THE RADIATION FIELD ON PRODUCT CONES

DEAN BASKIN AND JEREMY L. MARZUOLA

Abstract. We consider the wave equation on a product cone and find a joint asymptotic expansion for forward solutions near null and future infinities. The rates of decay seen in the expansion are the resonances of a hyperbolic cone on the "northern cap" of the compactification as computed exactly by the authors in [BM16]. The expansion treats an asymptotic regime not considered in the influential work of Cheeger and Taylor [CT82a, CT82b].

The main result follows the blueprint laid out in the asymptotically Minkowski setting [BVW15, BVW18]; the key new element consists of propagation estimates near the conic singularities. The proof of the propagation estimates builds on the work of Melrose–Vasy–Wunsch [MVW08] in the spacetime and on Gannot–Wunsch [GW18] in the semiclassical regime.

1. Introduction

In this paper we consider the long-time asymptotics of solutions of the wave equation on product cones. In particular, we find a complete (joint) asymptotic expansion for solutions near null infinity. The exponents in the expansion along null infinity are the resonances of the spectral family of the Laplacian on the "hyperbolic cone" living at the "northern cap" and were computed in a previous paper [BM16].

For a given compact connected Riemannian manifold (Z, k), we say that the cone $C(Z)$ over Z is the manifold with boundary

$$[0, \infty)_r \times Z,$$

equipped with the (singular) Riemannian metric

$$dr^2 + r^2 k.$$

We consider the wave equation

$$\begin{cases} \Box w = (D_t^2 - \Delta_{C(Z)})u = f \in C_c^\infty(\mathbb{R} \times C(Z)), \\ (w, \partial_t w)|_{t=0} \in C_c^\infty(C(Z)) \times C_c^\infty(C(Z)), \end{cases} \tag{1}$$

on $\mathbb{R} \times C(Z)$. In order to simplify matters, we always assume that $\Delta_{C(Z)}$ represents the Friedrichs extension of the Laplacian on $C(Z)$.

We compactify the spacetime $\mathbb{R} \times C(Z)$ to a manifold with corners we call M; in the case of a "phantom cone", this reduces to the compactification of the Minkowski metric considered in previous work [BVW15, BVW18]. The manifold with corners M has two boundary hypersurfaces: one, denoted mf corresponds to the "boundary at infinity", while the other, denoted cf, corresponds to the conic singularity.

The main result of this paper is to obtain the asymptotic behavior of the solution u near the light cone at infinity, which we denote S_+. In order to do this, we ultimately *blow up* S_+ in mf to obtain a third boundary hypersurface. Locally near the interior of this new front face (denoted $\mathcal{I}^+$), the blow-up amounts to introducing new coordinates $\rho = (1+t^2+r^2)^{-1/2}$, $s = t - r$, and z; the front face is given by $\rho = 0$.

In order to simplify the statement of the main theorem (i.e., to make a statement without compound asymptotics), we introduce the *Friedlander radiation field*, which is given in terms of $s = t - r$, r, and z by

$$\mathcal{R}_+[w](s,z) = \lim_{r\to\infty} r^{(n-1)/2} w(s+r, r, z),$$

i.e., by restricting an appropriate rescaling of the function to the new face. The function $\mathcal{R}_+[w]$ measures the radiation pattern seen by a distant observer and is an explicit realization of the Lax–Phillips translation representation as well as a generalization of the Radon transform.

Our main theorem can be stated in terms of the radiation field as s, the "lapse" parameter, tends toward infinity, and more generally, the compound asymptotics of the solution near the forward light cone.

Theorem 1.1. *Suppose u is a solution of the wave equation on a cone with smooth initial data compactly supported away from the conic singularity, i.e.,*

$$\Box w = 0 \quad \text{on } \mathbb{R}\times C(Z),$$
$$(w, \partial_t w)|_{t=0} \in C_c^\infty(C(Z)) \times C_c^\infty(C(Z)).$$

The radiation field $\mathcal{R}_+[w](s,z)$ of w admits an asymptotic expansion of the form

$$\mathcal{R}_+[w](s,z) \sim \sum_j \sum_{\kappa=0}^{m_j} a_{j,\kappa}(z) s^{i\sigma_j} (\log s)^\kappa$$

as $s \to +\infty$.

Moreover, w has a full asymptotic expansion away from cf*, with the compound asymptotics near $C_+ \cap \mathcal{I}^+$ given by*

$$w \sim r^{-\frac{n-1}{2}} \sum_j \sum_{\kappa\le m_j} \sum_{\ell=0}^{\infty} a_{j\kappa\lambda}(z) s^{-i\sigma_j} (\log s)^\kappa (s/r)^\ell.$$

In fact, the σ_j in the theorem are the resonances of the hyperbolic cone considered previously by the authors [BM16] and can be computed explicitly in terms of the eigenvalues μ_j^2 of Δ_k. Because each eigenvalue μ_j^2 leads to an entire family of resonances, it is easier to rename them $\sigma_{j,k}$ in terms of two parameters, which we call j and k. Here j refers to the eigenvalue in question and $k \in \mathbb{N} = \{0, 1, \dots\}$.

$$\sigma_{j,k} = -i\left(\frac{1}{2} + k + \sqrt{\left(\frac{n-2}{2}\right)^2 + \mu_j^2}\right) \tag{2}$$

provided that

$$\sqrt{\left(\frac{n-2}{2}\right)^2 + \mu_j^2} \notin \frac{1}{2} + \mathbb{Z}.$$

The resonance σ_j has the same multiplicity as the eigenvalue μ_j^2 of Δ_k.

Theorem 1.1 is an extension of the foundational work initiated by Cheeger–Taylor [CT82a, CT82b], though our aim is different. Cheeger and Taylor were more interested in the propagation of wavefront set for the wave equation on product cones; in particular their main aim was to show the existence (and calculate the symbol) of the diffracted wave arising from the metric singularity. In the process, they also found the asymptotic behavior of solutions of the wave equation away from $\mathcal{I}^+$; we recover their result in this region. Although in principle Theorem 1.1 can be recovered using the methods of Cheeger–Taylor [CT82a, CT82b] provided one could extend their asymptotic expansion uniformly to the boundary of the light cone, but we provide an alternative microlocal proof.

We note that the hypotheses of Theorem 1.1 may be relaxed somewhat; it is not strictly necessary that we consider the static wave equation on a product cone; we present this setting largely for pedagogical reasons and describe straightforward generalizations below (see Section 3). Although the argument simplifies in the product setting, the complications arising can be treated using more refined microlocal techniques. See for instance the previous papers [BVW15, BVW18] for relaxing the static hypothesis and Melrose–Vasy–Wunsch [MVW08] to relax the product hypothesis.

The novelty of this paper is at least twofold: not only do we contribute to the project of Cheeger–Taylor in a fashion that gives full a complete asymptotic description, but we find that cones provide an additional class of examples where the expansion of the radiation field can be computed explicitly.

1.1. **A sketch of the proof of Theorem 1.1.** To prove the main theorem, we adhere to the blueprint laid out in previous work of the first author [BVW15, BVW18], which in turn builds on the foundational work of Vasy [Vas13]. In particular, our aim is to reduce the problem of finding an asymptotic expansion to the inversion of a family of Fredholm operators on mf; the residues of the poles of this family generate the terms in the expansion. Showing that the family is Fredholm (and that the argument can begin) reduces to propagation of singularities arguments.

We begin with the solution of equation (1); by smoothly cutting off the solution for $t < 0$ we consider instead the forward solution of $\Box w = f'$, where $f' \in C_c^\infty(M)$ vanishes identically in a neighborhood of $\overline{C}_-$. We consider then the function $u = \rho^{-(n-1)/2} w$ and set

$$L = \rho^{-2} \rho^{-(n-1)/2} \Box \rho^{(n-1)/2},$$

so that u satisfies $Lu = f''$ for some other function $f'' \in C_c^\infty(M)$ vanishing near $\overline{C_-}$. A propagation of singularities argument (which will be proved in Section 6) shows that u is conormal to $S_\pm$.

We set $P_\sigma = \hat{N}(L)$ where $\hat{N}$ is the reduced normal operator, i.e., the family of operators on mf obtained by the Mellin transform in the normal variable ρ. We set $\widetilde{u}_\sigma$ and $\widetilde{f}_\sigma$ to be the Mellin transforms of u and f'', so that $\widetilde{u}_\sigma$ solves

$$P_\sigma \widetilde{u}_\sigma = \widetilde{f}_\sigma.$$

In general, one would expect additional correction terms, but the dilation invariance of the model problem simplifies the argument considerably. We show that we can propagate regularity from the past "radial points" of P_σ to the future ones. Away from the conic singularity, this argument is contained in the previous papers [BVW15, BVW18, Vas13]; the main missing piece is the propagation near the conic singularity (which we will prove in Section 7). This argument shows that P_σ is Fredholm on variable-order Sobolev-type spaces and P_σ^{-1} has finitely many poles in any horizontal strip. In fact, the poles of P_σ^{-1} can be identified with the resonances of the corresponding hyperbolic cone.

Once these pieces are in place, the argument from the prequel [BVW18] proves the main theorem. Since that argument simplifies slightly in the product context, we provide a sketch of it below (Section 8).

The next four sections are "preliminaries" to the main analysis contained in Sections 6, 7, and 8. Section 2 provides a brief review of the geometry of manifolds with corners and asymptotic expansions on them. Section 3 provides an introduction to the specific geometry we consider, while Section 4 presents the pseudodifferential calculi employed. Section 5 develops the function spaces in which the various arguments take place.

In the propagation arguments (and in the choice of coordinates in Section 3), we aim to match as closely as possible the notation of Melrose–Vasy–Wunsch [MVW08] to make reference to arguments easier. We further adopt the conventions that Δ is a non-negative operator, $D = \frac{1}{i}\partial$, m typically represents the order of a differential (or pseudodifferential) operator, and ℓ typically represents an index (or multi-index) for growth or decay.

Acknowledgments. The authors wish to thank Semyon Dyatlov, Oran Gannot, Peter Hintz, Rafe Mazzeo, Andras Vasy and Jared Wunsch and for valuable discussions. DB was supported in part by National Science Foundation (NSF) under NSF Grant DMS-1500646 and NSF CAREER Grant DMS-1654056. The research of JLM was supported by NSF Grant DMS-1312874 and NSF CAREER Grant DMS-1352353. Part of this work was done while the second author was on sabbatical at Duke University and the Mittag-Leffler Institute.

2. Basics of b-geometry

We begin by recalling results about analysis on manifolds with corners. Some of the discussion in the next few sections is adapated from the first author's previous work [BVW15, BVW18], while a more thorough discussion of b-geometry can be found in Melrose's book [Mel93, Chapter 4]. In the context of manifolds with corners, we refer the reader to Melrose's unpublished book [Mel96] and to Vasy's work [Vas08].

Throughout the paper we assume M is a compact $(n+1)$-dimensional manifold with corners and that X is a compact n-dimensional manifold with boundary. A function $\rho \in C^\infty(M)$ is a boundary defining function for a boundary hypersurface H of M if ρ vanishes simply at H and is non-vanishing elsewhere. A codimension k corner is the intersection of k boundary hypersurfaces of M.[1] Near a codimension k corner $H_1 \cap \cdots \cap H_k$, we may use

$$(\rho_1, \dots, \rho_k, y) \in [0,1)^k \times \mathbb{R}^{n+1-k}$$

as coordinates on M, where ρ_i is a boundary defining function for H_i and y are coordinates along the corner $H_1 \cap \cdots \cap H_k$.

The space of b-vector fields on M, denoted $\mathcal{V}_{\mathrm{b}}(M)$, is the space of smooth vector fields on M tangent to ∂M. Near a codimension k corner $H_1 \cap \cdots \cap H_k$, $\mathcal{V}_{\mathrm{b}}(M)$ is spanned over $C^\infty(M)$ by the vector fields $\rho_1\partial_{\rho_1}, \dots, \rho_k\partial_{\rho_k}, \partial_y$. The vector field $\rho_j\partial_{\rho_j}$ is called the b-*normal vector field* to the boundary hypersurface defined by ρ_j and is independent of choice of coordinates as an element of $\mathcal{V}_{\mathrm{b}}(M)/\rho\mathcal{V}_{\mathrm{b}}(M)$.

In fact, $\mathcal{V}_{\mathrm{b}}(M)$ is a Lie algebra and is the space of smooth sections of a vector bundle (called the b-tangent bundle) ${}^{\mathrm{b}}TM$ over M. The dual bundle of ${}^{\mathrm{b}}TM$ is ${}^{\mathrm{b}}T^*M$ and its sections are locally spanned (near a codimension k corner) over $C^\infty(M)$ by the 1-forms

[1] Our main applications involve corners of codimension no greater than two.

$d\rho_1/\rho_1, \dots, d\rho_k/\rho_k$ and dy. The b-cotangent bundle ${}^{\mathrm{b}}T^*M$ is equipped with a canonical 1-form, which can be written

$$\xi_1 \frac{d\rho_1}{\rho_1} + \cdots + \xi_k \frac{d\rho_k}{\rho_k} + \eta \cdot dy \tag{3}$$

in local coordinates near a codimension k corner. We further obtain the *fiber compactification* $\overline{{}^{\mathrm{b}}T^*}M$ of ${}^{\mathrm{b}}T^*M$ by radially compactifying each fiber. A defining function for the "boundary at infinity" of a fiber is given by

$$\nu = \left(\xi_1^2 + \cdots + \xi_k^2 + |\eta|^2\right)^{-1/2}$$

and near infinity we may use

$$\nu,\ \widehat{\xi} = \nu\xi,\ \widehat{\eta} = \nu\eta$$

as a redundant set of local coordinates on each fiber near $\{\nu = 0, \rho_1 = 0, \dots, \rho_k = 0\}$.[2] We let ${}^{\mathrm{b}}S^*M$ denote the boundary at infinity of $\overline{{}^{\mathrm{b}}T^*}M$, i.e., $\{\nu = 0\}$.

The b-cotangent bundle also inherits a canonical symplectic structure where the symplectic form is given by the exterior derivative of the canonical 1-form. (In other words, the natural symplectic structure on T^*M extends to ${}^{\mathrm{b}}T^*M$.) If we write covectors in ${}^{\mathrm{b}}T^*M$ in local coordinates as

$$\xi_1 \frac{d\rho_1}{\rho_1} + \cdots + \xi_k \frac{d\rho_k}{\rho_k} + \eta \cdot dy,$$

then the symplectic form is given by

$$d\xi_1 \wedge \frac{d\rho_1}{\rho_1} + \cdots + d\xi_k \wedge \frac{d\rho_k}{\rho_k} + d\eta \wedge dy. \tag{4}$$

As $\mathcal{V}_{\mathrm{b}}(M)$ is a Lie algebra, we may consider its universal enveloping algebra, denoted $\mathrm{Diff}_{\mathrm{b}}^*(M)$. Near the codimension k corner $H_1 \cap \cdots \cap H_k$ defined by $\{\rho_1 = 0, \dots, \rho_k = 0\}$, an operator $A \in \mathrm{Diff}_{\mathrm{b}}^m(M)$ has the form

$$A = \sum_{\alpha_1 + \cdots + \alpha_k + |\beta| \le m} a_{\alpha,\beta}(\rho_1, \dots, \rho_k, y)(\rho_1 D_{\rho_1})^{\alpha_1} \dots (\rho_k D_{\rho_k})^{\alpha_k} D_y^\beta, \tag{5}$$

where $a_{\alpha,\beta} \in C^\infty(M)$.

The semiclassical version of $\mathrm{Diff}_{\mathrm{b}}^m(M)$, denoted $\mathrm{Diff}_{\mathrm{b},h}^m(M)$, is similarly defined with a parametric dependence on a small parameter $h > 0$. In local coordinates, an operator $A \in \mathrm{Diff}_{\mathrm{b},h}^m(M)$ has the form

$$A = \sum_{\alpha_1 + \cdots + \alpha_k + |\beta| \le m} a_{\alpha,\beta}(\rho_1, \dots, \rho_k, y; h)(h\rho_1 D_{\rho_1})^{\alpha_1} \dots (h\rho_k D_{\rho_k})^{\alpha_k} D_y^\beta, \tag{6}$$

where $a_{\alpha,\beta} \in C^\infty(M)$ are bounded in h. In fact, we require $\mathrm{Diff}_{\mathrm{b},h}^*$ only in the context of manifolds with boundary.

We also introduce a multi-filtered version of $\mathrm{Diff}_{\mathrm{b}}^*(M)$. Suppose that $H_1, \dots, H_r$ are the boundary hypersurfaces of M and that ρ_i is a boundary defining function for H_i. Given a multi-index $\ell = (\ell_1, \dots, \ell_r) \in \mathbb{R}^r$, we obtain a multi-filtered algebra by setting

$$\mathrm{Diff}_{\mathrm{b}}^{m,\ell}(M) = \rho_1^{-\ell_1} \dots \rho_r^{-\ell_r} \,\mathrm{Diff}_{\mathrm{b}}^m(M) \equiv \rho^{-\ell}\,\mathrm{Diff}_{\mathrm{b}}^m(M).$$

[2]Strictly speaking, we should regard $(\widehat{\xi}, \widehat{\eta}) \in \mathbb{S}^n$ and then regard $(\nu, \widehat{\xi}, \widehat{\eta})$ as "polar coordinates" near infinity.

While the principal symbol of a differential operator specifies its high-frequency behavior, it does not capture the boundary asymptotics. At each boundary face, there is a dilation-invariant model operator, called the *normal operator*, that captures this behavior. The normal operator $N(A)$ of a b-differential operator A of the form above (5) at a codimension k boundary face H is the dilation-invariant operator given by freezing the coefficients of $\rho_i D_{\rho_i}$ and D_y at $\rho_1 = \cdots = \rho_k = 0$. In other words, $N(A) \in \mathrm{Diff}_{\mathrm{b}}^m([0,\infty)^k \times H)$ is given by

$$N(A) = \sum_{\alpha_1 + \cdots + \alpha_k + |\beta| \leq m} a_{\alpha,\beta}(0,\dots,0,y)(\rho_1 D_{\rho_1})^{\alpha_1} \dots (\rho_k D_{\rho_k})^{\alpha_k} D_y^\beta. \tag{7}$$

Just as the Fourier transform is useful in the study of approximately translation-invariant operators, the *Mellin transform* is useful in the study of approximately dilation-invariant operators. For the main application of this paper, we need only the Mellin transform associated to a single boundary *hypersurface* H. Suppose u is a distribution on M suitably localized near the boundary hypersurface H defined by ρ. The Mellin transform of u associated to H is defined by

$$\mathcal{M}_H u(\sigma, y) = \int_0^\infty \chi(\rho) u(\rho, y) \rho^{-i\sigma - 1}\, d\rho,$$

where χ is a smooth compactly supported function that is equal to 1 near $\rho = 0$.

