\varepsilon,xx}, h_{\varepsilon,xxx}$, and $h_{\varepsilon,xxxx}$ are uniformly convergent on any compact subset of P,

and hence (4.4) and (4.8). Furthermore, (4.10) and $H_\varepsilon \rightharpoonup H$ in $L^2(Q_T)$ tells us that $H = |h|^3(h_x + h_{xxx})$ on P.

Setting $f_\varepsilon = \frac{h_\varepsilon^2}{2} + S|h_\varepsilon|^3(h_{\varepsilon,x} + h_{\varepsilon,xxx})$, we have for any $\delta > 0$

$$\lim_{\varepsilon\to 0} \iint_{|h|>\delta} f_\varepsilon \phi_x \, dx\, dt = \iint_{|h|>\delta} f\phi_x \, dx\, dt. \tag{4.11}$$

On the other hand, we can choose $0 < \varepsilon \le \varepsilon_0$ small enough (dependent on δ) so that

$$\left| \iint_{|h|\le\delta} f_\varepsilon \phi_x \, dx\, dt \right| \le \iint_{|h|\le\delta} \frac{h_\varepsilon^2}{2} |\phi_x| \, dx\, dt + S \iint_{|h|\le\delta} |h_\varepsilon|^3 |(h_{\varepsilon,x} + h_{\varepsilon,xxx})\phi_x| \, dx\, dt$$
$$\le \iint_{|h|\le\delta} \frac{h_\varepsilon^2}{2} |\phi_x| \, dx\, dt + S \left(\iint_{|h|\le\delta} |h_\varepsilon|^3 \phi_x^2 dxdt \right)^{1/2} \left(\iint_{|h|\le\delta} |h_\varepsilon|^3 (h_{\varepsilon,x} + h_{\varepsilon,xxx})^2 dxdt \right)^{1/2}$$
$$\le C_S \delta^{3/2},$$

where C_S is independent of δ. Combining this fact with (4.11) implies that

$$\lim_{\varepsilon\to 0} \iint_{Q_T} f_\varepsilon \phi_x \, dx\, dt = \iint_P f\phi_x \, dx\, dt, \tag{4.12}$$

whence (4.6) follows. □

5. Non-Negativity of Solutions to Model I

Using similar techniques as in [BF90], we can give a non-negativity result for solutions constructed in section 4:

Theorem 6. *Let h be a weak solution to* (4.1) *as constructed in Theorem 1 with $h_0 > 0$. Further assume that $\int_\Omega h_0^{-1} \, dx < \infty$. Then $h \ge 0$. Furthermore, for each $T \in [0, \hat{T}]$, where $\hat{T}$ is a time of existence as constructed in section 4, the set $E_T = \{x \in \Omega : \ h(x,T) = 0\}$ is of measure zero. Also, $\int_\Omega \frac{dx}{h(x,t)}$ is uniformly bounded. Finally, if one further assumes that $\Omega \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then $\int_\Omega G_\varepsilon(h_\varepsilon(x,T)) \, dx$ is a monotonically decreasing function on T.*

Proof. Recall the definitions of g_ε and G_ε for $\varepsilon > 0$:

$$g_\varepsilon(s) = -\int_s^A \frac{dr}{|r|^3 + \varepsilon}, \quad G_\varepsilon(s) = -\int_s^A g_\varepsilon(r) \, dr,$$

where $A > \max |h_\varepsilon|$, which is a finite number by a priori estimates on h_ε. It follows by definition of G_ε that for $\varepsilon > 0$ we have

$$\begin{aligned}\int_\Omega G_\varepsilon(h_{0,\varepsilon}(x)) &< C_1 + \int_\Omega h_{0,\varepsilon}^{-1}\ dx, \text{ where } C_1 \text{ is independent of } \varepsilon\\ &\leq C_1 + \int_\Omega h_0^{-1}\ dx, \text{ as } h_{0,\varepsilon} \geq h_0\\ &< \infty, \text{ by hypothesis}\end{aligned}$$

and this bound is clearly independent of ε.