The Mellin conjugate of the operator $N(A)$ is known as the *reduced normal operator*.[3] For $N(A)$ given in the formula (7) above, the reduced normal operator is the family of operators on the boundary hypersurface H given by

$$\widehat{N}(A) = \sum_{j + |\beta| \leq m} a_{j,\beta}(0,y) \sigma^j D_y^\beta, \tag{8}$$

where the other ρ variables are included with the y.

The Mellin transform is especially useful in the study of asymptotic expansions in powers of ρ (the boundary defining function for the hypersurface) and $\log \rho$. We first discuss the case where M has only a single boundary hypersurface, i.e., when M is a manifold with boundary. In particular, we recall from Melrose [Mel93, Section 5.10] that if u is a distribution on a manifold with boundary, we write

$$u \in \mathcal{A}^E_{\mathrm{phg}}(M) \qquad (u \text{ is polyhomogeneous with index set } E)$$

if and only if u is conormal to ∂M (and, in particular, is smooth away from the boundary) and

$$u \sim \sum_{(z,k) \in E} \rho^{iz} (\log \rho)^k a_{z,k},$$

where $a_{z,k}$ are smooth functions on ∂M. Here the expansion should be interpreted as an asymptotic series as $\rho \to 0$ and E is an *index set* and therefore must satisfy[4]

- $E \subset \mathbb{C} \times \{0, 1, 2, \dots\}$,
- E is discrete,
- if $(z_j, k_j) \in E$ with $|(z_j, k_j)| \to \infty$, then $\operatorname{Im} z_j \to -\infty$,
- if $(z, k) \in E$, then $(z, l) \in E$ for all $l = 0, 1, \dots, k-1$, and
- if $(z, k) \in E$, then $(z - ij, k) \in E$ for all $j = 1, 2, \dots$.

[3] We only require this construction for differential operators, although it extends to b-pseudodifferential operators as well.

[4] We have adopted the index set conventions of Melrose's unpublished book [Mel96] rather than the other reference [Mel93] to remain consistent with the first author's previous work [BVW15, BVW18].

We refer the reader to the work of Melrose [Mel93, Section 5.10] as to why these conditions are natural. We occasionally use $z \in \mathbb{C}$ to denote the smallest index set containing $(z, 0)$. As an example, the functions that are smooth up to ∂M are polyhomogeneous with index set 0 (i.e., with index set $E = \{(-ij, 0) : j = 0, 1, 2, \dots\}$).

Distributions in $\mathcal{A}^{E}_{\mathrm{phg}}(M)$ can be characterized in two different ways: by the Mellin transform and by the application of scaling (or *radial*) vector fields. To see the former, we recall the characterization of this space given by Melrose [Mel93, Proposition 5.27]. For a given index set E, a distribution u lies in $\mathcal{A}^{E}_{\mathrm{phg}}(M)$ if and only if its Mellin transform is meromorphic with poles of order k only at points z for which $(z, k-1) \in E$ (together with appropriate decay estimates in σ).

Alternatively, we may test for polyhomogeneity by using radial vector fields. Let R denote the radial vector field ρD_ρ. We characterize $u \in \mathcal{A}^{E}_{\mathrm{phg}}(M)$ by the requirement that for all A, there is a γ_A with $\gamma_A \to +\infty$ as $A \to +\infty$ so that

$$\left(\prod_{(z,k)\in E\, \operatorname{Im} z > -A} (R - z) \right) u \in H_{\mathrm{b}}^{\infty, \gamma_A}(M). \tag{9}$$

Our main theorem concerns polyhomogeneity at several boundary hypersurfaces not on a manifold with boundary but on a manifold with codimension 2 corners. For convenience, we repeat here several remarks from the first author's previous work [BVW18]. In our setting, at a codimension 2 corner defined by $\{\rho_1 = \rho_2 = 0\}$, we have $\mathcal{E} = (E_1, E_2)$, where each E_j is an index set at the boundary hypersurface defined by ρ_j. Essentially, the idea is that u has an expansion at each boundary hypersurface with coefficients that are polyhomogeneous at the other. In other words, we have

$$u \in \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(M)$$

if and only if for each $j = 1, 2$ we have

$$u \sim \sum_{(z,k)\in E_j} a_{j,z,k} \rho^{iz} (\log \rho)^k \qquad \operatorname{mod} H_{\mathrm{b}}^{\infty, \gamma_j}(M),$$

where for each (z, k), the coefficients $a_{j,z,k}$ are smooth at the hypersurface defined by ρ_j and polyhomogeneous (with index set E_2 or E_1, depending on whether $j = 1$ or $j = 2$) at the other. Here $\gamma_1 = (+\infty, -A)$ and $\gamma_2 = (-A, +\infty)$, where A is some fixed number greater than $\sup\{\operatorname{Im} z \mid (z, k) \in E_j,\ j = 1, 2\}$.

When testing for polyhomogeneity at two (or more) boundary hypersurfaces, it suffices to test *individually* at each one with uniform estimates at the other. This result is a consequence of a characterization by multiple Mellin transforms (see Melrose [Mel96, Chapter 4] or the Appendix of the PhD thesis of Economakis [Eco93], which contains a proof by Mazzeo). In particular, we rely on the following proposition.

Proposition 2.1 (Mazzeo, Melrose). *Let R_j denote $\rho_j D_{\rho_j}$, the radial vector field at the boundary hypersurface defined by ρ_j. For $\mathcal{E} = (E_1, E_2)$, a distribution u lies in $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(M)$ if and only if for each $j = 1, 2$ there are fixed weights γ'_j at the other boundary hypersurfaces and, for all A, there is a $\gamma_{j,A}$ with $\gamma_{j,A} \to +\infty$ as $A \to +\infty$, so that*

$$\left(\prod_{(z,k)\in E_j\, \operatorname{Im} z > -A} (R_j - z) \right) u \in H_{\mathrm{b}}^{\infty, \gamma_{j,A}, \gamma'_j}(M), \tag{10}$$

where $\gamma_{j,A}$ is the weight at the boundary hypersurface defined by ρ_j and γ'_j encodes the weights at the other boundary hypersurfaces.

In other words, applying the test above (9) at the boundary hypersurface H_i defined by ρ_i improves the decay at H_i at no cost to the growth or decay at the other boundary hypersurfaces. Note that there is no requirement that the coefficients in the expansion (or the remainder) are polyhomogeneous. Indeed, their polyhomogeneity follows automatically when the condition (10) is imposed at all boundary hypersurfaces.

3. Conic geometry

Our primary concern is the wave equation on a cone, so we describe this setting in detail. Remark 3.1 describes natural extensions to this setting in which our main results still hold.

Suppose (Z, k) is a compact, connected $(n-1)$-dimensional Riemannian manifold. We say that the metric cone $C(Z)$ over Z is the manifold

$$\mathbb{R}^+ \times Z = (0,\infty)_r \times Z$$

equipped with the warped product metric

$$dr^2 + r^2 k.$$

This metric is singular and incomplete at $r = 0$; we refer to the natural boundary $\{0\} \times Z$ as the *cone point*.[5]

Our main result concerns solutions of the wave equation on the spacetime $M^\circ = \mathbb{R}_t \times C(Z)$, which is equipped with the Lorentzian metric

$$g = -dt^2 + dr^2 + r^2 k.$$

We may regard M° as the interior of a compact manifold with corners. For clarity, we first describe this compactification in the $(1+1)$-dimensional setting (i.e., when Z is a single point). We will return to this familiar example throughout the manuscript as an illustration of our methods (though of course Theorem 1.1 is trivial in this case).

We compactify $\mathbb{R}_t \times (0,\infty)_r$ by stereographic projection to a quarter-sphere $\mathbb{S}^2_{++}$ as depicted in Figure 1. In other words, the map $\mathbb{R}_t \times (0,\infty)_r \to \mathbb{S}^2 \subset \mathbb{R}^3$ given by

$$(t, r) \mapsto \frac{(t, r, 1)}{\sqrt{1 + t^2 + r^2}}$$

sends M° to the interior of the quarter-sphere given by

$$\mathbb{S}^2_{++} = \{(z_1, z_2, z_3) \in \mathbb{S}^2 \subset \mathbb{R}^3 \mid z_2 \geq 0,\ z_3 \geq 0\}.$$

The quarter-sphere $\mathbb{S}^2_{++}$ is a manifold with corners and has two boundary hypersurfaces defined by the boundary defining functions z_2 and z_3. We let cf (or the *conic face*) be the hypersurface defined by the function

$$z_2 = \frac{r}{\sqrt{1 + t^2 + r^2}}$$

and we let mf (or the *main face*) be the face defined by

$$z_3 = \frac{1}{\sqrt{1 + t^2 + r^2}}.$$

[5]We regard the conic singularity as being purely metric; we can think of it has having been previously resolved.

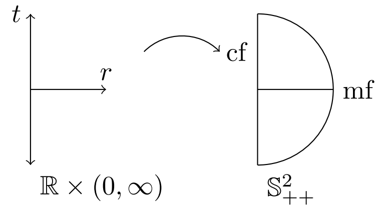

FIGURE 1. The compactification of $\mathbb{R} \times (0, \infty)$ to $\mathbb{S}^2_{++}$

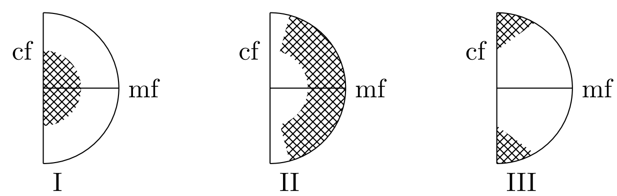

FIGURE 2. Regions I, II, and III in $\mathbb{S}^2_{++}$

Having defined the smooth structure on this compactification, it is often convenient to work with other equivalent boundary defining functions in different regions. We define regions I, II, and III (the shaded regions in Figure 2) as follows: We let region I denote a fixed neighborhood in $\mathbb{S}^2_{++}$ bounded away from mf; region II is a neighborhood bounded away from cf; finally, region III is a neighborhood of the corner $\mathrm{cf} \cap \mathrm{mf}$. For concreteness, we can take region I to be given by $|t|, r \leq 10$, region II to be $r \geq 2, r \geq |t|/2$, and region III to be $|t| \geq 2, |t| \geq r/2$.

We now describe several convenient coordinate systems valid in different regions. For notational convenience, we will always use ρ to denote a defining function for mf and x to denote a defining function for cf. In region I (where we are bounded away from mf), it is convenient to take $x = r$, while in region II (where we are bounded away from cf), we can take $\rho = 1/r$. Finally, in region III (which is the source of most of the new technical work in this document), it is often convenient to take $\rho = 1/t$ and $x = r/t$. *Because polyhomogeneity is independent of the choice of (equivalent) boundary defining function, we typically use whichever boundary defining functions are most convenient at the time.*

As another way of understanding the smooth structure of the compactification, we also introduce "almost global" coordinates that are valid away from $t = r = 0$. We introduce $\rho \in [0, 1)$ and $\theta \in [0, \pi]$ defined by

$$t = \frac{1}{\rho} \cos\theta, \quad r = \frac{1}{\rho} \sin\theta.$$

One can then take $x = \sin\theta$ as the defining function for cf. In these "almost global" coordinates the metric g takes the form

$$g = -\cos 2\theta \frac{d\rho^2}{\rho^4} - 2\sin 2\theta \frac{d\theta}{\rho} \frac{d\rho}{\rho^2} + \cos 2\theta \frac{d\theta^2}{\rho^2}.$$

By introducing $v = \cos 2\theta$, we have

$$g = -v \frac{d\rho^2}{\rho^4} + \frac{dv}{\rho} \frac{d\rho}{\rho^2} + \frac{v}{4(1 - v^2)} \frac{dv^2}{\rho^2},$$

and so the metric in region II has the same form as the short-range asymptotically Minkowski metrics introduced by the first author and collaborators [BVW15].

Near the corner (region III), if we instead use $\rho = 1/t$ and $x = r/t$, the metric has the form

$$g = -(1-x^2)\,\frac{d\rho^2}{\rho^4} - 2x\,\frac{dx}{\rho}\frac{d\rho}{\rho^2} + \frac{dx^2}{\rho^2}.$$

For the more general case of $M^\circ = \mathbb{R} \times C(Z)$, we take M to be the closure of the image of M° under the map $\mathbb{R} \times (0,\infty) \times Z \to \mathbb{S}^2 \times Z$ given by

$$(t, r, z) \mapsto \left(\frac{(t, r, 1)}{\sqrt{1 + t^2 + r^2}}, z \right).$$

In other words, we take $M = \mathbb{S}^2_{++} \times Z$ to be the compactification of M° to a manifold with corners.

In region I, the metric is the spacetime metric on a conic manifold as was studied by Melrose–Wunsch [MW04a] (and later by Melrose–Vasy–Wunsch [MVW08]). In region II, the metric has the form

$$g = -v\,\frac{d\rho^2}{\rho^4} + \frac{dv}{\rho}\frac{d\rho}{\rho^2} + \frac{v}{4(1-v^2)}\,\frac{dv^2}{\rho^2} + \frac{1-v}{2}\,\frac{k}{\rho^2}, \tag{11}$$

which is again a short-range asymptotically Minkowski metric.

Near the corner (region III), in terms of $\rho = 1/t$ and $x = r/t$, the metric has the form

$$g = -(1-x^2)\,\frac{d\rho^2}{\rho^4} - 2x\,\frac{dx}{\rho}\frac{d\rho}{\rho^2} + \frac{dx^2}{\rho^2} + x^2\,\frac{k}{\rho^2}. \tag{12}$$

This metric is a hybrid of a Lorentzian scattering metric (in that it is built from 1-forms of the type $d\rho/\rho^2$ and α/ρ) and a conic-type metric (in that it degenerates as $x \to 0$).

Remark 3.1. There are a number of natural extensions to the product cone setting that require little additional work. All of the results and proofs in this manuscript apply to the setting where g is a Lorentzian metric on $M = \mathbb{S}^2_{++} \times Z$ that is

(1) a spacetime conic metric (so that the results of Melrose–Wunsch [MW04a] apply) in region I,
(2) a (long-range or short-range) asymptotically Minkowski metric in region II, and
(3) a hybrid in region III. In other words, in region III, we demand that g is built out of $\frac{d\rho}{\rho^2}$, $\frac{dx}{\rho}$, and $\frac{dz}{\rho}$ and that its leading order behavior as $x \to 0$ (in terms of these objects) is
$$-\frac{d\rho^2}{\rho^4} + \frac{dx^2}{\rho^2} + x^2\frac{k}{\rho^2}.$$

3.1. **The radiation field blow-up.** In this section we recall from previous work [BVW15, BVW18] the construction of the manifold with corners on which the radiation field naturally lives.

Consider the submanifold given by $S = \{v = \rho = 0\}$ in terms of the almost global coordinates on M. This submanifold naturally splits into two pieces according to whether $\pm t > 0$ near the component. We denote these two pieces $S_\pm$. The complement of S in mf consists of three regions. The region C_0 consists of those points in mf where $v < 0$, while the region in mf where $v > 0$ has two components, denoted $C_\pm$ according to whether $\pm t > 0$ nearby.

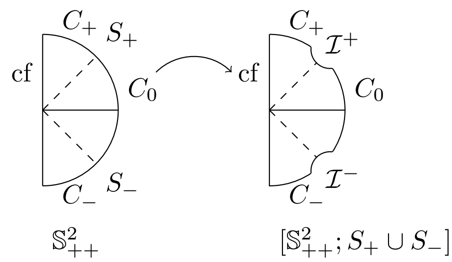

FIGURE 3. A schematic view of the radiation field blow-up. The lapse function s increases along $\mathcal{I}^+$ towards C_+.

We now *blow up* S in M by replacing it with its inward pointing spherical normal bundle.[6] In the product cone setting, this is equivalent to blowing up a pair of points in $\mathbb{S}^2_{++}$ and then taking the product with Y. This process replaces M with a new manifold $\overline{M} = [M; S]$ on which polar coordinates around the submanifold are smooth; the structure of this manifold with corners depends only on the submanifold S (and not on the particular choice of defining functions v and ρ). The blow-up is equipped with a natural blow-down map $\bar{M} \to M$, which is a diffeomorphism on the interior.

The new space $\overline{M}$ is again a manifold with corners and has six boundary hypersurfaces: the closure of the lifts of the interiors of C_0 and $C_\pm$ to $\overline{M}$, which are again denoted by C_0 and $C_\pm$; the lift of cf, again denoted cf, and two new boundary hypersurfaces consisting of the pre-image of the future and past components of S under the blow-down map. These two new hypersurfaces are denoted $\mathcal{I}^\pm$. Moreover, $\mathcal{I}^\pm$ is naturally a fiber bundle over $S_\pm$ with fibers diffeomorphic to intervals. Indeed, the interior of the fibers is naturally an affine space (i.e., $\mathbb{R}$ acts by translations, but there is no natural origin). Figure 3 depicts this blow-up construction. Given v and ρ, the fibers of the interior of $\mathcal{I}^\pm$ in $\overline{M}$ can be identified with $\mathbb{R} \times Z$ via the coordinate $s = v/\rho$.