Multiplying (2.1) by $g_\varepsilon(t)$ and integrating over Q_T, we see that

$$\begin{aligned}\int_\Omega G_\varepsilon(h_\varepsilon(x,T))\ dx - \int_\Omega G_\varepsilon(h_{0,\varepsilon}(x))\ dx &= \iint_{Q_T} g_\varepsilon(h) h_t\ dx\, dt\\ &= -\iint_{Q_T} \left[\frac{h_\varepsilon^2}{2} + S\left(|h_\varepsilon|^3 + \varepsilon\right)\left(h_{\varepsilon,x} + h_{\varepsilon,xxx}\right)\right]_x g_\varepsilon(h)\ dx\, dt\\ &= -\iint_{Q_T} h_\varepsilon g_\varepsilon(h_\varepsilon) h_{\varepsilon,x} - S(h_{\varepsilon,x} + h_{\varepsilon,xxx}) h_{\varepsilon,x}\ dx\, dt\\ &= -\iint_{Q_T} h_\varepsilon h_{\varepsilon,x} g_\varepsilon(h_\varepsilon)\ dx\ dt + S\iint_{Q_T} h_{\varepsilon,x}^2\ dx\ dt - S\iint_{Q_T} h_{\varepsilon,xx}^2\ dx\ dt,\end{aligned}$$

where the last two equalities follow by integrating by parts and using the fact that

$$\frac{\partial}{\partial x} g_\varepsilon(h_\varepsilon) = \frac{h_{\varepsilon,x}}{|h_\varepsilon|^3 + \varepsilon}.$$

Hence, we have

$$\begin{aligned}&\int_\Omega G_\varepsilon(h_\varepsilon(x,T))\ dx + S\iint_{Q_T} h_{\varepsilon,xx}^2\ dx\, dt\\ &\qquad = -\iint_{Q_T} h_\varepsilon g_\varepsilon(h_\varepsilon) h_{\varepsilon,x}\ dx\ dt + S\iint_{Q_T} h_{\varepsilon,x}^2\ dx\, dt + \int_\Omega G_\varepsilon(h_{0,\varepsilon}(x))\ dx.\end{aligned}$$

We can define

$$F_\varepsilon(s) := \int_0^s r g_\varepsilon(r)\ dr.$$

Using the fundamental theorem of calculus, it is clear that

$$\int_\Omega h_\varepsilon g_\varepsilon(h_\varepsilon) h_{\varepsilon,x}\ dx = \int_\Omega \left[F_\varepsilon(h_\varepsilon)\right]_x\ dx = F_\varepsilon(h_\varepsilon)\Big|_{\partial\Omega}.$$

The periodic boundary conditions on h_ε implies that this integral is zero, and hence

$$\int_\Omega G_\varepsilon(h_\varepsilon(x,T))\ dx + S\iint_{Q_T} h_{\varepsilon,xx}^2\ dx\ dt = \int_\Omega G_\varepsilon(h_{0,\varepsilon})\ dx + S\iint_{Q_T} h_{\varepsilon,x}^2\ dx\ dt.$$

If we have that $\int_\Omega h_{\varepsilon,x}^2(x,t)\ dx$ is uniformly bounded for $0 \leq T \leq \hat{T}$, then it follows that $\int_\Omega G_\varepsilon(h_\varepsilon(x,T))\ dx$ and $\iint_{Q_T} h_{\varepsilon,xx}^2\ dx\ dt$ are bounded for all $0 \leq T \leq \hat{T}$.

If we further assume that $\Omega \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we can apply the Poincaré inequality to h_x with $\int_\Omega h_x = 0$ by the periodic boundary conditions. As a result, we see that

$$\int_\Omega G_\varepsilon(h_\varepsilon(x,T))\ dx + \left(1 - \left(\frac{|\Omega|}{\pi}\right)^2\right)\iint_{Q_T} h_{\varepsilon,xx}^2 \leq \int_\Omega G_\varepsilon(h_{0,\varepsilon})\ dx.$$

From this inequality, we see that $\int_\Omega G_\varepsilon(h_\varepsilon(x,T))\ dx$ is a decreasing function on T.

Suppose, toward a contradiction, that there a point $(x_0, t_0) \in Q_{\hat{T}}$ so that $h(x_0,t_0) < 0$. Because $h_\varepsilon \to h$ uniformly in $Q_{\hat{T}}$, there is $\delta > 0$ and $\varepsilon_0 > 0$ so that for every $0 < \varepsilon \leq \varepsilon_0$ and every $x \in \Omega$ satisfying $|x - x_0| < \delta$, we have $h_\varepsilon(x,t_0) < -\delta$. However, this implies

$$G_\varepsilon(h_\varepsilon(x,t_0)) = -\int_{h_\varepsilon(x,t_0)}^A g_\varepsilon(s)\ ds \geq -\int_{-\delta}^0 g_\varepsilon(s)\ ds.$$

Note that this lower bound tends to $-\int_{-\delta}^{0} g_0(s)\ ds$ as $\varepsilon \to 0$. However, we have that $g_0(s) = \infty$ for $s < 0$, so the integral on the right is infinite. This implies that

$$\lim_{\varepsilon\to 0}\int_\Omega G_\varepsilon(h_\varepsilon(\cdot,T))\ dx = \infty,$$

which is a contradiction. Hence, $h \geq 0$ in Q_T.

Now, suppose toward a contradiction that there is a t_0 in $[0,\hat{T}]$ so that $\mathrm{meas}(E_{t_0}) > 0$. Then because $h_\varepsilon \to h$ uniformly, there is a modulus of continuity $\sigma(\varepsilon) > 0$ so that $h_\varepsilon(x,t_0) < \sigma(\varepsilon)$ for $x \in E_{t_0}$. This implies that for $x \in E_{t_0}$ and $\delta > 0$, we have

$$\begin{aligned} G_\varepsilon(h_\varepsilon(x,t_0)) &= -\int_{h_\varepsilon(x,t_0)}^{A} g_\varepsilon(s)\ ds \geq -\int_{\sigma(\varepsilon)}^{A} g_\varepsilon(s)\ ds \\ &\geq -\int_{\delta}^{A} g_\varepsilon(s)\ ds \to -\int_{\delta}^{A} g_0(s)\ ds \end{aligned}$$

provided that ε is taken small enough so that $\sigma(\varepsilon) < \delta$. It is also easy to show that

$$-\int_\delta^A g_0(s)\ ds \geq c\delta^{-1},$$

where $A = \max|h_\varepsilon|$, which is uniformly bounded for $0 \leq T \leq \hat{T}$. This bound implies

$$\int_\Omega G_\varepsilon(h_\varepsilon(x,t_0))\ dx \geq c\delta^{-1}\mathrm{meas}(E_{t_0})$$

which tends to infinty as δ (and hence ε) go to zero. This is a contradiction.

Finally, note that by definition of $G_0(s)$, we have that for (x,t) such that

$$h(x,t) > 0,\ \lim_{\varepsilon\to 0} G_\varepsilon(h_\varepsilon(x,t)) = G_0(h(x,t)).$$

Because E_t has measure zero for every $0 \leq t \leq \hat{T}$, it follows that this limit is valid for almost all x in Ω. Then observe that uniform convergence of the integrand for positive s yields

$$\begin{aligned} G_0(s) = \lim_{\varepsilon\to 0} G_\varepsilon(s) &= \lim_{\varepsilon\to 0}\int_s^A\int_r^A \frac{1}{\rho^3+\varepsilon}\ d\rho dr \\ &= \int_s^A\int_r^A \frac{1}{\rho^3}\ d\rho dr = \frac{1}{2s} - \frac{1}{A} + \frac{s}{2A^2}. \end{aligned}$$

In particular, for $h(x,t) > 0$ we have

$$G_0(h(x,t)) = \frac{1}{h(x,t)} - \frac{1}{A} + \frac{h(x,t)}{2A^2}.$$

Integrating over Ω yields

$$\int_\Omega \frac{dx}{h(x,t)} = \int_\Omega G_0(h(x,t))\ dx + \frac{|\Omega|}{A} - \frac{M}{2A^2}.$$

Finally, an application of Fatou's lemma and the fact that the measure of E_T is zero for each $T \in [0,\hat{T}]$ implies that $\int_\Omega \frac{dx}{h(x,t)}$ is uniformly bounded. □

6. Model II

6.1. **Local in Time Theory.** We now discuss the model given by (1.3). As with Model I, (1.3) is degenerate if h vanishes, so we must regularize the problem by analyzing

$$\left\{\begin{aligned} &h_{\varepsilon,t} = -2h_\varepsilon^2 h_{\varepsilon,x} - S\left[\left(|h_\varepsilon|^3+\varepsilon\right)\left(h_{\varepsilon,x}+h_{\varepsilon,xxx}\right)\right]_x = 0 \text{ in } Q_T, \\ &h_\varepsilon(x,0) = h_{0,\varepsilon}(x) \in C^{4+\gamma}(\Omega) \text{ for some } \gamma\in(0,1), \\ &\partial_x^j h_\varepsilon(-a,\cdot) = \partial_x^j h_\varepsilon(a,\cdot) \text{ for } t\in(0,T),\ j=0,1,2,3. \end{aligned}\right. \tag{6.1}$$