In our setting, Friedlander's argument [Fri80, Fri01] shows that for solutions u of $\Box u = 0$ with smooth, compactly supported initial data, the restriction

$$\mathcal{R}_\pm[u](s, y) = \rho^{-\frac{n-1}{2}} u|_{\mathcal{I}^\pm}$$

is well-defined and smooth. This is *Friedlander's radiation field*.[7]

Friedlander's argument motivates the definition below of the operator

$$L = \rho^{-2-(n-1)/2} \Box_g \rho^{(n-1)/2},$$

and its reduced normal operator $P_\sigma = \widehat{N}(L)$. Because changing the boundary defining functions by a smooth non-vanishing multiple changes L and P_σ by an element of $\mathrm{Diff}^1_{\mathrm{b}}$, we freely work with whichever forms of the boundary defining functions are most convenient. In particular, for the main argument of Section 8 we use coordinates in which the metric takes the form in equation (11), while in Sections 6 and 7 we use the coordinates in which the metric has the form (12).

For later reference, we record the forms of the operators in regions II and III. In region II, using the same coordinates as in the form of the metric there (11), we can record (where

[6]The reader may wish to consult Melrose's book [Mel93] for more details on the blow-up construction.

[7]Note that our definition differs from Friedlander's by the absence of a derivative.

throughout we use the notation $D = \frac{1}{i}\partial$)

$$
\begin{aligned}
L = {} & v(\rho D_\rho)^2 - 4(1-v^2)\rho D_\rho D_v - 4v(1-v^2)D_v^2 - \frac{2}{1-v}\Delta_z \\
& - ((n-1)+(n+1)v)\, i\rho D_\rho + 2\left(2-(n-1)v-(n-3)v^2\right) iD_v \\
& - 2\left(\frac{n-1}{2}\right)^2 - (n-1)\left(\frac{n+3}{2}\right) v.
\end{aligned} \tag{13}
$$

We also record the form of P_σ here:

$$
\begin{aligned}
P_\sigma = {} & -4v(1-v^2)D_v^2 - \frac{2}{1-v}\Delta_z + 2\left(2-(n-1)v-(n-3)v^2\right) iD_v \\
& - 4(1-v^2)\sigma D_v + v\sigma^2 - ((n-1)+(n+1)v)\, i\sigma \\
& - 2\left(\frac{n-1}{2}\right)^2 - (n-1)\left(\frac{n+3}{2}\right) v.
\end{aligned} \tag{14}
$$

In region III, where the metric has the form in equation (12), we write

$$
\begin{aligned}
L &= (\rho D_\rho + xD_x)^2 - ni(\rho D_\rho + xD_x) - D_x^2 + \tfrac{(n-1)i}{x} D_x - \tfrac{1}{x^2}\Delta_z - \tfrac{n^2-1}{4}, \\
P_\sigma &= (xD_x + \sigma)^2 - ni(xD_x + \sigma) - D_x^2 + \tfrac{(n-1)i}{x} D_x - \tfrac{1}{x^2}\Delta_z - \tfrac{n^2-1}{4}.
\end{aligned} \tag{15}
$$

3.2. **Broken bicharacteristics.** In the main propagation result of Section 6, we require a small amount of the edge calculus machinery (namely, the bundles and the differential operators) introduced by Mazzeo [Maz91]. We specialize our description to the specific setting in which we work, though the calculus applies in much more general settings. For a more detailed description, we refer the reader to Mazzeo [Maz91]. In an abuse of notation, we use the term "edge" to refer to objects that behave as edge objects near cf and as b-objects at mf.

Our use of the edge machinery is limited to a neighborhood of the boundary hypersurface cf corresponding to the conic singularities. This boundary hypersurface is the total space of a (trivial) fiber bundle:

$$
\begin{array}{ccc}
Z & \text{———} & \mathrm{cf} \\
 & & \downarrow \\
 & & I
\end{array}
$$

Here I is a compactification of $\mathbb{R}$; t is locally a coordinate on the interior of I, while ρ provides a coordinate near the boundary of I.

The set of edge-vector fields, denoted $\mathcal{V}_{\mathrm{e}}$, consists of those b-vector fields tangent to the leaves of the fibration. In local coordinates (ρ, x, z), where x is the boundary defining function for cf and z is a coordinate along Z, $\mathcal{V}_{\mathrm{e}}$ is spanned over C^∞ by

$$x\partial_x, \quad x(\rho\partial_\rho), \quad \text{and } \partial_z.$$

The Lie algebra $\mathcal{V}_{\mathrm{e}}$ is the space of smooth sections of a vector bundle (called the e-tangent bundle) ${}^{\mathrm{e}}TM$ over M.[8]

[8]Strictly speaking, as a global object we are considering a mixed edge-b-tangent bundle, but our results are essentially local so we do not stress this point.

We let $\operatorname{Diff}_{\mathrm{e}}^*(M)$ denote the universal enveloping algebra of $\mathcal{V}_{\mathrm{e}}(M)$. An element $A \in \operatorname{Diff}_{\mathrm{e}}^m(M)$ near $\mathrm{cf} \cap \mathrm{mf}$ has the form

$$A = \sum_{j+k+|\beta|\leq m} a_{jk\beta}(\rho, x, z)(x\rho D_\rho)^j (xD_x)^k D_z^\beta,$$

where the $a_{jk\beta}$ are smooth on M. In region III above, the wave operator is an element of $\rho^2 x^{-2} \operatorname{Diff}_{\mathrm{e}}^2(M)$; this relationship is exploited below in Section 6.

Canonical coordinates on ${}^{\mathrm{e}}T^*M$ induced by coordinates (ρ, x, z) are $(\rho, x, z, \tau, \xi, \zeta)$, which corresponds to writing covectors as

$$\tau \frac{d\rho}{x\rho} + \xi \frac{dx}{x} + \zeta \cdot dz.$$

We then obtain a bundle map $\pi : {}^{\mathrm{e}}T^*M \to {}^{\mathrm{b}}T^*M$ given in these coordinates by

$$\pi(\rho, x, z, \tau, \xi, \zeta) = (\rho, x, z, \tau, x\xi, x\zeta).$$

In other words, the map π is given by $\omega \mapsto x\omega$, which is an isomorphism ${}^{\mathrm{e}}T^*M \to {}^{\mathrm{b}}T^*M$ away from $x = 0$.

Away from $x = 0$, the bicharacteristics (geodesics lifted to the b-cotangent bundle) of L are the integral curves of the b-Hamilton vector field of the b-principal symbol of L. Because (M, g) is incomplete (owing to the conic singularity in $C(Z)$), we must clarify what we mean by bicharacteristics hitting the cone point. The aim of this subsection is to describe one notion of bicharacteristic flow through the singularity at cf. As we are interested in wave equations, we restrict our attention to *null bicharacteristics*, i.e., those lying in the characteristic set of L.

We start by defining the *compressed cotangent bundle* by

$${}^{\mathrm{b}}\dot{T}^*M = \pi({}^{\mathrm{e}}T^*M)/Z, \quad \dot{\pi} : {}^{\mathrm{e}}T^*M \to {}^{\mathrm{b}}\dot{T}^*M.$$

The quotient by Z acts only over the boundary; the topology is given by the quotient topology. Observe that ${}^{\mathrm{b}}\dot{T}^*_{\mathrm{cf}}M$ can be identified with ${}^{\mathrm{b}}T^*I$. More explicitly, in terms of coordinates $(\rho, x, z, \tau, \xi, \zeta)$ on ${}^{\mathrm{e}}T^*M$, $\pi({}^{\mathrm{e}}T^*_{\mathrm{cf}}M)$ is given by points of the form $(\rho, 0, z, \tau, 0, 0)$. After the quotient, ρ and τ provide coordinates on ${}^{\mathrm{b}}\dot{T}^*_{\partial M}M$.

Observe that $x^2 L \in \operatorname{Diff}_{\mathrm{e}}^2(M)$; near $\mathrm{mf} \cap \mathrm{cf}$, its edge-principal symbol is

$$\sigma_{\mathrm{e}}(x^2 L) = (\tau + x\xi)^2 - \xi^2 - |\zeta|^2 - x^2 \frac{n^2 - 1}{4}.$$

In an abuse of notation (but following Melrose–Vasy–Wunsch [MVW08, Section 7]), we can introduce

$$\pi({}^{\mathrm{e}}S^*M) = \left(\pi({}^{\mathrm{e}}T^*M) \setminus 0\right)/\mathbb{R}^+ \subset {}^{\mathrm{b}}S^*M,$$
$$\dot{\pi}({}^{\mathrm{e}}S^*M) = \left(\dot{\pi}({}^{\mathrm{e}}T^*M) \setminus 0\right)/\mathbb{R}^+ \subset {}^{\mathrm{b}}\dot{S}^*M,$$

where ${}^{\mathrm{b}}S^*M$ and ${}^{\mathrm{e}}S^*M$ are quotients of their respective cotangent bundles by the natural scaling action and ${}^{\mathrm{b}}\dot{S}^*M = {}^{\mathrm{b}}S^*M/Z$.

Because cf is noncharacteristic, nonzero covectors in the edge-characteristic set of x^2L are mapped to nonzero covectors by π and $\dot{\pi}$. We can thus define the *compressed characteristic set*

$$\dot{\Sigma} = \dot{\pi}(\Sigma),$$

where $\Sigma \subset {}^{\mathrm{e}}S^*M$ is the edge-characteristic set of x^2L. Over $x = 0$, $\dot{\Sigma} = {}^{\mathrm{b}}\dot{S}^*_{\{x=0\}}M$.

Just as in Melrose–Vasy–Wunsch [MVW08, Section 7], we define the hyperbolic subset of $\pi({}^{\mathrm{e}}S^*_{\mathrm{cf}}M)$ by

$$\mathcal{H} = \{q \in \pi({}^{\mathrm{e}}S^*_{\mathrm{cf}}M) : \# \left(\pi^{-1}(q) \cap \Sigma\right) \geq 2\}.$$

Observe that in fact $\mathcal{H} = \pi({}^{\mathrm{e}}S^*_{\mathrm{cf}}M)$ and so there is no need to define the elliptic or glancing parts of this set. The cooresponding set in ${}^{\mathrm{b}}\dot{S}^*_{\mathrm{cf}}M$ is given by

$$\dot{\mathcal{H}} = \mathcal{H}/Z.$$

For a general definition of generalized broken bicharacteristics, we refer to Melrose–Vasy–Wunsch [MVW08, Definition 7.5]. In the present context, they can instead be described more simply. Away from cf, they consist of lifts of maximally extended light-like geodesics to ${}^{\mathrm{b}}S^*M$ (or, alternatively, to ${}^{\mathrm{e}}S^*M$). Near the cone points, they are concatenations of bicharacteristics that are continuous as functions to $\dot{\Sigma}$.

In particular, near cf, the generalized broken bicharacteristics are concatenations of (lifts of) light-like geodesics entering and exiting the same conic singularity; the continuity condition requires only that they enter and leave "at the same time" (i.e., the ρ-coordinates agree) and with the same "time momentum" (i.e., the same value of τ). More precisely, a simple ODE analysis shows that null bicharacteristics enter ${}^{\mathrm{e}}S^*_{\mathrm{cf}}M$ with coordinates

$$(\rho_0, 0, z_0, \tau_0, \xi_0, 0),$$

where $\tau_0^2 = \xi_0^2$. These bicharacteristics then leave ${}^{\mathrm{e}}S^*_{\mathrm{cf}}M$ from the point

$$(\rho_0, 0, z_1, \tau_0, -\xi_0, 0),$$

where $z_1 \in Z$.[9]

4. Pseudodifferential operators

We may now describe the spaces of b-pseudodifferential operators we employ. In the bulk M, we rely on the homogenous b-calculus associated to a manifold with corners. On $X = \mathrm{mf}$ we use the semiclassical b-calculus on a manifold with boundary.

4.1. The homogeneous b-calculus.

We now briefly describe the spaces Ψ^m_{b}, $\Psi^m_{\mathrm{b}\infty}$, and $\Psi^{m,\ell}_{\mathrm{b}}$ of b-pseudodifferential operators on M. Rather than provide detailed definitions and proofs, we instead provide a list of their properties and refer the reader to Melrose's unpublished book [Mel96] and Vasy's paper [Vas08] for details.

Our discussion in this section is specialized to a neighborhood of $\mathrm{mf} \cap \mathrm{cf}$ (region III) in M; the relevant results in region I can be quoted, while the results in region II can be recovered by assuming that x is bounded away from 0.

The space of b-pseudodifferential operators $\Psi^*_{\mathrm{b}}(M)$ is the "quantization" of the Lie algebra $\mathcal{V}_{\mathrm{b}}(M)$ and formally consists of operators of the form

$$b\left(\rho, x, z, \rho D_\rho, x D_x, D_z\right),$$

where b is a classical symbol (i.e., it is smooth on ${}^{\mathrm{b}}T^*M$ and has an asymptotic expansion in increasing powers of ν). In terms of coordinates (ρ, x, z) near the corner $\mathrm{mf} \cap \mathrm{cf}$, we may

[9] In other words, the direction in which the bicharacteristic leaves the cone point has no relation to the direction in which it entered. In the parlance of Melrose–Wunsch [MW04b], these are the "diffractive" bicharacteristics.

write an explicit quantization of the symbol b by

$$\mathrm{Op}(b)u(\rho, x, z) = \frac{1}{(2\pi)^{n+1}} \int\int e^{i(\rho-\rho')\underline{\tau}+i(x-x')\underline{\xi}+i(z-z')\cdot\underline{\zeta}} \phi\left(\frac{\rho-\rho'}{\rho}\right)\phi\left(\frac{x-x'}{x}\right)\psi(z) \cdot b(\rho, x, z, \rho\underline{\tau}, x\underline{\xi}, \underline{\zeta}) u(\rho', x', z')\, d\underline{\tau}\, d\underline{\xi}\, d\underline{\zeta}\, d\rho'\, dx'\, dz',$$

where $\phi \in C_c^\infty((-1/2, 1/2))$ is identically 1 near 0, ψ localizes to a region where the local coordinates on z are valid, and the integrals in ρ' and x' are over $[0, \infty)$.

We further define a multi-filtered algebra $\Psi_{\mathrm{b}}^{m,\ell}(M) = \rho^{-\ell}\Psi_{\mathrm{b}}^m(M)$. Thus the index ℓ refers only to the filtration in ρ; any filtration in x will be made more explicit in the text in later sections.

Similar to the results from Melrose–Vasy–Wunsch [MVW08], for our regularization arguments in Section 6 we require a slightly larger algebra we call $\Psi_{\mathrm{b}\infty}^*(M)$. It is defined in the same way but with symbols satisfying Kohn–Nirenberg estimates (rather than having complete asymptotic expansions).

The algebra $\Psi_{\mathrm{b}}^{m,\ell}(M)$ satisfies the following properties:

i. If $A \in \Psi_{\mathrm{b}}^{m,\ell}(M)$ (or $\Psi_{\mathrm{b}\infty}^{m,\ell}$), then A defines continuous maps $A : \dot{C}^\infty(M) \to \dot{C}^\infty(M)$ and $A : C^{-\infty}(M) \to C^{-\infty}(M)$.

ii. The principal symbol of a b-differential operator extends continuously to give a map

$$\sigma_{\mathrm{b},m,\ell} : \Psi_{\mathrm{b}}^{m,\ell}(M) \to \rho^{-\ell} C^\infty({}^{\mathrm{b}}S^*M).$$

The principal symbol map is multiplicative, i.e., $\sigma(AB) = \sigma(A)\sigma(B)$.

In the case of $\Psi_{\mathrm{b}\infty}^m$, the principal symbol takes values in the quotient space

$$S^m({}^{\mathrm{b}}T^*M)/S^{m-1}({}^{\mathrm{b}}T^*M),$$

which in the case of classical symbols can be identified with $C^\infty({}^{\mathrm{b}}S^*M)$.

The principal symbol captures the top order behavior of elements of $\Psi_{\mathrm{b}}^{m,\ell}(M)$. In other words, the following sequence is exact:

$$0 \to \Psi_{\mathrm{b}}^{m-1,\ell}(M) \to \Psi_{\mathrm{b}}^{m,\ell}(M) \to \rho^{-\ell}C^\infty({}^{\mathrm{b}}S^*M) \to 0.$$

(In the case of $\Psi_{\mathrm{b}\infty}$, the latter space must be replaced by the quotient S^m/S^{m-1}.)

iii. There is a quantization map $\mathrm{Op}_{\mathrm{b}} : \rho^{-\ell}S^m({}^{\mathrm{b}}T^*M) \to \Psi_{\mathrm{b}}^{m,\ell}(M)$ so that

$$\sigma_{\mathrm{b},m,\ell}(\mathrm{Op}_{\mathrm{b}}(a)) = a$$

as an element of $\rho^{-\ell}S^m({}^{\mathrm{b}}T^*M)/\rho^{-\ell}S^{m-1}({}^{\mathrm{b}}T^*M)$. A similar statement holds for $\Psi_{\mathrm{b}\infty}$ when the symbol space only satisfies Kohn–Nirenberg estimates.

iv. The algebras $\Psi_{\mathrm{b}}^{m,\ell}$ and $\Psi_{\mathrm{b}\infty}^m$ are closed under adjoints, and

$$\sigma(A^*) = \overline{\sigma(A)}.$$

v. If $A \in \Psi_{\mathrm{b}}^{m,\ell}(M)$ and $B \in \Psi_{\mathrm{b}}^{m',\ell'}(M)$, then $[A, B] = AB - BA \in \Psi_{\mathrm{b}}^{m+m'-1,\ell+\ell'}(M)$ and

$$\sigma_{\mathrm{b},m+m'-1,\ell+\ell'}(i[A, B]) = \{\sigma(A), \sigma(B)\},$$

where the right hand side denotes the Poisson bracket induced by the symplectic structure on ${}^{\mathrm{b}}T^*M$.

vi. Elements of $\Psi_{\mathrm{b}}^0(M)$ are bounded on L^2. In particular, given $A \in \Psi_{\mathrm{b}}^0(M)$, there is an $A' \in \Psi_{\mathrm{b}}^{-1}(M)$ so that

$$\|Au\|_{L^2} \leq 2 \sup|\sigma(A)|\, \|u\|_{L^2} + \|A'u\|_{L^2}.$$

We further require the notion of a *basic operator* (reintroduced below in Section 6). As in Melrose–Vasy–Wunsch [MVW08, Section 9], we say a symbol $a \in C^\infty({}^{\mathrm{b}}T^*M)$ is *basic* if it is constant on the fibers above ${}^{\mathrm{b}}\dot{T}^*M$, i.e., in terms of local coordinates $\partial_z a = 0$ at $\{x = 0, \underline{\xi} = 0, \underline{\zeta} = 0\}$. The quantization of such a symbol is also called basic.

We now recall from Melrose–Vasy–Wunsch [MVW08, Lemma 8.6] how the b-calculus interacts with $1/x$, D_x and $\frac{1}{x}D_{z_j}$.

Lemma 4.1 (cf. [MVW08, Lemma 8.6]). *If $A \in \Psi_{\mathrm{b}}^m(M)$, then there are $B \in \Psi_{\mathrm{b}}^m(M)$ and $C \in \Psi_{\mathrm{b}}^{m-1}(M)$ depending continuously on A so that*

$$i[D_x, A] = B + CD_x,$$

with $\sigma(B) = \partial_x(\sigma(A))$ and $\sigma(C) = \partial_{\underline{\xi}}(\sigma(A))$.

In particular, given $C = i[1/x, A]x$, we have $C \in \Psi_{\mathrm{b}}^{m-1}(M)$ and

$$\sigma_{m-1}(C) = H_{x^{-1}}\sigma(A) = \partial_{\underline{\xi}}\sigma(A).$$

If, in addition, A is a basic operator, then

$$i\left[\frac{1}{x}D_{z_j}, A\right] = B_j + C_j D_x + \sum_k E_{jk}\frac{1}{x}D_{z_k} + \frac{1}{x}F_j,$$

with $B_j \in \Psi_{\mathrm{b}}^m(M)$, C_j, E_{jk}, $F_j \in \Psi_{\mathrm{b}}^{m-1}(M)$ depending continuously on A, and

$$\partial_{z_j}(\sigma(A)) + \underline{\zeta}_k \partial_{\underline{\xi}}(\sigma(A)) = x\sigma(B_j) + \underline{\xi}\sigma(C_j) + \sum_k \underline{\zeta}_k \sigma(E_{jk}).$$

4.2. **The semiclassical b-calculus.** In this section we briefly describe some properties satisfied by the semiclassical b-calculus $\Psi_{\mathrm{b}}^*(X)$ described by Gannot–Wunsch [GW18, Section 3]. We refer to that paper for details. Throughout this section we assume X is an n-dimensional manifold with boundary.[10]

An explicit quantization procedure on $\mathbb{R}^n_+ = [0,\infty)\times\mathbb{R}^{n-1}$ is given by fixing $\phi \in C_c^\infty((-1/2, 1/2))$ so that $\phi(s) = 1$ near $s = 0$. Given $a \in S_h^m({}^{\mathrm{b}}T^*\mathbb{R}^n_+)$, define $\mathrm{Op}_{\mathrm{b},h}(a)$ by

$$\mathrm{Op}_{\mathrm{b},h}(a)u(x,z) = \frac{1}{(2\pi)^n}\int e^{\frac{i}{h}\left((x-x')\underline{\xi} + (z-z')\cdot\underline{\zeta}\right)}\phi\left(\frac{x-x'}{x}\right) a(x, z, x\underline{\xi}, \underline{\zeta}) u(x', z')\, d\underline{\xi}\, d\underline{\zeta}\, dx'\, dy'.$$

As in the homogeneous case, the space of semiclassical b-pseudodifferential operators on X satisfies the following properties:

i. Each $A \in \Psi_{\mathrm{b},h}(X)$ maps $\dot{C}^\infty(X) \to \dot{C}^\infty(X)$ and $C^{-\infty}(X) \to C^{-\infty}(X)$.
ii. There is a principal symbol map $\sigma_{\mathrm{b},h} : \Psi_{\mathrm{b},h}^m(X) \to S^m({}^{\mathrm{b}}T^*X)/hS^{m-1}({}^{\mathrm{b}}T^*X)$ so that the sequence

$$0 \to h\Psi_{\mathrm{b},h}^{m-1}(X) \to \Psi_{\mathrm{b},h}^m(X) \to S^m({}^{\mathrm{b}}T^*X)/hS^{m-1}({}^{\mathrm{b}}T^*X) \to 0$$

is exact. Moreover, this map is multiplicative.
iii. There is a (non-canonical) quantization map $\mathrm{Op}_{\mathrm{b},h} : S^m({}^{\mathrm{b}}T^*X) \to \Psi_{\mathrm{b},h}^m(X)$ so that if $a \in S^m({}^{\mathrm{b}}T^*X)$ then

$$\sigma(\mathrm{Op}_{\mathrm{b},h}(a)) = a$$

as an element of $S^m({}^{\mathrm{b}}T^*X)/hS^{m-1}({}^{\mathrm{b}}T^*X)$.

[10] We employ the semiclassical calculus only on mf, accounting for the shift in dimension.

iv. The algebra $\Psi^*_{\mathrm{b},h}(M)$ is closed under adjoints (with repsect to a fixed density on X) and

$$\sigma(A^*) = \overline{\sigma(A)}.$$

v. If $A \in \Psi^m_{\mathrm{b},h}(X)$ and $B \in \Psi^{m'}_{\mathrm{b},h}(X)$, then $[A,B] \in h\Psi^{m+m'-1}_{\mathrm{b},h}(X)$ and has principal symbol

$$\sigma\left(\frac{i}{h}\,[A,B]\right) = \{\sigma(A), \sigma(B)\}\,,$$

where the Poisson bracket is taken with respect to the symplectic structure on ${}^{\mathrm{b}}T^*X$.

vi. Each $A \in \Psi^0_{\mathrm{b},h}(X)$ extends to a bounded operator on L^2 and there exists $A' \in \Psi^{-\infty}_{\mathrm{b},h}(X)$ so that

$$\|Au\|_{L^2} \leq 2\sup|\sigma(A)|\,\|u\|_{L^2} + O(h^\infty)\,\|A'u\|_{L^2}$$

for each $u \in L^2$.

As in the homogeneous setting, we define a basic operator to be the quantization of a symbol a with $\partial_z a = 0$ at $\{x = 0, \underline{\xi} = 0, \underline{\zeta} = 0\}$. We also use the semiclassical analogue of Lemma 4.1, with proof essentially identical to the proof in the homogeneous setting.

Lemma 4.2 (cf. [MVW08, Lemma 8.6] and [GW18, Lemma 3.6])**.** *If $A \in \Psi^m_{\mathrm{b},h}(X)$ has compact support in a coordinate patch, there are $B \in \Psi^m_{\mathrm{b},h}(X)$ and $C \in \Psi^{m-1}_{\mathrm{b},h}(X)$ so that*

$$\frac{i}{h}\,[hD_x, A] = B + C(hD_x)$$

with $\sigma(B) = \partial_x\sigma(A)$ and $\sigma(C) = \partial_{\underline{\xi}}\sigma(A)$.

In particular, given $C = i[1/x, A]x$, we have $C \in \Psi^{m-1}_{\mathrm{b}}(M)$ and

$$\sigma_{m-1}(C) = H_{x^{-1}}\sigma(A) = \partial_{\underline{\xi}}\sigma(A).$$

If, in addition, A is a basic operator, then

$$\frac{i}{h}\left[\frac{h}{x}D_{z_j}, A\right] = B_j + C_j(hD_x) + \sum_k E_{jk}\frac{h}{x}D_{z_k} + \frac{h}{x}F_j$$

with $B_j \in \Psi^m_{\mathrm{b},h}(X)$, $C_j, E_{jk}, F_i \in \Psi^{m-1}_{\mathrm{b},h}(X)$ and

$$\partial_{z_j}\sigma(A) + \underline{\zeta}_k\partial_{\underline{\xi}}\sigma(A) = x\sigma(B_j) + \underline{\xi}\sigma(C_j) + \sum_k \underline{\zeta}_k\sigma(E_{jk}).$$

5. Function spaces and wavefront sets

5.1. b**-Sobolev spaces.** We start by defining the "standard" b-Sobolev spaces on M and X, referring the reader to other references for more details.

Here and throughout the paper we fix a "b-density", which is a non-vanishing density that, near the codimension 2 corner $\mathrm{mf} \cap \mathrm{cf}$ has the form

$$f(\rho, x, z)\left|\frac{d\rho}{\rho}\frac{dx}{x}\,dz\right|$$

where f is smooth and positive everywhere (down to the corner). We define $L^2_{\mathrm{b}}(M)$to be the space of functions on M that are square-integrable with respect to this (fixed) density. We use $H^m_{\mathrm{b}}(M)$ to denote the Sobolev space of order m relative to the function space $L^2_{\mathrm{b}}(M)$ and the algebras $\mathrm{Diff}^m_{\mathrm{b}}(M)$ and $\Psi^m_{\mathrm{b}}(M)$. In particular, for $m \geq 0$, if $A \in \Psi^m_{\mathrm{b}}(M)$ is any fixed elliptic operator, then $u \in H^m_{\mathrm{b}}(M)$ if and only if $u \in L^2_{\mathrm{b}}(M)$ and $Au \in L^2_{\mathrm{b}}(M)$.[11] For $m < 0$, the space $H^m_{\mathrm{b}}(M)$ is defined as the dual space of $H^{-m}_{\mathrm{b}}(M)$ with respect to the $L^2_{\mathrm{b}}(M)$ pairing. For $\ell = (\ell_1, \dots, \ell_r)$, we let

$$H^{m,\ell}_{\mathrm{b}}(M) = \rho_1^{\ell_1} \dots \rho_r^{\ell_r} H^m_{\mathrm{b}}(M)$$

denote the corresponding weighted spaces. A similar definition (with a similarly defined density) applies to characterize the space $H^m_{\mathrm{b}}(X)$.

The spaces $H^{\infty,\ell}_{\mathrm{b}}$ are the spaces of distributions conormal to the boundary (possibly with different boundary weights). These conormal spaces can be characterized without reference to microlocal methods by the iterated regularity condition: a function u lies in $H^{\infty,\ell}_{\mathrm{b}}(M)$ if and only if for all N and all $V_1, \dots, V_n \in \mathcal{V}_{\mathrm{b}}(M)$,

$$V_1 \dots V_N u \in \rho^\ell L^2_{\mathrm{b}}(M).$$

5.2. **Domains.** Although in the previous section we defined Sobolev spaces relative to a b-density, the proofs of the propagation statements in Section 6 below more naturally employ a density associated to the conic structure of the problem, i.e., the density associated to the (Lorentzian) metric $\rho^2 g$, which in local coordinates has the form

$$\frac{x^{n-1}\sqrt{k}}{\rho}\, d\rho\, dx\, dz.$$

The L^2 space defined by this density is the one used in the definition of the Friedrichs extensions of our various operators.

We now characterize the domains of the operators (defined above in equation (15)) L and P_σ in region III (and its restriction to $X = \mathrm{mf}$). In our propagation theorem in the "bulk" (Section 6), the domain of L provides the background with respect to which we measure regularity. In the semiclassical regime (Section 7) the propagation of semiclassical wavefront set is measured with respect to the domain of P_σ (or, rather, its semiclassical analogue).

The domain of Δ on $C(Z)$ informs our understanding of the domain of L on M as well as the domain of P_σ on $X = \mathrm{mf}$. We let $\mathcal{D}$ denote the Friedrichs domain of the Laplacian on $C(Z)$ and $\mathcal{D}'$ the domain of $\Delta^{-1/2}$ so that $\Delta : \mathcal{D} \to \mathcal{D}'$. In other words, $\mathcal{D}$ is the closure of $\dot{C}^\infty((0,\infty) \times Z)$ with repsect to the quadratic form $\|du\|^2 + \|u\|^2$. (Here both norms are the L^2 norm with respect to the conic metric.) We recall now a central result concerning this domain.

Lemma 5.1 ([MVW08, Lemma 5.2])**.** *If* $\dim Z > 1$*, then there is some* C *so that for all* $v \in \dot{C}^\infty((0,\infty) \times Z)$*,*

$$\left\|x^{-1} v\right\|^2 + \left\|x^{-1} D_z v\right\|^2 + \|D_x v\|^2 \le C \|v\|^2_{\mathcal{D}}.$$

When the dimension of Z is 1, i.e., when the link is a circle, then this Lemma is false. Modifications similar to those used by Melrose–Vasy–Wunsch [MVW08, Section 10] allow us to recover the propagation results of Sections 6 and 7.

The domain of L is induced by the domain $\mathcal{D}$ above and is denoted $\widetilde{\mathcal{D}}$. The analogue of $\mathcal{D}'$ we denote $\widetilde{\mathcal{D}}'$. Because L is essentially the D'Alembertian for the metric $\rho^2 g$, we measure

[11] If m is a positive integer, H^m_{b} can be characterized in terms of $\mathrm{Diff}^m_{\mathrm{b}}(M)$. A characterization for other values of m then follows by interpolation and duality.

size with respect to this Lorentzian metric. In particular, we may take as a representative of this norm

$$\|u\|_{\widetilde{\mathcal{D}}}^2 = \|du\|_{L^2(\rho^2 g)}^2 + \|u\|_{L^2(\rho^2 g)}^2 ,$$

where du now denotes the differential on M. In particular, the $\widetilde{\mathcal{D}}$ norm controls the L^2 norms of $\rho D_\rho u$, $D_x u$, and $x^{-1} D_z u$. As solutions of wave equations do not typically lie in L^2 in time, we also use the "weighted domains", denoted $\rho^{-\ell}\widetilde{\mathcal{D}}$, which consist of those u for which $v = \rho^\ell u \in \widetilde{\mathcal{D}}$. The $\rho^{-\ell}\widetilde{\mathcal{D}}$ norm of u is defined to be the $\widetilde{\mathcal{D}}$ norm of $\rho^\ell u$.

On M, one of the main uses for the domain $\widetilde{\mathcal{D}}$ is as the background with respect to which we measure regularity. To that end, for fixed $\ell \in \mathbb{R}$ and $m \geq 0$, we define the finite order conormal space $H^{m,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(M)$ to consist of those $u \in \rho^\ell \widetilde{\mathcal{D}}$ with $Au \in \rho^\ell \widetilde{\mathcal{D}}$ for some elliptic $A \in \Psi^m_{\mathrm{b}}$. In other words, $H^{m,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}$ consists of distributions conormal to cf of finite order m relative to $\rho^\ell\widetilde{\mathcal{D}}$. These spaces do not depend on the choice of A (as in the work of Vasy [Vas08, Remark 3.6]).

On the boundary $X = \mathrm{mf}$, we characterize the domain of P_σ near the conic singularity. As P_σ differs from $-\Delta$ by an element of $\mathrm{Diff}^2_{\mathrm{b}}$, Lemma 5.1 applies to P_σ as well, so in an abuse of notation we use $\mathcal{D}$ to denote the domain of P_σ as well.

The main propagation result of Section 7 is semiclassical, so we introduce a rescaled version of the domain norm, denoted $\mathcal{D}_h$ associated to the operator $P_h = h^2 P_\sigma$, where $h = |\sigma|^{-1}$. For u supported near ∂X, this norm is given by

$$\|u\|_{\mathcal{D}_h}^2 = \|h\, du\|^2 + \|u\|^2,$$

where the L^2 norms are taken with respect to the conic metric $dx^2 + x^2 k$ near the cone point on X.[12] As in Lemma 5.1, the $\mathcal{D}_h$ norm controls the L^2 norms of $h\, D_x u$ and $h\, x^{-1} D_z u$. The dual of $\mathcal{D}_h$ we again denote $\mathcal{D}_h'$.

It is worth pointing out that the characterization of domains stemming from Lemma 5.1 shows that if $\chi \in C^\infty_c$ localizes near $\mathrm{mf} \cap \mathrm{cf}$, then the map $u \mapsto \chi u$, viewed as a map from the domain to L^2 (with respect to the relevant metric) is compact. By duality, the corresponding map from L^2 to the dual of the domain is also compact. This observation is used in the Fredholm statement proved in Section 7. When $\dim Z = 1$, the characterization of the Friedrichs domain given by Melrose–Wunsch [MW04a, Proposition 3.1] also shows the compactness of these inclusions.

As we aim eventually to reduce problems on M to problems on the boundary hypersurface $X = \mathrm{mf}$, we record the following lemma relating the two domains. The proof of the lemma with $\widetilde{\mathcal{D}}$ replaced by a Sobolev space H^k is standard; the proof for $\widetilde{\mathcal{D}}$ proceeds identically.

Lemma 5.2 (cf. [BVW15, Lemma 2.3])**.** *If $u \in \rho^{-\ell}\widetilde{\mathcal{D}}$ and $\chi \in C^\infty(\mathbb{R})$, then the Mellin transform in ρ of $\chi(x)u$ is a holomorphic function for $\operatorname{Im}\sigma > \ell$ taking values in $L^\infty_{\operatorname{Im}\sigma} L^2_{\Re\sigma}(\mathbb{R}_{\Re\sigma}; \mathcal{D})$.*

5.3. **Variable order Sobolev spaces.** As in the prequels [BVW15, BVW18], we wish in Section 7 to propagate regularity from S_- to S_+ in $X = \mathrm{mf}$. In order to do this, the family $\widetilde{w}_\sigma$ must be more regular than a threshold regularity at S_- and less regular than the threshold at S_+. Because the two thresholds are equal, we employ variable-order Sobolev spaces.

We therefore must define a smooth regularity function $s : {}^{\mathrm{b}}S^*\mathrm{mf} \to \mathbb{R}$. The characteristic set of P_σ in ${}^{\mathrm{b}}S^*\mathrm{mf}$ has two components $\Sigma_\pm$; the integral curves of the Hamilton flow in $\Sigma_\pm$

[12] In fact, P_σ is a rescaled conjugate of the Laplacian on a hyperbolic cone, but we do not need to exploit this fact here.

tend to $S_\pm$ as the parameter tends to $+\infty$. The sets $\Lambda^\pm = N^* S_\pm \subset {}^{\mathrm{b}}S^*\mathrm{mf}$ are the radial sets at future/past infinity, respectively.