One can prove local in time energy identities and estimates for Model I that are essentially identical to those proved for Model I. Mirroring the work done in section 3 with (6.1), we prove uniform a priori control of norms of h_ε in $H^1(\Omega)$, $L^\infty(\Omega)$, $C_x^{1/2}(\Omega)$ and $C_t^{1/8}[0,T_{\mathrm{loc}}]$.

Theorem 6.3 [Eid69] tells us that for each $\varepsilon > 0$ there is a solution h_ε to (6.1) on Q_{τ_ε}, where $\tau_\varepsilon > 0$. The a priori control listed above allows us to apply Theorem 9.3 (p. 316) and Corollary 2 (p. 213) [Eid69] in order to extend each solution h_ε to $Q_{T_{\mathrm{loc}}}$.

As in section 4, we then use the uniform boundedness and Hölder continuity in order to apply the Arzelà-Ascoli lemma as we take ε to zero. Then writing the problem

$$\begin{cases} h_t = -2h^2 h_x - S\left[|h|^3(h_x + h_{xxx})\right]_x = 0 \text{ in } Q_{T_{\mathrm{loc}}}, \\ h(x,0) = h_0(x) \in H^1(\Omega), \\ \partial_x^j h(-a,\cdot) = \partial_x^j h(a,\cdot) \text{ for } t \in (0,T_{\mathrm{loc}}),\ j = 0,1,2,3, \end{cases} \tag{6.2}$$

and a definition comparable to Definition 1, we prove that the limit as ε to zero (along a subsequence) satisfies such a definition. Finally, we move forward to prove that this limit is also non-negative as in section 5, which proves that it is a weak solution to (1.3).

6.2. **Global in Time Estimates on $\Omega \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.** In this section, we assume that $\Omega \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We use non-negativity of solutions to (1.3) and (6.1) in order to write them, respectively, as

$$\begin{cases} h_t + S[h^3(h + h_{xx} + \frac{2}{3S}x)_x]_x = 0 \text{ in } Q_T, \\ h(x,0) = h_0(x) \in H^1(\Omega), \\ \partial_x^j h(-a,\cdot) = \partial_x^j h(a,\cdot) \text{ for } t \in (0,T),\ j = 0,1,2,3, \\ \frac{2}{3S}x\left(h + h_{xx} + \frac{2}{3S}x\right)_x\Big|_{-a}^{a} \equiv 0, \end{cases} \tag{6.3}$$

and

$$\begin{cases} h_{\varepsilon,t} + S[(h_\varepsilon^3 + \varepsilon)(h_\varepsilon + h_{\varepsilon,xx} + \frac{2}{3S}x)_x]_x = 0 \text{ in } Q_T, \\ h_\varepsilon(x,0) = h_{0,\varepsilon}(x) \in C^{4+\gamma}(\Omega), \\ \partial_x^j h_\varepsilon(-a,\cdot) = \partial_x^j h_\varepsilon(a,\cdot) \text{ for } t \in (0,T),\ j = 0,1,2,3, \\ \frac{2}{3S}x\left(h_\varepsilon + h_{\varepsilon,xx} + \frac{2}{3S}x\right)_x\Big|_{-a}^{a} \equiv 0, \end{cases} \tag{6.4}$$

where the boundary conditions have been added. Note that it follows by a similar argument as in section 5 that for sufficiently small $\varepsilon > 0$, we must have $h_\varepsilon \geq 0$ on Q_{T_ε}, legitimizing (6.4). Now, we provide a uniform $H^1(\Omega)$ bound on $h_\varepsilon(\cdot,T)$, independent of $\varepsilon > 0$ and $T > 0$.