We fix a future regularity function $s_{\mathrm{ftr}} : {}^{\mathrm{b}}S^*\mathrm{mf} \to \mathbb{R}$ satisfying the following:

i. s_{ftr} is constant near $\Lambda^\pm$ and $s_{\mathrm{ftr}} \equiv 1$ in a neighborhood of the conic singularity $\partial\mathrm{mf}$;
ii. Away from $\partial\mathrm{mf}$, s_{ftr} is decreasing along the H_p-flow on Σ_+ and increasing on Σ_-;
iii. s_{ftr} is less than the threshold exponent on Λ^+ and greater than the threshold exponent on Λ^-.

By increasing s_{ftr} near Λ^- (or decreasing it near Λ^+), we can arrange that 1 lies between $s_{\mathrm{ftr}}|_{\Lambda^+}$ and $s_{\mathrm{ftr}}|_{\Lambda^-}$ that the second and third requirements are compatible with the first.

Note that s_{ftr} is implicitly a function of σ; the thresholds at $\Lambda^\pm$ are also σ-dependent. Indeed, as in the previous paper [BVW15, Section 5], the thresholds are given by

$$\frac{1}{2} + \operatorname{Im}\sigma.$$

We further define $s^*_{\mathrm{past}} = -s_{\mathrm{ftr}} + 1$. With these functions in hand, we can define (as in [BVW15, Appendix A]) the variable order Sobolev spaces $H^{s_{\mathrm{ftr}}}$ and $H^{s^*_{\mathrm{past}}}$ away from the conic singularity $\partial\mathrm{mf}$. Recall that the standard hyperbolic propagation estimates still hold, provided that the order is decreasing along the flow.

We now fix a partition of unity $\phi, 1-\phi \in C^\infty$ so that ϕ is supported near the conic singularities $\partial\mathrm{mf}$ where $s = 1$, and $1-\phi = 0$ in a neighborhood of $\partial\mathrm{mf}$. We now define the spaces

$$\mathcal{Y}^{s_{\mathrm{ftr}}-1} = \{u = (1-\phi)u_1 + \phi u_2 : u_1 \in H^{s_{\mathrm{ftr}}-1},\ u_2 \in L^2\},$$
$$\mathcal{Y}^{s^*_{\mathrm{past}}-1} = \{u = (1-\phi)u_1 + \phi u_2 : u_1 \in H^{s^*_{\mathrm{past}}-1},\ u_2 \in \mathcal{D}'\},$$

where we have abused notation slightly – the spaces $\mathcal{Y}^{s_{\mathrm{ftr}}-1}$ and $\mathcal{Y}^{s^*_{\mathrm{past}}-1}$ differ by how they look near the conic singularity. As $s_{\mathrm{ftr}} = 1$ near the cone points, $Y^{s_{\mathrm{ftr}}-1}$ should behave like L^2 there; $s^*_{\mathrm{past}} = 0$ near the cone point, so $Y^{s^*_{\mathrm{past}}-1}$ should stand in for H^{-1} there. We equip these two spaces with the norms

$$\|u\|^2_{\mathcal{Y}^{s_{\mathrm{ftr}}-1}} = \|(1-\phi)u\|^2_{H^{s_{\mathrm{ftr}}-1}} + \|\phi u\|^2_{L^2},$$
$$\|u\|^2_{\mathcal{Y}^{s^*_{\mathrm{past}}-1}} = \|(1-\phi)u\|^2_{H^{s^*_{\mathrm{past}}-1}} + \|\phi u\|^2_{\mathcal{D}'}.$$

To define the semiclassical versions of the norms, we replace the Sobolev part of the norm with a semiclassical Sobolev norm and replace the $\mathcal{D}'$ part of the norm with the $\mathcal{D}'_h$ norm.

We now define the $\mathcal{X}$ spaces, again relying on the localizer ϕ:

$$\mathcal{X}^{s_{\mathrm{ftr}}} = \{u = (1-\phi)u_1 + \phi u_2 : u_1 \in H^{s_{\mathrm{ftr}}},\ u_2 \in \mathcal{D},\ P_\sigma u \in \mathcal{Y}^{s_{\mathrm{ftr}}-1}\},$$
$$\mathcal{X}^{s^*_{\mathrm{past}}} = \{u = (1-\phi)u_1 + \phi u_2 : u_1 \in H^{s^*_{\mathrm{past}}},\ u_2 \in L^2,\ P_\sigma u \in \mathcal{Y}^{s^*_{\mathrm{past}}-1}\}.$$

Note first that we have abused notation in the same way as in the definitions of the $\mathcal{Y}$ spaces. Observe also that the condition on $P_\sigma u$ in the definition of the $\mathcal{X}$ spaces is independent of σ, as $P_{\sigma_1} - P_{\sigma_2} \in \mathrm{Diff}^1_{\mathrm{b}}$ (or $\mathrm{Diff}^1_{\mathrm{b},h}$ in the semiclassical setting). The norms on these spaces are given by

$$\|u\|^2_{\mathcal{X}^{s_{\mathrm{ftr}}}} = \|(1-\phi)u\|^2_{H^{s_{\mathrm{ftr}}}} + \|\phi u\|^2_{\mathcal{D}} + \|P_\sigma u\|^2_{\mathcal{Y}^{s_{\mathrm{ftr}}-1}},$$
$$\|u\|^2_{\mathcal{X}^{s^*_{\mathrm{past}}}} = \|(1-\phi)u\|^2_{H^{s^*_{\mathrm{past}}}} + \|\phi u\|^2_{L^2} + \|P_\sigma u\|^2_{\mathcal{Y}^{s^*_{\mathrm{past}}-1}},$$

with the semiclassical analogues obtained in the same way as for the $\mathcal{Y}$ spaces.

Observe that with our choices of s_{ftr} and s^*_{past}, the dual of $\mathcal{Y}^{s_{\mathrm{ftr}}-1}$ is the space of distributions of the form $\phi u_1 + (1-\phi)u_2$, where $u_1 \in H^{s^*_{\mathrm{past}}}$ and $u_2 \in L^2$. Similarly, the dual of $\mathcal{Y}^{s^*_{\mathrm{past}}-1}$ consists of those distributions $\phi u_1 + (1-\phi)u_2$ with $u_1 \in H^{s_{\mathrm{ftr}}}$ and $u_2 \in \mathcal{D}$.

5.4. **Wavefront sets.** In this subsection we describe our microlocal characterization of regularity, the wavefront set. We define the notion only for the problem on the bulk M. Though it would be natural to define similar (homogeneous and semiclassical) notions on $X = \mathrm{mf}$, the new propagation estimates are directly stated in terms of operators and so we avoid introducing definitions on X.

Definition 5.3. Let $u \in H^{s,r}_{\mathrm{b},\widetilde{\mathcal{D}}}$ for some $s, r \in \mathbb{R}$ and $s \geq 0$ (so that $u \in \widetilde{\mathcal{D}}$) and let $m \geq 0$. We say $q \in {}^{\mathrm{b}}T^*M \setminus 0$ is not in $\mathrm{WF}^{m,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(u)$ if there is some $A \in \Psi^{m,\ell}_{\mathrm{b}}(M)$ elliptic at q so that $Au \in \widetilde{\mathcal{D}}$.

For $m = \infty$, we say q is not in $\mathrm{WF}^{\infty,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(u)$ if there is some $A \in \Psi^0_{\mathrm{b}}$ so that $Au \in H^{\infty,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}$.

Because this notion of wavefront set is only defined for $u \in \bigcup_{s\geq 0, r\in\mathbb{R}} H^{s,r}_{\mathrm{b},\widetilde{\mathcal{D}}}$, we call these distributions $\widetilde{\mathcal{D}}$*-admissible*. Given an operator $A \in \Psi_{\mathrm{b}}(M)$, we also will use a notion of operator wavefront set, $\mathrm{WF}'_{\mathrm{b}}(A)$, where we have $\mathrm{WF}'_{\mathrm{b}}(A) \subset {}^{\mathrm{b}}\dot{S}^*M$, with properties as in [MVW08, Section 3].

That there are non-trivial elements in this class follows immediately by energy conservation (indeed, polynomially growing energy estimates would suffice).

Lemma 5.4. *If u is the forward solution of $Lu = f$, where $f \in C^\infty_c(M^\circ)$, then there is some $\ell \in \mathbb{R}$ so that $u \in \rho^{-\ell}\widetilde{\mathcal{D}}$.*

6. Propagation of singularities in the bulk

The main aim of this section is to show that if $u \in \rho^\ell \widetilde{\mathcal{D}}$ satisfies $Lu \in C^\infty_c$, $u \equiv 0$ for $t < 0$, then $u \in H^{s,\gamma}_{\mathrm{b},\widetilde{\mathcal{D}}}$ for some s and γ. As in previous work [BVW15, Section 4], the same argument can be augmented to show that in fact u possesses iterated regularity under the module generated by $\rho\partial_\rho$, $\rho\partial_v$, $v\partial_v$, and ∂_z.

Throughout this section we use Q_j to denote those first-order differential operator we encounter that are not in Ψ^1_{b}. In particular, we set $Q_0 = 1/x$, $Q_1 = D_x$, and $Q_j = \frac{1}{x}D_{z_j}$ for local coordinates $z_2, \dots, z_n$ on Z.

The results of the previous work [BVW15, Section 4] apply here away from $x = 0$; the propagation of singularities results of Melrose–Wunsch [MW04a] establish the result away from $\rho = 0$. We must thus only show the following two propositions:

Proposition 6.1 (cf. [MVW08, Proposition 8.10 and Theorem 8.11])**.** *If $u \in \rho^\ell\widetilde{\mathcal{D}}$ then, in a neighborhood of* $\mathrm{mf} \cap cf$*,*

$$\mathrm{WF}^{m,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(u) \subset \mathrm{WF}^{m-1,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(Lu) \cup \dot{\pi}({}^{\mathrm{e}}S^*M) \quad \text{and} \quad \mathrm{WF}^{m,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(u) \setminus \mathrm{WF}^{m,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(Lu) \subset \dot{\Sigma}.$$

In particular, if $Lu \in C^\infty_c(M)$, then $\mathrm{WF}^{s,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(u) \subset \dot{\Sigma}$ for all s.

Proposition 6.2 (cf. [MVW08, Theorem 9.7])**.** *Suppose u is an admissible solution of $Lu = f \in C^\infty_c(M)$, and suppose that $U \subset \dot{\Sigma}$ is a neighborhood of $q_0 \in \dot{\mathcal{H}}$. Then*

$$U \cap \{-\underline{\hat{\xi}} < 0\} \cap \mathrm{WF}^{s,\ell}(u) = \emptyset \implies q_0 \notin \mathrm{WF}^{s,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(u),$$

where we note that away from $x = 0$, $\mathrm{WF}^{s,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}} u = \mathrm{WF}^{s,\ell} u$.

Taken together, Propositions 6.1 and 6.2 establish the diffractive theorem (cf. [MVW08, Theorem 9.12]):

Theorem 6.3. *If $u \in \rho^\ell \widetilde{\mathcal{D}}$ and $Lu \in C^\infty_c$, then near $\mathrm{cf} \cap \mathrm{mf}$,*

$$\mathrm{WF}_{\mathrm{b},\widetilde{\mathcal{D}}}(u) \subset \dot{\Sigma}$$

is a union of maximally extended generalized broken bicharacteristics of L in $\dot{\Sigma}$.

Away from the cone points, *exactly* the same estimates as in the previous work [BVW15, Section 4] apply, so Theorem 6.3 yields the following corollary:

Corollary 6.4. *If $u \in \rho^\ell \widetilde{\mathcal{D}}$ satisfies $Lu \in C^\infty_c$ and $u \equiv 0$ for $t < 0$, then there are $s, \gamma \in \mathbb{R}$ with $s + \ell < 1/2$ so that $u \in H^{s,\gamma}_{\mathrm{b},\widetilde{\mathcal{D}}}$. Moreover, u possesses module regularity with respect to this space.*

The rest of this section is devoted to the proofs of Propositions 6.1 and 6.2.

6.1. **Elliptic regularity.** Throughout this subsection and the next, we work only near $\mathrm{mf} \cap \mathrm{cf}$ and assume all operators and distributions are localized in a fixed neighborhood near this set. In particular, *we always assume we are localized to a region with $x \le 1/4$.* As mentioned earlier, we abuse notation and use the symbol $\mathrm{Diff}_{\mathrm{e}}$ to denote differential operators that are edge-like at cf (i.e., in x) and otherwise b-like at mf (i.e., in ρ). We measure L^2 with respect to the density for the metric $\rho^2 g$; in local coordinates this has the form

$$\frac{x^{n-1}\sqrt{k}}{\rho}\, d\rho\, dx\, dz.$$

With respect to this density, we observe that L has the following form:

$$L = (\rho D_\rho + x D_x)^* (\rho D_\rho + x D_x) - D_x^* D_x - (\frac{1}{x}\nabla_z)^* (\frac{1}{x}\nabla_z) - \frac{n^2-1}{4}.$$

As in the work of Melrose–Vasy–Wunsch [MVW08], we set

$$x^{-k} \mathrm{Diff}^k_{\mathrm{e}} \Psi^m_{\mathrm{b}} \subset x^{-k} \Psi^{k+m}_{\mathrm{b}}$$

to be the span of the products QA with $Q \in x^{-k}\mathrm{Diff}^k_{\mathrm{e}}$ and $A \in \Psi^m_{\mathrm{b}}$. It is also generated by the products AQ and so the union

$$\bigcup_{k,m} x^{-k} \mathrm{Diff}^k_{\mathrm{e}} \Psi^m_{\mathrm{b}}$$

is a bigraded ring closed under adjoints with respect to any b-density.

Following the notation of [MVW08], we say a symbol $a \in C^\infty({}^{\mathrm{b}}T^*M)$ is *basic* if it is constant on the fibers above ${}^{\mathrm{b}}\dot{T}^*M$, i.e., in terms of local coordinates, $\partial_z a = 0$ at $\{x = 0, \underline{\xi} = 0, \underline{\zeta} = 0\}$. An operator with such a symbol is also called basic.

The main elliptic estimate follows from the following two lemmas:

Lemma 6.5 (cf. [MVW08, Lemma 8.8])**.** *Suppose that $K \subset U \subset {}^{\mathrm{b}}S^*M$ with K compact and U open, and suppose further that $\mathcal{A} = \{A_r \in \Psi^{s-1}_{\mathrm{b}} : r \in (0,1]\}$ is a basic family bounded in $\Psi^s_{\mathrm{b}\infty}$ with $\mathrm{WF}'_{\mathrm{b}}(\mathcal{A}) \subset K$. Then there are $G \in \Psi^{s-1/2}_{\mathrm{b}}$ and $\tilde{G} \in \Psi^{s+1/2}_{\mathrm{b}}$ with $\mathrm{WF}'_{\mathrm{b}}(G)$,*

$\mathrm{WF}'_{\mathrm{b}}(\tilde{G}) \subset U$, *and* $C_0 > 0$ *so that for all* $r \in (0,1]$ *and* $u \in \widetilde{\mathcal{D}}$ *with* $\mathrm{WF}^{s-1/2}_{\mathrm{b},\widetilde{\mathcal{D}}}(u) \cap U = \emptyset$, $\mathrm{WF}^{s+1/2}_{\mathrm{b},\widetilde{\mathcal{D}}}(Lu) \cap U = \emptyset$, *we have*

$$\left| \int \left(|d_{x,z} A_r u|^2 + \frac{n^2-1}{4} |A_r u|^2 - |(\rho\partial_\rho + x\partial_x)u|^2 \right) \frac{x^{n-1}\sqrt{k}}{\rho} \, d\rho \, dx \, dz \right|$$
$$\leq C_0 \left(\|u\|^2_{\widetilde{\mathcal{D}}} + \|Gu\|^2_{\widetilde{\mathcal{D}}} + \|Lu\|^2_{\widetilde{\mathcal{D}}'} + \left\| \tilde{G} L u \right\|^2_{\widetilde{\mathcal{D}}'} \right).$$

Lemma 6.6 (cf. [MVW08, Lemma 8.9])**.** *With the same hypotheses as in Lemma 6.5, there are* $G \in \Psi^{s-1/2}_{\mathrm{b}}$ *and* $\tilde{G} \in \Psi^s_{\mathrm{b}}$ *with* $\mathrm{WF}'_{\mathrm{b}}(G)$, $\mathrm{WF}'_{\mathrm{b}}(\tilde{G}) \subset U$ *and* $C_0 > 0$ *so that for all* $\epsilon > 0$, *and* $u \in \widetilde{\mathcal{D}}$ *with* $\mathrm{WF}^{s-1/2}_{\mathrm{b},\widetilde{\mathcal{D}}}(u) \cap U = \emptyset$ *and* $\mathrm{WF}^{s}_{\mathrm{b},\widetilde{\mathcal{D}}}(Lu) \cap U = \emptyset$, *we have*

$$\left| \int \left(|d_{x,z} A_r u|^2 + \frac{n^2-1}{4} |A_r u|^2 - |(\rho\partial_\rho + x\partial_x)u|^2 \right) \frac{x^{n-1}\sqrt{k}}{\rho} \, d\rho \, dx \, dz \right|$$
$$\leq \epsilon \left(\|d_{x,z} A_r u\|^2_{L^2} + \|\rho\partial_\rho A_r u\|^2_{L^2} \right) + C_0 \left(\|u\|^2_{\widetilde{\mathcal{D}}} + \|Gu\|^2_{\widetilde{\mathcal{D}}} + \epsilon^{-1} \|Lu\|^2_{\widetilde{\mathcal{D}}'} + \left\| \tilde{G} L u \right\|^2_{\widetilde{\mathcal{D}}'} \right).$$

Proof. By the hypothesis on u, we know $A_r u \in \widetilde{\mathcal{D}}$ for $r \in (0,1]$, so we have that

$$\langle L A_r u, A_r u \rangle = \|(\rho\partial_\rho + x\partial_x)u\|^2_{L^2} - \|\partial_x A_r u\|^2_{L^2} - \left\| \frac{1}{x} \nabla_k A_r u \right\|^2_{L^2} - \frac{n^2-1}{4} \|A_r u\|_{L^2}.$$

The proofs of these two lemmas are now identical to the ones given by Melrose–Vasy–Wunsch [MVW08] with ∂_t replaced by $\rho\partial_\rho + x\partial_x$. □

At this stage we record a corollary that will be useful in the next subsection:

Corollary 6.7. *Under the hypotheses of Lemma 6.6, we can estimate the domain norm of* $A_r u$ *by*

$$\|A_r u\|_{\widetilde{\mathcal{D}}} \leq C \left(\|u\|_{\widetilde{\mathcal{D}}} + \|Gu\|_{\widetilde{\mathcal{D}}} + \|Lu\|_{\widetilde{\mathcal{D}}'} + \left\| \tilde{G} L u \right\|_{\widetilde{\mathcal{D}}'} + \|(\rho\partial_\rho + x\partial_x) A_r u\|_{L^2} \right).$$

In particular, this corollary allows us to replace factors of Q_i with the b-differential operator $\rho\partial_\rho + x\partial_x$ at the cost of the terms on the right side of Lemma 6.6.

The proof of Proposition 6.1 further yields another useful corollary:

Corollary 6.8. *Under the hypotheses of Lemma 6.6, if we further suppose that* $\mathrm{WF}'_{\mathrm{b}}(\mathcal{A}) \cap \dot{\Sigma} = \emptyset$, *then*

$$\|A_r u\|_{\widetilde{\mathcal{D}}} \leq C \left(\|u\|_{\widetilde{\mathcal{D}}} + \|Gu\|_{\widetilde{\mathcal{D}}} + \|Lu\|_{\widetilde{\mathcal{D}}'} + \left\| \tilde{G} L u \right\|_{\widetilde{\mathcal{D}}'} \right).$$

Proof of Proposition 6.1. The proposition follows by the same argument as in Melrose–Vasy–Wunsch [MVW08, Proposition 8.10] with ∂_t replaced by $\rho\partial_\rho + x\partial_x$. Because the proof simplifies a bit in our setting, we sketch it here.

Suppose $q \in {}^{\mathrm{b}}S^*M \setminus \dot{\pi}({}^{\mathrm{e}}S^*M)$. We assume inductively that $q \notin \mathrm{WF}^{s-1/2,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(u)$ and need to show that $q \notin \mathrm{WF}^{s,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(u)$. Let $A \in \Psi^{s,\ell}_{\mathrm{b}}$ be basic and such that

i. $\mathrm{WF}'_{\mathrm{b}}(A) \cap \mathrm{WF}^{s-1/2,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(u) = \emptyset$,
ii. $\mathrm{WF}'_{\mathrm{b}}(A) \cap \mathrm{WF}^{s,\ell}_{\mathrm{b},\widetilde{\mathcal{D}}}(Lu) = \emptyset$, and

iii. $\mathrm{WF}'_{\mathrm{b}}(A)$ is a subset of a small neighborhood U of q on which $1 < C(\hat{\underline{\xi}}^2 + |\hat{\underline{\zeta}}|^2)$.

We now introduce $\Lambda_r \in \Psi_{\mathrm{b}}^{-2}$ for $r > 0$ with symbol $(1 + r(\underline{\tau}^2 + \underline{\xi}^2 + |\underline{\zeta}|^2))^{-1}$ so that Λ_r are uniformly bounded in Ψ_{b}^0 and $\Lambda_r \to \mathrm{Id}$ as $r \to 0$. We set $A_r = \Lambda_r A$ so that for $r > 0$ we have

$$\sigma(A_r) = \frac{a}{1 + r(\underline{\tau}^2 + \underline{\xi}^2 + |\underline{\zeta}|^2)}$$

and $A_r \rho^\ell$ and $\rho^{-\ell} u$ satisfy the hypotheses of the previous lemma.

If A is supported in $x < \delta$, then

$$\begin{aligned} \delta^{-2} \left\| x \partial_x A_r u \right\|_{L^2}^2 &\leq \left\| \partial_x A_r u \right\|_{L^2}^2, \\ \delta^{-2} \left\| \partial_{z_j} A_r u \right\|_{L^2}^2 &\leq \left\| \frac{1}{x} \partial_{z_j} A_r u \right\|_{L^2}^2. \end{aligned}$$

Near q, for $\delta > 0$ sufficiently small, we define and observe

$$\begin{aligned} I &= \frac{1-\epsilon}{2\delta^2} \left(\left\| x \partial_x A_r \right\|^2 + \sum_j \left\| \partial_{z_j} A_r u \right\|^2 \right) - (1+\epsilon) \| (\rho \partial_\rho + x \partial_x) A_r u \|^2 \\ &= \left\| B A_r u \right\|^2 + \langle F A_r u, A_r u \rangle, \end{aligned}$$

where $B \in \Psi_{\mathrm{b}}^1$, $F \in \Psi_{\mathrm{b}}^1$, and the symbol of B is given by

$$\sigma(B) = (((1-\epsilon)/2\delta^2)(\underline{\xi}^2 + |\underline{\zeta}|^2) - (1+\epsilon)(\underline{\tau} + \underline{\xi})^2)^{1/2}$$

(and is therefore elliptic on U for small enough δ and ϵ). Because F is order 1 and $A_r \in \widetilde{\mathcal{D}}$, the second term is uniformly bounded in r.

By the previous lemma, we know that the difference

$$\left\| \partial_x A_r u \right\|^2 + \left\| \frac{1}{x} \nabla_k A_r u \right\|^2 + \frac{n^2 - 1}{4} \| A_r u \|^2 - \| (\rho \partial_\rho + x \partial_x) u \|^2 - \epsilon \| d A_r u \|^2$$

is uniformly bounded in r. This is bounded below by

$$I + \frac{1-\epsilon}{2} \left(\left\| \partial_x A_r u \right\|^2 + \left\| \frac{1}{x} \nabla_k A_r u \right\|^2 \right),$$

and so we deduce that

$$\frac{1-\epsilon}{2} \left(\left\| \partial_x A_r u \right\|^2 + \left\| \frac{1}{x} \nabla_k A_r u \right\|^2 \right) + \left\| B A_r u \right\|^2$$

is uniformly bounded. Extracting weak limits shows that $dAu \in L^2$ and proves the proposition. $\square$

6.2. **Hyperbolic propagation.** The aim of this section is to prove Proposition 6.2. We establish the proposition by a positive commutator estimate; the positivity we seek arises from

$$\hat{\underline{\xi}} = \frac{1}{|\underline{\tau}|} \underline{\xi}.$$

Indeed, for $p_0 = \sigma(L)$, the Hamilton vector field of p_0 satisfies

$$\frac{1}{2}H_{p_0}(-\hat{\underline{\xi}}) = \frac{1}{|\underline{\tau}|}\left(\frac{\underline{\xi}^2}{x^2} + \frac{|\underline{\zeta}|^2}{x^2}\right).$$

We follow Melrose–Vasy–Wunsch [MVW08] (which itself closely mirrors Vasy [Vas08]) in this section. First, we define the two auxiliary functions

$$\omega = x^2 + |\rho - \rho_0|^2$$

and

$$\phi = -\hat{\underline{\xi}} + \frac{1}{\beta^2}\delta\omega.$$

As long as we assume $\rho < 1$ and that $\omega < \delta$, we can bound

$$\frac{1}{|\underline{\tau}|}H_{p_0}\omega = O\left(\sqrt{\omega}\left(\frac{\hat{\underline{\xi}}^2}{x^2} + \frac{|\hat{\underline{\zeta}}|^2}{x^2} + 1\right)^{1/2}\right).$$

In fact, the $\hat{\underline{\zeta}}$ term is unnecessary, but we include it as it does no harm.

We now fix $\chi_0 \in C^\infty(\mathbb{R})$ supported in $[0,\infty)$ with $\chi_0(s) = \exp(-1/s)$ for $s > 0$ so that $\chi_0'(s) = s^{-2}\chi_0(s)$. Take $\chi_1 \in C^\infty(\mathbb{R})$ supported in $[0,\infty)$ to be equal to 1 on $[1,\infty)$ and so that $\chi_1' \geq 0$ is compactly supported in $(0,1)$. Finally, for a given c_1, take $\chi_2 \in C_c^\infty(\mathbb{R})$ supported in $[-2c_1, 2c_1]$ and identically 1 on $[-c_1, c_1]$. We insist as well that all cut-offs and their derivatives have smooth square roots up to sign.

We now define a basic test symbol a given by

$$a = \chi_0\left(1 - \frac{\phi}{\delta}\right)\chi_1\left(\frac{-\hat{\underline{\xi}}}{\delta} + 1\right)\chi_2\left(\hat{\underline{\xi}}^2 + |\hat{\underline{\zeta}}|^2\right).$$

As in Melrose–Vasy–Wunsch [MVW08] and Gannot–Wunsch [GW18], we can arrange that a is well-localized:

Lemma 6.9. *Given any neighborhood U of $q_0 \in \dot{\mathcal{H}}$ in ${}^{\mathrm{b}}\dot{S}^*M$ and any $\beta > 0$, there are $\delta > 0$ and $c_1 > 0$ so that a is supported in U for all $0 < \delta < \delta_0$.*

We choose a basic operator $B \in \Psi_{\mathrm{b}}^{1/2}$ with

$$b = \sigma(B) = |\tau|^{1/2}\delta^{-1/2}(\chi_0\chi_0')^{1/2}\chi_1\chi_2,$$

so that b^2 occurs as a factor whenever derivatives of a land on χ_0. We further choose $C \in \Psi_{\mathrm{b}}^0$ with principal symbol

$$\sigma(C) = \frac{\sqrt{2}}{|\underline{\tau}|}|\underline{\tau} + \underline{\xi}|\psi,$$

where $\psi \in S^0({}^{\mathrm{b}}T^*M)$ is identically 1 on the support of the symbol of B.

With precisely the same argument in Melrose–Vasy–Wunsch [MVW08], we can now write the commutator of A^*A and L in a nice form:

Lemma 6.10 (cf. [MVW08, Lemma 9.6 and Theorem 9.7])**.** *There is a $\delta_0 > 0$ so that for all $0 < \delta < \delta_0$, the commutator of L and A^*A is given by*

$$i[A^*A, L] = R'L + B^*\left(C^*C + R_0 + \sum_j R_jQ_j + \sum_{j,k} Q_j^*R_{jk}Q_k\right)B + R'' + E' + E'',$$

where the terms enjoy the following properties:

- *all factors are microlocalized near* q_0*,*
- $R_0 \in \Psi_{\mathrm{b}}^0$*,* $R', R_j \in \Psi_{\mathrm{b}}^{-1}$*,* $R_{jk} \in \Psi_{\mathrm{b}}^{-2}$*,*
- $E', E'' \in x^{-2} \operatorname{Diff}_{\mathrm{e}}^2 \Psi_{\mathrm{b}}^{-1}$*,* $R'' \in x^{-2} \operatorname{Diff}_{\mathrm{e}}^2 \Psi_{\mathrm{b}}^{-2}$*,*
- *the symbols* r_0*,* r_j*, and* r_{jk} *of* R_0*,* R_j*,* R_{jk} *are supported in* $\{\omega \leq 9\delta^2\beta\}$*,*
- r_0*,* $\underline{\tau} r_j$*, and* $\underline{\tau}^2 r_{jk}$ *are bounded by both*
$$C_2\left(1 + \frac{1}{\beta^2\delta}\right) \quad \text{and} \quad 3C_2(\delta\beta + \beta^{-1}),$$
- $\mathrm{WF}_{\mathrm{b}}'(E') \subset \underline{\xi}^{-1}((0,\infty)) \cap U$*,*
- $\mathrm{WF}_{\mathrm{b}}'(E'') \cap \dot{\Sigma} = \emptyset$*.*

Proof. This representation follows from the computations in the proof of Theorem 9.7 as in [MVW08] relying heavily on the carefully chosen structure of the operators A and B. However, for convenience we remind the reader here of the origins of each term.

Our choice of the function χ_0 ensures that when derivatives fall on χ_0, we get the contributions between B^* and B. The positive term, C^*C, arises from a copy of $|\underline{\tau} + \underline{\xi}|^2/|\underline{\tau}|^2$ appearing from exchanging the leading order term in $a\partial_{\underline{\xi}} a$ with L, which also leads to the $R'L$ term.

When derivatives fall on χ_1, we get contributions to E' and when they fall on χ_2, we get contributions to E''. Commuting Q_j vector fields through B, also leads to contributions to E' and E''.

Because the above is largely a computation involving principal symbols, we also need the R'' term; this can be further used to absorb other lower order commutation terms. □

Lemma 6.11. *Given* $\epsilon > 0$*, there is a* $\delta_1 \in (0, \delta_0)$ *so that for all* $0 < \delta < \delta_1$ *and all* $v \in \widetilde{\mathcal{D}}$*,*
$$\begin{aligned} &|\langle R_0 Bv, Bv\rangle| + \sum_j |\langle R_j Q_j Bv, v\rangle| + \sum_{j,k} |\langle Q_j R_{jk} Q_k Bv, Bv\rangle| \\ &\qquad \leq \epsilon \left\|Bv\right\|_{L^2}^2 + C\left\|R'Bv\right\|_{L^2}^2 + C\left(\|u\|_{\widetilde{\mathcal{D}}} + \|Gu\|_{\widetilde{\mathcal{D}}} + \|Lu\|_{\widetilde{\mathcal{D}}'} + \left\|\tilde{G}Lu\right\|_{\widetilde{\mathcal{D}}'}\right)\end{aligned}$$
for some $R' \in \Psi_{\mathrm{b}}^{-1}$*.*

We typically do not require ϵ to be very small; it is used to absorb the $\|Bv\|^2$ term into the left side of an estimate later.

Proof. We rely on the symbol estimates of Lemma 6.10, together with the observation that if $A \in \Psi_{\mathrm{b}}^0$, then there is a $A' \in \Psi_{\mathrm{b}}^{-1}$ so that for all $u \in L^2$,
$$\|Au\| \leq \sup|\sigma_0(A)|\|v\| + C\|A'v\|.$$

We begin with the term involving R_0. By the observation above, there is an $R_0' \in \Psi_{\mathrm{b}}^{-1}$ so that
$$\begin{aligned}|\langle R_0 Bv, Bv\rangle| &\leq \|R_0 Bv\|_{L^2}\|Bv\|_{L^2} \\ &\leq 3C_2(\delta\beta + \beta^{-1})\left\|B_v\right\|_{L^2}^2 + \left\|R_0' Bv\right\|_{L^2}\|Bv\|_{L^2}.\end{aligned}$$

We now turn our attention to R_i and R_{ij}. Let $T_1 \in \Psi_{\mathrm{b}}^1$ be elliptic and $T_{-1} \in \Psi_{\mathrm{b}}^{-1}$ an elliptic parametrix for T_1 so that $T_1 T_{-1} = I + F$, where $F \in \Psi_{\mathrm{b}}^{-\infty}$. We first observe that we may find an $R_j' \in \Psi^{-1}$ so that

$$
\begin{aligned}
|\langle R_j Q_j Bv, Bv\rangle| &\leq \|R_j T_1 T_{-1} Q_j Bv\|_{L^2} \|Bv\|_{L^2} + \|R_j F Q_j Bv\|_{L^2} \|Bv\|_{L^2} \\
&\leq 3C_2(\delta\beta + \beta^{-1}) \|T_{-1} Q_j Bv\|_{L^2} \|Bv\|_{L^2} + \|R_j T_{-1} Q_j Bv\|_{L^2} \|Bv\|_{L^2} \\
&\leq 3C_2(\delta\beta + \beta^{-1}) \|T_{-1} Bv\|_{\widetilde{\mathcal{D}}} \|Bv\|_{L^2} + \left\|R_j' T_{-1} Q_j Bv\right\|_{L^2} \|Bv\|_{L^2} \\
&\leq 3C(\delta\beta + \beta^{-1}) \|(\rho\partial_\rho + x\partial_x) T_{-1} Bv\|_{L^2} \|Bv\|_{L^2} + \left\|R_j' T_{-1} Q_j Bv\right\|_{L^2} \|Bv\|_{L^2} \\
&\quad + C\left(\|u\|_{\widetilde{\mathcal{D}}} + \|Gu\|_{\widetilde{\mathcal{D}}} + \|Lu\|_{\widetilde{\mathcal{D}}'} + \left\|\tilde{G}Lu\right\|_{\widetilde{\mathcal{D}}'}\right).
\end{aligned}
$$

As $(\rho\partial_\rho + x\partial_x)T_{-1} \in \Psi_{\mathrm{b}}^0$, the first part of the first term is bounded by a (fixed) multiple of $\|Bv\|_{L^2}$.

A similar argument applies to the R_{ij} terms, finally yielding an estimate of the form

$$
\begin{aligned}
|\langle R_0 Bv, Bv\rangle| &+ \sum_i |\langle R_j Q_j Bv, Bv\rangle| + \sum_{j,k} |\langle Q_j^* R_{jk} Q_k Bv, Bv\rangle| \\
&\leq C(\delta\beta + \beta^{-1}) \|Bv\|_{L^2}^2 + \frac{\epsilon}{2} \|Bv\|_{L^2}^2 + C\|R' Bv\|_{L^2}^2 \\
&\qquad + C\left(\|u\|_{\widetilde{\mathcal{D}}} + \|Gu\|_{\widetilde{\mathcal{D}}} + \|Lu\|_{\widetilde{\mathcal{D}}'} + \left\|\tilde{G}Lu\right\|_{\widetilde{\mathcal{D}}'}\right),
\end{aligned}
$$

where $R' \in \Psi_{\mathrm{b}}^{-1}$. Choosing $\beta > 0$ large enough and then $\delta > 0$ small enough finishes the proof. □

We now finish the proof of Proposition 6.2.

Proof of Proposition 6.2. We first consider the case with $\ell = 0$. Suppose $s < \sup\{s' : q_0 \notin \mathrm{WF}_{\mathrm{b},\widetilde{\mathcal{D}}}^{s'} u\}$ so we may assume $\mathrm{WF}_{\mathrm{b},\widetilde{\mathcal{D}}}^{s} u \cap U = \emptyset$. Our aim is to show $q_0 \notin \mathrm{WF}_{\mathrm{b},\widetilde{\mathcal{D}}}^{s+1/2} u$.