Lemma 7. *Suppose h_ε is a solution to* (6.4). *Then $||h_\varepsilon(\cdot,T)||_{H^1(\Omega)}$ is uniformly bounded for all $T > 0$ and $\varepsilon > 0$ sufficiently small.*

Proof. Suppose $h := h_\varepsilon$ is a solution to (6.4). Then multiplying (6.4) by $\left(h + h_{xx} + \frac{2}{3S}x\right)$ and integrating over Ω yields

$$0 = \int_\Omega \left[hh_t + h_{xx}h_t + \frac{2}{3S}xh_t + S\left[\left(h^3 + \varepsilon\right)\left(h + h_{xx} + \frac{2}{3S}x\right)_x\right]_x \left(h + h_{xx} + \frac{2}{3S}x\right)\right] dx.$$

Integrating by parts and using the boundary conditions prescribed in (6.4), we obtain

$$0 = \frac{1}{2}\frac{d}{dt}\int_\Omega \left(h_x^2 - h^2 - \frac{4}{3S}xh\right) dx + S\int_\Omega \left(h^3 + \varepsilon\right)\left(h + h_{xx} + \frac{2}{3S}x\right)_x^2 dx.$$

Integrating in time from 0 to T and applying the fundamental theorem of calculus, it follows that

$$\tilde{\mathcal{E}}(h(\cdot,T)) + S\iint_{Q_T} \left(h^3 + \varepsilon\right)\left[\left(h + h_{xx} + \frac{2}{3S}x\right)_x\right]^2 dx = \tilde{\mathcal{E}}(h_{0\varepsilon}), \tag{6.5}$$

where $\tilde{\mathcal{E}}(h) := \frac{1}{2}\int_\Omega (h_x^2 - h^2 - \frac{4}{3S}xh)\ dx$. From (6.5), it follows that

$$\int_\Omega h_x^2(x,T)\ dx \leq 2\tilde{\mathcal{E}}(h_{0,\varepsilon}) + \int_\Omega h^2(x,t)\ dx + \frac{4}{3S}\int_\Omega xh(x,t)\ dx.$$

Using integration parts with periodic boundary conditions and applying the Cauchy-Schwarz inequality to the right-hand integral, one obtains

$$\int_\Omega h_x^2(x,T)\ dx \le 2\tilde{\mathcal{E}}(h_{0,\varepsilon}) + \int_\Omega h^2(x,T)\ dx + \frac{2}{3S}\left(\int_\Omega x^4\ dx\right)^{1/2}\left(\int_\Omega h^2\ dx\right)^{1/2}. \tag{6.6}$$

Recalling the Poincaré inequality, we have

$$\int_\Omega h^2\ dx \le \left(\frac{|\Omega|}{\pi}\right)^2 \int_\Omega h_x^2\ dx + \frac{M_\varepsilon^2}{|\Omega|}. \tag{6.7}$$

Applying (6.7) to (6.6) and integrating, we find that

$$\left[1-\left(\frac{|\Omega|}{\pi}\right)^2\right]\int_\Omega h_x^2(x,T) \le 2\tilde{\mathcal{E}}(h_{0,\varepsilon}) + \frac{M_\varepsilon^2}{|\Omega|} + \frac{2}{3S}\left(\frac{2a^5}{5}\right)^{1/2}\left(\int_\Omega h_x^2(x,T)\ dx\right)^{1/2}. \tag{6.8}$$

An application of Cauchy's inequality yields

$$\left[1-\left(\frac{|\Omega|}{\pi}\right)^2\right]\int_\Omega h_x^2(x,T)\ dx \le \delta\int_\Omega h_x^2(x,T)\ dx + 2\tilde{\mathcal{E}}(h_{0,\varepsilon}) + \frac{M_\varepsilon^2}{|\Omega|} + C(\delta),$$

where we can choose $\delta = \frac{1}{2}\left[1-\left(\frac{|\Omega|}{\pi}\right)^2\right] > 0$. It follows that

$$\int_\Omega h_x^2(x,T)\ dx \le \delta^{-1}\left[2\tilde{\mathcal{E}}(h_{0,\varepsilon}) + \frac{M_\varepsilon^2}{|\Omega|} + C(\delta)\right] =: A, \tag{6.9}$$

i.e. $\int_\Omega h_{\varepsilon,x}^2(x,T)\ dx$ is uniformly bounded, indepedent of T and ε. The Poincaré inequality again implies that $\int_\Omega h_\varepsilon^2(x,T)\ dx$ is uniformly bounded so that $||h_\varepsilon(\cdot,T)||_{H^1(\Omega)}$ is uniformly bounded. The Sobolev embedding theorem yields a uniform bound on $||h_\varepsilon||_{L^\infty(Q_\infty)}$, where $Q_\infty = \Omega\times(0,\infty)$. □

One can use the arguments in section 3.3 to show that $h_\varepsilon \in C_{x,t}^{1/2,1/8}(Q_\infty)$ uniformly. Again, it follows by the arguments in sections 4 and 5 that taking ε to zero yields a non-negative weak solution to (6.3) in Q_∞.