As we measure regularity with respect to $\widetilde{\mathcal{D}}$, we know that if $B \in \Psi_{\mathrm{b}}^s$ localizes to U, then Bu, $Q_i Bu$, and $\rho\partial_\rho Bu$ all lie in L^2. By the hypothesis and Corollary 6.7, it suffices to control $\rho\partial_\rho u$ at q_0. In particular, it suffices to find a b-pseudodifferential operator B of order $s+3/2$ elliptic at q_0 for which $Bu \in L^2$, explaining the apparent shift in order (by 1) below.

Let A, B, and C be in the discussion preceding Lemma 6.10, and Λ_r be a quantization of

$$|\tau|^{s+1}(1 + r|\tau|^2)^{-(s+1)/2}, \quad r \in [0,1],$$

and set $A_r = A\Lambda_r \in \Psi_{\mathrm{b}}^0$ for $r > 0$, so that A_r is uniformly bounded in $\Psi_{\mathrm{b}\infty}^{s+1}$. We may further arrange that $[L, \Lambda_r] = 0$.

By the commutator calculation in Lemma 6.10, we may write

$$
\begin{aligned}
i\langle [A_r^* A_r, L]u, u\rangle = &\|CB\Lambda_r u\|^2 + \langle R' L\Lambda_r u, \Lambda_r u\rangle + \langle R_0 B\Lambda_r u, \Lambda_r u\rangle \\
&+ \sum_j \langle R_j Q_j B\Lambda_r u, B\Lambda_r u\rangle + \sum_{j,k} \langle R_{jk} Q_j B\Lambda_r u, Q_k B\Lambda_r u\rangle \\
&+ \langle R''\Lambda_r u, \Lambda_r u\rangle + \langle (E' + E'')\Lambda_r u, \Lambda_r u\rangle.
\end{aligned} \tag{16}
$$

As $u \in \widetilde{\mathcal{D}}$, the following pairings are well-defined:

$$
\begin{aligned}
\langle [A_r^* A_r, L]u, u\rangle &= \langle A_r^* A L u, u\rangle - \langle L A_r^* A_r u, u\rangle \\
&= \langle A_r L u, A_r u\rangle - \langle A_r u, A_r L u\rangle.
\end{aligned}
$$

Because Lu is residual, these terms are uniformly bounded in r and so we may estimate $\|CB\Lambda_r u\|^2$ by the absolute values of all other terms in equation (16). The second term is uniformly bounded because Lu is residual, while the next three terms are estimated by Lemma 6.11. The R'' term is bounded by the regularity hypothesis of u on U, while the E'' term is bounded by elliptic regularity. Finally, the E' term is bounded by the hypothesis of the theorem. We therefore can find a constant $\mathcal{C}$ independent of r so that

$$\|CB\Lambda_r u\|^2 \leq \mathcal{C} + \epsilon\|B\Lambda_r u\|^2 + \mathcal{C}\left(\|R'Bv\|^2 + \|u\|^2_{\widetilde{\mathcal{D}}} + \|Gu\|^2_{\widetilde{\mathcal{D}}} + \|Lu\|^2_{\widetilde{\mathcal{D}}'} + \left\|\tilde{G}Lu\right\|^2_{\widetilde{\mathcal{D}}'}\right),$$

where $G \in \Psi_{\mathrm{b}}^{s+1/2}$, $\tilde{G} \in \Psi_{\mathrm{b}}^{s+1}$ are supported in U. Another application of Corollary 6.7 shows that $\|CB\Lambda_r u\|$ (and the rest of the right hand side) controls $\|B\Lambda_r u\|$. The other terms on the right are uniformly bounded by the assumed regularity of u, so we can extract a subsequence and conclude that $B\Lambda_0 u \in L^2$, so that $q_0 \notin \mathrm{WF}^{s+1/2}_{\mathrm{b},\widetilde{\mathcal{D}}}(u)$.

A similar argument to the one sketched by Melrose–Vasy–Wunsch [MVW08] (which is based on a similar argument of Hörmander [H0̈9, 24.5.1] shows that in fact q_0 has a neighborhood U' for which $\mathrm{WF}_{\mathrm{b},\widetilde{\mathcal{D}}}(u) \cap U' = \emptyset$ for all s and therefore $q_0 \notin \mathrm{WF}^{\infty}_{\mathrm{b},\widetilde{\mathcal{D}}}(u)$.

To finish the proof, we now suppose instead that $\ell \neq 0$, i.e., that $u \in \rho^{-\ell}\widetilde{\mathcal{D}}$. Because $Lu \in \dot{C}^\infty$, $\tilde{L}v \in \dot{C}^\infty$ as well, where $v = \rho^\ell u \in \widetilde{\mathcal{D}}$ and $\tilde{L} = \rho^\ell L \rho^{-\ell}$. Because L and $\tilde{L}$ differ only by an element of $\mathrm{Diff}^1_{\mathrm{b}}$, the same proof applies to v. □

7. Propagation of singularities in the boundary

The aim of this section is to establish a natural setting in which the P_σ operator is Fredholm for each σ, allowing us to rigorously evaluate contour integrals on compact intervals in σ. This will be stated formally as Proposition 7.3 below. The main missing ingredient is the semiclassical propagation estimate for P_σ near the cone point. As in the previous section, throughout this section we use Q_i to denote the first-order-differential operators not lying in the semiclassical b-calculus. In particular, we let $Q_0 = 1/x$, $Q_1 = hD_x$, and the remaining Q_i denote $\frac{h}{x}D_{z_i}$ for local coordinates $z_2, \dots, z_n$ on Z.

We record the form of P_σ and its semiclassical rescaling by $h = |\sigma|^{-1}$, $\lambda = \sigma/|\sigma|$ (which, in an abuse of notation, we call P_h):

$$\begin{aligned} P_\sigma &= (xD_x + \sigma)^*(xD_x + \sigma) - D_x^* D_x - (\frac{1}{x}\nabla_z)^*(\frac{1}{x}\nabla_z) \\ &\quad - \frac{n^2-1}{4} - 2i(\operatorname{Im}\sigma)(xD_x + \sigma), \\ P_h = h^2 P_\sigma &= (hxD_x + \lambda)^*(hxD_x + \lambda) - (hD_x)^*(hD_x) - (\frac{h}{x}\nabla_z)^*(\frac{h}{x}\nabla_z) \\ &\quad - h^2\frac{n^2-1}{4} - 2i(\operatorname{Im}\lambda)(hD_x + \lambda). \end{aligned}$$

As we are only ever concerned with $\operatorname{Im}\sigma \in [a,b]$, we observe that $\lambda = \pm 1 + O(h)$.

As in the bulk, the propagation arguments from previous work [BVW15, Section 5] establish the required estimates away from the singular points. The aim for this section is therefore to establish the following two propositions:

Proposition 7.1 (cf. [GW18, Proposition 5.1]). *Suppose $A, G \in \Psi^0_{\mathrm{b},h}$ with A basic satisfy $\mathrm{WF}'_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b}}(G)$ and $\mathrm{WF}'_{\mathrm{b},h}(A) \cap \dot{\Sigma} = \emptyset$. Then there is a constant C so that*

$$\|Au\|_{\mathcal{D}_h} \leq C\,\|GP_h u\|_{\mathcal{D}'_h} + Ch\,\|Gu\|_{\mathcal{D}_h} + O(h^\infty)\,\|u\|_{\mathcal{D}_h}$$

for all $u \in \mathcal{D}_h$.

Proposition 7.2 (cf. [GW18, Proposition 5.8])**.** *If* $G \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ *is elliptic at* $\{(0, z, 0, 0) : z \in Z\}$ *then there are* $\mathcal{Q}, \mathcal{Q}_1 \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ *with* $\mathcal{Q}$ *elliptic at* $\{(0, z, 0, 0) : z \in Z\}$ *and*

$$\mathrm{WF}'_{\mathrm{b},h}\, \mathcal{Q} \subset \mathrm{ell}_{\mathrm{b}}(G),$$
$$\mathrm{WF}'_{\mathrm{b},h}\, \mathcal{Q}_1 \subset \mathrm{ell}_{\mathrm{b}}(G) \cap \{-\underline{\xi} > 0\},$$

so that for all $u \in \mathcal{D}_h$,

$$\|\mathcal{Q}u\|_{\mathcal{D}_h} \leq \frac{C}{h} \|GPu\|_{\mathcal{D}'_h} + C \|\mathcal{Q}_1 u\|_{\mathcal{D}_h} + Ch \|Gu\|_{\mathcal{D}_h} + O(h^\infty) \|u\|_{\mathcal{D}_h}.$$

One advantage of our choice of order function s_{ftr} is that we can measure regularity at the conic singularity with respect to the domain $\mathcal{D}_h$ and rely on previously known results (as in [BVW15, Appendix A]) to handle the variable order regularity away from the cone point.

Having proved Propositions 7.1 and 7.2, we can put them together with standard propagation of singularities estimates and radial points estimates *exactly* as in the prequel [BVW15, Section 5], yielding an estimate of the form

$$\|u\|_{\mathcal{X}_h^{s_{\mathrm{ftr}}}} \leq \frac{C}{h} \|Pu\|_{\mathcal{Y}_h^{s_{\mathrm{ftr}}-1}} + Ch \|u\|_{\mathcal{X}_h^{s_{\mathrm{ftr}}}} + O(h^\infty) u,$$

with an analogous estimate for P^* (with s_{ftr} replaced by s^*_{past}). Here the $\mathcal{Q}_1$ term (which measures semiclassical singularities propagating to the cone point) can be absorbed into the final $O(h^\infty)$ term by using that u is trivial near S_-. In particular, for small enough h, P is invertible. Together with the compactness of the inclusions $\mathcal{D} \to L^2$ and $L^2 \to \mathcal{D}'$, we obtain the following proposition proved analogously to [Vas13, Section 2.6]. In [Vas13], the author uses complex absorbing potential, while here as in [BVW15, BVW18] the variable coefficient Sobolev spaces give the necessary Fredholm structure. It is key that with respect to σ we construct a holomorphic family of operators, which is clear when since have P_σ as written in (15) is an operator valued polynomial in σ.

Proposition 7.3. *The family* P_σ *has the following mapping properties:*

(1) $P_\sigma : \mathcal{X}^{s_{\mathrm{ftr}}} \to \mathcal{Y}^{s_{\mathrm{ftr}}-1}$ *and* $P^*_\sigma : \mathcal{X}^{s^*_{\mathrm{past}}} \to \mathcal{Y}^{s^*_{\mathrm{past}}-1}$ *are Fredholm.*

(2) The operators P_σ *form a holomorphic Fredholm family on these spaces in*

$$\mathbb{C}_{s_+,s_-} = \{\sigma \in \mathbb{C} : s_+ < \frac{1}{2} + \operatorname{Im}\sigma < s_-\},$$

with $s_{\mathrm{ftr}}|_{\Lambda^\pm} = s_\pm$. *The formal adjoint* P^*_σ *is antiholomorphic in the same region.*

(3) P_σ^{-1} *has only finitely many poles in each strip* $a < \operatorname{Im}\sigma < b$.

(4) For all a *and* b, *there is a constant* C *so that*

$$\left\|P_\sigma^{-1}\right\|_{\mathcal{Y}^{s_{\mathrm{ftr}}-1}_{|\sigma|^{-1}} \to \mathcal{X}^{s_{\mathrm{ftr}}}_{|\sigma|^{-1}}} \leq C\langle \Re\sigma\rangle^{-1}$$

on $a < \operatorname{Im}\sigma < b$, $|\Re\sigma| > C$, *with a similar estimate holding for* $(P^*_\sigma)^{-1}$.

7.1. **Elliptic regularity.** As in Section 6.1, we assume all operators and distributions are supported in $x < 1/4$.

Lemma 7.4. *Suppose $A, G \in \Psi^0_{\mathrm{b},h}$ with A basic satisfy $\mathrm{WF}'_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b}}(G)$. There is a constant C so that*

$$\int \left(h^2|dAu|^2 - |(hxD_x + \lambda)Au|^2\right) x^{n-1}\sqrt{k}\, dx\, dz$$
$$\leq \epsilon \left\|Au\right\|^2_{\mathcal{D}_h} + \frac{C}{\epsilon}\left\|GPu\right\|^2_{\mathcal{D}'_h} + Ch\left\|Gu\right\|^2_{\mathcal{D}_h} + O(h^\infty)\left\|u\right\|^2_{\mathcal{D}_h}$$

for all $u \in \mathcal{D}_h$.

Proof. Integration by parts shows that if $v \in \mathcal{D}_h$, then

$$\int \left(h^2|dv|^2 - |(hxD_x + \lambda)v|^2 + 2i(\operatorname{Im}\lambda)((xD_x + \lambda)v)\overline{v}\right) = \langle P_h v, v\rangle,$$

where the pairing on the right side is the pairing of $\mathcal{D}_h$ with $\mathcal{D}'_h$.

We apply this to $v = Au \in \mathcal{D}_h$ and then first estimate

$$\langle APu, Au\rangle + \langle [P, A]u, u\rangle - 2i(\operatorname{Im}\lambda)\langle (hxD_x + \lambda)Au, Au\rangle.$$

The first term is estimated by Cauchy–Schwarz:

$$|\langle APu, Au\rangle| \leq \frac{1}{4\epsilon}\left\|APu\right\|^2_{\mathcal{D}'_h} + \epsilon\left\|Au\right\|^2_{\mathcal{D}}.$$

Microlocal elliptic regularity lets us estimate APu in terms of GPu. Because $\operatorname{Im}\lambda = O(h)$, the final term is bounded by

$$Ch\left(\left\|hxD_x Au\right\|^2 + \left\|Au\right\|^2\right).$$

The extra factor of h means that both of these terms can be absorbed into the $h\left\|Gu\right\|^2_{\mathcal{D}_h}$ term.

We now turn to the term involving $[P, A]$. Recall from Melrose–Vasy–Wunsch [MVW08, Lemma 8.6] that for a basic operator $A \in \Psi^0_{\mathrm{b},h}$, we may write

$$[P, A] = h\sum_{j,k} Q_j^* B_{jk} Q_k + h\sum_j B_j Q_j + B,$$

where Q_i refer to $1/x$, D_x, and $\frac{1}{x}D_{z_j}$, $B_{jk} \in \Psi^{-1}_{\mathrm{b},h}$, $B_j \in \Psi^0_{\mathrm{b},h}$, and $B \in \Psi^1_{\mathrm{b},h}$. We may then estimate each of the terms by $h\left\|Gu\right\|^2_{\mathcal{D}_h}$. □

We record a useful corollary for the hyperbolic section:

Corollary 7.5. *If A and G are as above, then there are constants C_0 (independent of A) and C so that*

$$\left\|Au\right\|_{\mathcal{D}_h} \leq C_0\left\|Au\right\|_{L^2} + C\left(\left\|GPu\right\|_{\mathcal{D}'_h} + h\left\|Gu\right\|_{\mathcal{D}_h}\right) + O(h^\infty)\left\|u\right\|_{\mathcal{D}_h}.$$

Proof. As $x < 1/4$, we can bound $-|(hxD_x + \lambda)Au|^2$ below by

$$-\frac{1}{4}|hD_x Au|^2 - |\lambda|^2|A_u|^2.$$

The first of these terms can be absorbed into the first term on the left in Lemma 7.4, while the second is moved to the right side. □

Exactly as in Gannot–Wunsch [GW18, Lemma 5.4], we can further improve this estimate by an iterative argument:

Lemma 7.6. *If A and G are as above, then*

$$\|Au\|_{\mathcal{D}_h} \lesssim \|GP_h u\|_{\mathcal{D}'_h} + \|Gu\|_{L^2} + O(h^\infty)\|u\|_{\mathcal{D}_h}.$$

Proposition 7.1 follows immediately from the following lemma:

Lemma 7.7. *Suppose $A, G \in \Psi^0_{\mathrm{b},h}$ with A basic satisfy $\mathrm{WF}'_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b}}(G)$. If A is supported in $\{x < \delta/\sqrt{2}\}$ and $\{(\underline{\xi} + z)^2 < \frac{1}{2}\delta^{-2}(\underline{\xi}^2 + |\underline{\zeta}|^2)\}$, then*

$$\|Au\|_{\mathcal{D}_h} \le C\|GPu\|_{\mathcal{D}'_h} + Ch\|Gu\|_{\mathcal{D}_h} + O(h^\infty)\|u\|_{\mathcal{D}_h}.$$

Proof. Since A is supported in $\{x < \delta/\sqrt{2}\}$, we know

$$\int \left(\delta^{-2}|hxdAu|^2 - |(hxD_x + \lambda)Au|^2\right) \le \int \left(|hD_x Au|^2 - |(hxD_x + \lambda)Au|^2\right).$$

Our other hypothesis on the support of A shows that we can find B with $\mathrm{WF}'_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b}}(B)$ so that given the operator

$$\tilde{Z} = \delta^{-2}\left((hxD_x)^*(hxD_x) + (\frac{h}{x}\nabla_z)^*(\frac{h}{x}\nabla_z) - (hxD_x + z)^*(hxD_x + z)\right) - (B^*B + hF)$$

we have

$$\mathrm{WF}_{\mathrm{b},h}(\tilde{Z}) \cap \mathrm{WF}'_{\mathrm{b},h}(A) = \emptyset.$$

Integrating by parts and using Lemma 7.4 shows that

$$\|BAu\|^2_{L^2} + \int \frac{1}{2}h^2|dAu|^2 \le \epsilon\|Au\|^2_{\mathcal{D}_h} + \frac{C}{\epsilon}\|GPu\|^2_{\mathcal{D}'_h} + Ch\|Gu\|^2_{\mathcal{D}_h} + Ch\|FAu\|_{L^2}\|Au\|_{L^2} + O(h^\infty)\|u\|^2_{\mathcal{D}_h}.$$

As B is elliptic on $\mathrm{WF}'_{\mathrm{b},h}(A)$, the the left side controls $\|Au\|_{\mathcal{D}_h}$, while the right side is controlled by

$$\epsilon\|Au\|^2_{\mathcal{D}_h} + C\|GPu\|^2_{\mathcal{D}'_h} + Ch\|Gu\|^2_{\mathcal{D}_h} + O(h^\infty)\|u\|^2_{\mathcal{D}_h}.$$

Absorbing the first term into the left side then finishes the proof. □

7.2. **Hyperbolic propagation.** In this subsection we prove Proposition 7.2. As in Section 6.2, we introduce an operator A with symbol given by

$$a = \chi_0(2 - \phi/\delta)\chi_1(2 - \underline{\xi}/\delta)\chi_2(\underline{\xi}^2 + |\underline{\zeta}|^2),$$

where χ_i are the same functions as in that section and $\phi = -\underline{\xi} + \frac{1}{\beta^2\delta}x^2$. Recall that χ_2 is supported in $[-2c_1, 2c_1]$ and is identically one on $[-c_1, c_1]$, so we think of a as being determined by the three parameters c_1, β, and δ.