7. Extending the Results and Future Work

An immediate extension of the results for Model I and Model II is local in time existence of weak solutions to the problem represented by the mixed model:

$$\begin{cases} h_t = -\alpha h h_x - (1-\alpha)h^2 h_x - S\left[h^3(h_x + h_{xxx})\right]_x \text{ in } Q_T, \\ h(x,0) = h_0(x) \in H^1(\Omega), \\ \partial_x^j h(-a,\cdot) = \partial_x^j h(a,\cdot) \text{ for } t\in(0,T),\ j=0,1,2,3, \end{cases} \tag{7.1}$$

where $\alpha\in[0,1]$. How to apply the local in time theory is very straightforward. Similarly, the work of section 3.2 should provide long-time estimates for a regularization of (7.1). Following the steps set forth in sections 4 and 5 provide the existence of weak solutions to (7.1).

As discussed in the introduction, both Model I and Model II are limits of long-wave models discussed, from both the experimental and numerical standpoints, in [CFL+12, Ogr13, COO14, COO17, CMOV16]. One of such models is given below

$$\begin{cases} \mu R_t = \rho g f_1(R;b)R_x + \frac{\gamma}{16R}\left[f_2(R;b)(R_x + R^2 R_{xxx})\right]_x, \text{ in } Q_T = \Omega\times(0,T) \\ R(x,0) = R_0(x) \in H^1(\Omega) \\ \partial_x^j R(a,t) = \partial_x^j R(-a,t), \text{ for } t\in(0,T) \text{ and } j=0,1,2,3,, \end{cases} \tag{7.2}$$

where μ,ρ,g are all parameters of interest, b is the radius of the cylinder, and f_1 and f_2 are functions given by

$$f_1(R;b) = \frac{1}{2}\left[R^2 - b^2 - 2R^2\log\left(\frac{R}{b}\right)\right] \tag{7.3}$$

$$f_2(R;b) = -\frac{b^4}{R^2} + 4b^2 - 3R^2 + 4R^2 \log\left(\frac{R}{b}\right). \tag{7.4}$$

The next step is to apply the local in time energy methods used in section 3 to the corresponding long-wave models. Because the degeneracies in the models in [CMOV16] are more complicated than the simply polynomials in (1.2) and (1.3), the regularizations must be defined more meticulously. Defining weak solutions appropriately and adapting the nonnegativity arguments of section 5 will demonstrate the existence of weak solutions to the models.

Acknowledgements: The authors thank Roberto Camassa and Reed Ogrosky for many helpful conversations about this family of thin film models. JLM and SS were supported in part by the NSF through JLM's NSF CAREER Grant DMS-1352353.

References

[Ben66] D.J. Benney, *Long waves in liquid films*, J. Math. Phys. **45** (1966), 150–155.

[BF90] F. Bernis and A. Friedman, *Higher order nonlinear degenerate parabolic equations*, J. of Diff. Eq. **83** (1990), 179–206.

[BJN+13] B. Barker, M. Johnson, P. Noble, L.M. Rodrigues, and K. Zumbrun, *Nonlinear modulational stability of periodic traveling-wave solutions of the generalized Kuramoto-Sivashinsky equation*, Physica D: Nonlinear Phenomena **258** (2013), 11–46.

[BP00] A.L. Bertozzi and M.C. Pugh, *Finite-time blow-up of solutions of some long-wave unstable thin film equations*, Indiana Univ. Math. J. **49** (2000), no. 4, 1323–1366.

[CFL+12] Roberto Camassa, M Gregory Forest, Long Lee, H Reed Ogrosky, and Jeffrey Olander, *Ring waves as a mass transport mechanism in air-driven core-annular flows*, Physical Review E **86** (2012), no. 6, 066305.

[CMOV16] R. Camassa, J.L. Marzuola, H.R. Ogrosky, and N. Vaughn, *Traveling waves for a model of gravity-driven film flows in cylindrical domains*, Physica D: Nonlinear Phenomena **333** (2016), 254–265.

[COO14] R. Camassa, H.R. Ogrosky, and J. Olander, *Viscous film flow coating the interior of a vertical tube. part i. gravity-driven flow*, J. Fluid Mech. **745** (2014), 682–715.

[COO17] Roberto Camassa, H Reed Ogrosky, and Jeffrey Olander, *Viscous film-flow coating the interior of a vertical tube. part 2. air-driven flow*, Journal of Fluid Mechanics **825** (2017), 1056–1090.

[CPT10] M. Chugunova, M.C. Pugh, and R.M. Taranets, *Nonnegative solutions for a long-wave unstable thin film equation with convection*, J. of Math. Anal. **42** (2010), no. 4, 1826–1853.

[CT12] M. Chugunova and R.M. Taranets, *Qualitative analysis of coating flows on a rotating horizontal cylinder*, Inter. J. of Diff. Eq. **2012** (2012), no. Article ID 570283, 30 pages.

[CT16] ——, *Blow-up with mass concentration for the long-wave unstable thin-film equation*, Applicable Anal. **95** (2016), no. 5, 944–962.

[Eid69] S.D. Eidel'man, *Parabolic systems*, North-Holland Publishing Company and Wolters-Noordhoof Publishing, 1969.

[ES95] M. Eres and L. Schwartz, *Modeling of coating flows on curved surfaces*, J. Engrg. Math. **29** (1995), 91–103.

[ESW97] M. Eres, L. Schwartz, and D. Weidner, *Simulation of coating layer evolution and drop formation on horizontal cylinders*, J. Colloid Interface Sci. **187** (1997), 243–258.

[Eva98] L. C. Evans, *Partial differential equations*, American Mathematical Society, 1998.

[Fre92] A.L. Frenkel, *Nonlinear theory of strongly undulating thin films flowing down vertical cylinders*, Europhys. Lett. **18** (1992), 583–588.

[Gia99] L. Giacomelli, *A fourth-order degenerate parabolic equation describing thin viscous flows over an incline plane*, Applied Math Letters **12** (1999), 107–111.

[Gyö71] I. Györi, *A generalisation of Bellman's inequality for Stieltjes integrals and a uniqueness theorem*, Studia. Sci. Math. Hungar. **6** (1971), 137–145.

[HR93] T. Hocherman and P. Rosenau, *On KS-type equations describing the evolution and rupture of a liquid interface*, Physica D: Nonlinear Phenomena **67** (193), 113–125.

[Ker94] V.I. Kerchman, *Strongly nonlinear interfacial dynamics in core-annular flows*, J. Fluid Mech. **290** (1994), 131–166.

[KF94] V.I. Kerchman and A.L. Frenkel, *Interactions of coherent structures in a film flow: Simulations of a highly nonlinear evolution equation*, Theoret. Comput. Fluid Dynamics **6** (1994), 235–254.

[Lin74] S.P. Lin, *Finite amplitude side-band stability of a viscous film*, J. Fluid Mech. **63** (1974), 417–429.

[LL75] S.P. Lin and W.C. Liu, *Instability of film coating of wires andtubes*, AIChE **24** (1975), 775–782.

[Ogr13] H Reed Ogrosky, *Modeling liquid film flow inside a vertical tube*, Ph.D. thesis, The University of North Carolina at Chapel Hill, 2013.

[SGKM95] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov, *Blow-up in quasilinear parabolic equations*, Walter de Gruyter and Co., 1995.

[Tar06] R. Taranets, *Propagation of perturbations in thin capillary film equations with nonlinear diffusion and convection*, Sib. Math. J. **47** (2006), 751–766.

E-mail address: marzuola@math.unc.edu

Department of Mathematics, University of North Carolina at Chapel Hill, CB#3250 Phillips Hall, Chapel Hill, NC 27599

E-mail address: swygerts@live.unc.edu

Department of Mathematics, University of North Carolina at Chapel Hill, CB#3250 Phillips Hall, Chapel Hill, NC 27599

E-mail address: taranets_r@yahoo.com

Institute of Applied Mathematics and Mechanics of the NASU, 1, Dobrovolśkogo Str., 84100, Sloviansk, Ukraine