We also choose a basic operator $B \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ with symbol

$$b = \frac{2}{\sqrt{\delta}}(\chi_0\chi_0')^{1/2}\chi_1\chi_2,$$

so that factors of B arise when the derivative lands on χ_0 in A.

As in that section (and Melrose–Vasy–Wunsch [MVW08] or Gannot–Wunsch [GW18, Lemma 5.9]), the symbol a is well-localized:

Lemma 7.8. *Given any neighborhood U of $\{(0, z, 0, 0) : z \in Z\}$ in ${}^{\mathrm{b}}T^*\mathrm{mf}$ and any $\beta > 0$, there are $\delta_0 > 0$ and $c_1 > 0$ so that a is supported in U for all $0 < \delta < \delta_0$.*

We compute now the commutator of P with A^*A in much the same way as in Melrose–Vasy–Wunsch [MVW08]:

Lemma 7.9 (cf. [MVW08, Lemma 9.6 and Theorem 9.7]). *Let $Q_0 = 1/x$, $Q_1 = D_x$ and Q_i denote the remaining $\frac{1}{x}D_z$. There is a $\delta_0 > 0$ so that for all $0 < \delta < \delta_0$, the commutator of P and A^*A is given by*

$$\frac{i}{h}\left[P, A^*A\right] = -B_0 P + B^*\left(C^*C + R_0 + \sum_j R_j Q_j + \sum_{j,k} Q_j^* R_{jk} Q_k\right) B + E + E'' + hR',$$

where the terms enjoy the following properties:

- $C = (hxD_x + z)$,
- $\sigma(B_0) = 2\partial_{\underline{\xi}}(a^2)$,
- $R_0, R_j, R_{jk} \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ *satisfy*
$$|\sigma(R_*)| \leq C_1(\delta\beta + \beta^{-1}),$$
- $E', E'', R' \in x^{-2}\,\mathrm{Diff}^2_{\mathrm{b},h}\,\Psi^{\mathrm{comp}}_{\mathrm{b},h}$ *satisfy*
$$\mathrm{WF}'_{\mathrm{b},h}(E') \subset \{-\underline{\xi} > 0\}, \quad \mathrm{WF}'_{\mathrm{b},h}(E'') \cap \Sigma = \emptyset.$$

Proof. We use Lemma 4.2 to commute A^*A through P, using that A is basic. The main term arising from the commutator then reproduces the main terms in P; indeed, it is of the form

$$B^*\left((hD_x)^*(hD_x) + \frac{h^2}{x^2}\Delta_z\right)B.$$

We use the form of the operator to exchange this term for B_0P and B^*C^*CB. The other terms in the expression arise in the same way as in Melrose–Vasy–Wunsch [MVW08] (explained above in the proof of Lemma 6.10. □

Lemma 7.10. *For any $\epsilon > 0$, there are $\beta > 0$ and $\delta_1 \in (0, \delta_0)$ so that*

$$|\langle R_0 Bu, Bu\rangle| + \sum_j |\langle R_j Q_j Bu, Bu\rangle| + \sum_{j,k} |\langle Q_j^* R_{jk} Q_k Bu, Bu\rangle| \leq \epsilon \left\|Bu\right\|^2_{\mathcal{D}_h} + O(h^\infty)\left\|u\right\|^2_{\mathcal{D}_h}.$$

As with Section 6.2, we typically do not need ϵ to be especially small; our aim is to absorb that term into the left side in the propagation argument.

Proof. As in the proof of Lemma 6.11, we rely on the symbol estimates in Lemma 7.9. Indeed, we bound

$$\begin{aligned}\left\|R_* v\right\|_{L^2} &\leq 2\sup|\sigma_{\mathrm{b},h}(R_*)|\left\|v\right\|_{L^2} + O(h^\infty)\left\|v\right\|_{L^2} \\ &\leq 2C_1(\delta\beta + \beta^{-1})\left\|v\right\|_{L^2} + O(h^\infty)\left\|v\right\|_{L^2}.\end{aligned}$$

We now fix $\beta > 0$ sufficiently large and then take $\delta_1 \in (0, \delta_0)$ sufficiently small to make $2C_1(\delta\beta + \beta^{-1}) < \epsilon/3$.

We now consider the individual terms. For the R_0 term, we apply the above inequality with $v = Bu$ and then use Cauchy–Schwarz. The R_j and R_{jk} terms are nearly identical:

$$\begin{aligned}\left|\langle Q_j^* R_{jk} Q_k Bu, Bu\rangle\right| &= |\langle R_{jk} Q_k Bu, Q_j Bu\rangle| \\ &\leq 2C_1(\delta\beta + \beta^{-1})\left\|Bu\right\|^2_{\mathcal{D}_h} \leq \epsilon\left\|Bu\right\|^2_{\mathcal{D}_h}.\end{aligned}$$

□

We now finish the proof of Proposition 7.2.

Proof. Given $u \in \mathcal{D}_h$, we apply Lemma 7.9 to write

$$\begin{aligned}-\frac{2}{h}\operatorname{Im}\langle APu, Au\rangle &= \frac{i}{h}\langle [A^*A, P]u, u\rangle \\ &= \|CBu\|_{L^2}^2 + \langle R_0 Bu, Bu\rangle + \sum_j \langle R_j Q_j Bu, Bu\rangle \\ &\quad + \sum_{j,k}\langle R_{jk} Q_k Bu, Q_j Bu\rangle + \langle E'u, u\rangle + \langle E''u, u\rangle + h\langle R'u, u\rangle - \langle B_0 Pu, u\rangle.\end{aligned}$$

We note that A, B, and CB preserve $\mathcal{D}_h$, while B_0 preserves $\mathcal{D}_h'$.

By Corollary 7.5 and the ellipticity of C on $\mathrm{WF}'_{\mathrm{b},h}(B)$,

$$c_0 \|Bu\|_{\mathcal{D}_h}^2 \leq \|CBu\|_{L^2}^2 + C\|GPu\|_{\mathcal{D}_h'}^2 + Ch\|Gu\|_{\mathcal{D}_h}^2 + O(h^\infty)\|u\|_{\mathcal{D}_h}^2,$$

where $c > 0$ is independent of β and δ and G is elliptic on $\mathrm{WF}'_{\mathrm{b},h}(B)$.

We now suppose $G \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ is elliptic on $\mathrm{WF}'_{\mathrm{b},h}(B)$ and $\mathcal{Q}_1 \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ is elliptic on $\mathrm{WF}'_{\mathrm{b},h}(E')$ with $\mathrm{WF}'_{\mathrm{b},h}(\mathcal{Q}_1) \subset \mathrm{ell}_{\mathrm{b}}(G) \cap \{-\underline{\xi} > 0\}$. Applying Lemma 7.10 yields an estimate of the form

$$\begin{aligned}\frac{c_0}{2}\|Bu\|_{\mathcal{D}_h}^2 \leq \frac{2}{h}|\langle APu, Au\rangle| + C\|GPu\|_{\mathcal{D}_h'}^2 + Ch\|Gu\|_{\mathcal{D}_h}^2 \\ + |\langle (E' + E'')u, u\rangle| + h|\langle R'u, u\rangle| + |\langle B_0 Pu, u\rangle| + O(h^\infty)\|u\|_{\mathcal{D}_h}^2.\end{aligned}$$

We can estimate the E' term by $\mathcal{Q}_1$ using microlocal elliptic regularity and the E'' term by Proposition 7.1. We can therefore estimate the second line by

$$\frac{C}{h}\|GPu\|_{\mathcal{D}_h'}^2 + Ch\|Gu\|_{\mathcal{D}_h}^2 + C\|\mathcal{Q}_1 u\|_{\mathcal{D}_h}^2 + O(h^\infty)\|u\|_{\mathcal{D}_h}^2.$$

Because $\mathrm{WF}'_{\mathrm{b},h}(A) \subset \mathrm{ell}_{\mathrm{b}}(G)$, we can further estimate

$$\frac{2}{h}|\langle APu, Au\rangle| \leq \frac{C}{h^2\epsilon}\|GPu\|_{\mathcal{D}_h}^2 + C\epsilon\|Au\|_{\mathcal{D}_h}^2 + O(h^{|infty})\|u\|_{\mathcal{D}_h}^2.$$

By construction, $\chi_0'(s) = s^2\chi_0'(s)$ for $s > 0$ and so

$$a = (2 - \phi/\delta)(\chi_0\chi_0')^{1/2}\chi_1\chi_2 = \frac{1}{2}\delta^{1/2}(2 - \phi/\delta)b.$$

We may therefore write $A = FB + hF'$ for some $F, F' \in \Psi^{\mathrm{comp}}_{\mathrm{b},h}$ in order to estimate Au by Bu. Putting the above together yields the estimate

$$\|Bu\|_{\mathcal{D}_h} \leq \frac{C}{h}\|GPu\|_{\mathcal{D}_h'} + C\|\mathcal{Q}_1 u\|_{\mathcal{D}_h} + Ch^{1/2}\|Gu\|_{\mathcal{D}_h} + O(h^\infty)\|u\|_{\mathcal{D}_h}.$$

An iterative argument exactly as in the end of the proof of the analogous proposition in Gannot–Wunsch [GW18, Proposition 5.8] finishes the proof. □

8. Proof of Theorem 1.1

This section is devoted to a sketch of the proof of the main theorem. The outline of the proof is the same as in the asymptotically Minkowski setting [BVW15, BVW18]; the key missing steps involve propagation near the conic singularities and are formalized in Corollary 6.4 and Proposition 7.3. Indeed, with those missing steps, the same approach as in the asymptotically Minkowski setting ([BVW15, Section 9] and [BVW18, Section 9]) applies here, essentially verbatim.

We let ρ denote a defining function for mf and x denote a defining function for cf. (Near S_+, the primary region of interest, we can take $\rho = t^{-1}$.)

We consider the equation

$$\Box_g w = f'$$

on M°, but then rescale and conjugate to rewrite it as

$$Lu = f,$$

where

$$\begin{aligned} L &\equiv \rho^{-(n-1)/2-2} \Box_g \rho^{(n-1)/2}, \\ u &= \rho^{-(n-1)/2} w \in C^{-\infty}(M), \\ f &= \rho^{-(n-1)/2-2} f' \in \dot{C}^\infty(M). \end{aligned}$$

Under this conjugation, L becomes a "wedge-b-differential operator", i.e., it is a b-differential operator (in the sense of Melrose [Mel93]) at mf and is a wedge-type operator at cf. Moe precisely, $L \in x^{-2}\operatorname{Diff}^2_{\mathrm{e}}(M)$.

Due to the scaling invariance (in the variable ρ) of the metric, we observe that $N(L) = L$. This observation simplifies the analysis as we are ultimately able to avoid dealing with remainder terms.[13] For convenience, we recall from Section 3.1 the form of the operator L in both the (ρ, v, y) and (ρ, x, y) coordinate systems.

Near S_+, in the "almost global" (ρ, v, y) coordinate system,

$$\begin{aligned} L = {} & v(\rho D_\rho)^2 - 4(1-v^2)\rho D_\rho D_v - 4v(1-v^2)D_v^2 - \frac{2}{1-v}\Delta_z \\ & - ((n-1) + (n+1)v)\, i\rho D_\rho + 2\left(2 - (n-1)v - (n-3)v^2\right) iD_v \\ & - 2\left(\frac{n-1}{2}\right)^2 - (n-1)\left(\frac{n+3}{2}\right) v. \end{aligned}$$

where Δ_h is the non-negative Laplacian on the link (Z, h).

We also record the form of the operator in the coordinates (ρ, x, z), where $\rho = 1/t$ and $x = r/t$:

$$L = (\rho D_\rho + x D_x)^2 - ni(\rho D_\rho + x D_x) - D_x^2 + \frac{(n-1)i}{x} D_x - \frac{1}{x^2}\Delta_z - \frac{n^2-1}{4}.$$

After applying the Mellin transform to the identity $Lu = f$, we obtain

$$P_\sigma \widetilde{u}_\sigma = \widetilde{f}_\sigma,$$

where $P_\sigma = \widehat{N}(L)$ is the reduced normal operator of L. As w vanishes near $\overline{C_-}$, we may arrange that $\widetilde{u}_\sigma$ also vanishes there.

Because our solution u in the bulk is not typically conormal to the boundary hypersurface cf, the Mellin transformed solution $\widetilde{u}_\sigma$ is not typically conormal to $x = 0$ on mf. As a result, we must introduce a slightly modified version of the standard cornomal spaces. To do this,

[13] If we instead perturb the spacetime metric, the remainder terms can be handled as in the asymptotically Minkowski setting [BVW15, BVW18].

we fix a smooth cutoff function ϕ on $X = \mathrm{mf}$ so that $\phi \equiv 1$ near $\partial X = \mathrm{mf} \cap \mathrm{cf}$ and $\phi \equiv 0$ near $S_+ \cup S_-$.

Definition 8.1. Suppose $\widetilde{u}$ is a distribution on $X = \mathrm{mf}$. We say that $\widetilde{u} \in I^{(s)}(S_+)$ if

1. $\widetilde{u}$ is smooth away from S_+ and $\partial X = \mathrm{mf} \cap \mathrm{cf}$,
2. $(1 - \phi)\widetilde{u} \in H^s(\mathrm{mf})$,
3. $\phi\widetilde{u} \in \mathcal{D}$, and
4. $(1 - \phi)\widetilde{u}$ is conormal to S_+ relative to H^s.

In other words, $\widetilde{u} \in I^{(s)}(S_+)$ if $\widetilde{u} \in \mathcal{D}$ near ∂X and, away from $x = 0$, $\widetilde{u} \in H^s(X)$ remains in $H^s(X)$ after repeated application of vector fields vanishing on S_+.

Because we use these spaces many times, we recall the same compact notation used in the previous paper [BVW18]. In what follows $\mathcal{H}(\Omega)$ refers to the space of holomorphic functions on the domain $\Omega \subset \mathbb{C}$.

Definition 8.2. For $\varsigma, s \in \mathbb{R}$, we let $\mathbb{C}_\varsigma$ be the upper half-plane $\operatorname{Im}\sigma > -\varsigma$ and then define

$$\mathcal{B}(\varsigma, s) = \mathcal{H}(\mathbb{C}_\varsigma) \cap \langle\sigma\rangle^{-\infty} L^\infty L^2(\mathbb{R}; I^{(s)}(S_+)).$$

In other words, $\mathcal{B}(\varsigma, s)$ consists of those g_σ holomorphic in $\sigma \in \mathbb{C}_\varsigma$ taking values in $I^{(s)}(\Lambda^+)$ such that for each seminorm on $I^{(s)}(S_+)$,

$$\int_{-\infty}^{\infty} \|g_{\mu+i\nu}\|_\bullet^2 \langle\mu\rangle^{2k}\, d\mu$$

is uniformly bounded in $\nu > -\varsigma$.

Further complicating matters, we in fact allow elements of $\mathcal{H}(\mathbb{C}_\varsigma)$ to take values in σ-dependent Sobolev spaces.

Observe that because $f \in \dot{C}^\infty(M)$, we have

$$\widetilde{f}_\sigma \in \mathcal{B}(C, s') \text{ for all } C, s'.$$

One consequence of Corollary 6.4 is the following proposition:

Proposition 8.3. *There are ς_0, s so that*

$$\widetilde{u}_\sigma \in \mathcal{B}(\varsigma_0, s - 0).$$

Proof. Because $\rho^{(n-1)/2} w$ lies in some $H_{\mathrm{b}}^{s,\gamma}(M)$, we have

$$\widetilde{w}_\sigma \in \mathcal{H}(\mathbb{C}_{\varsigma_0}) \cap \langle\sigma\rangle^{\max(0,-s)} L^\infty L^2(\mathbb{R}, H^s), \tag{17}$$

where $\varsigma_0 = \gamma - (n-1)/2$. By reducing s, we may assume that $s + \gamma < 1/2$, so as to be able to apply the regularity results of Corollary 6.4. We may also arrange that $\widetilde{w}_\sigma$ vanishes in a neighborhood of $\overline{C}_-$ in ∂M because w vanishes near $\overline{C}_-$ in M.

Corollary 6.4 implies that (away from cf) w is conormal to S_+ and so by [BVW15, Lemma 2.3],

$$\widetilde{w}_\sigma \in \mathcal{B}(\varsigma_0, -\infty).$$

Interpolating with equation (17) yields the result. □

This allows us to start the iterative procedure. Because

$$P_\sigma \widetilde{u}_\sigma = \widetilde{f}_\sigma,$$

our aim is to invert P_σ and employ a contour shifting argument (as $\widetilde{w}_\sigma$ is holomorphic in a half-plane) to enlarge the domain of meromorphy of $\widetilde{w}_\sigma$.

By Proposition 7.3, P_σ^{-1} forms a meromorphic family in any half-plane. Because $\widetilde{f}_\sigma$ is entire, we now write $\widetilde{u}_\sigma = P_\sigma^{-1}\widetilde{f}_\sigma$ to see that $\widetilde{u}_\sigma$ is meromorphic in any upper half-plane. More precisely, after shifting the contour by N units, we have

$$\begin{aligned}\widetilde{w}_\sigma &\in \mathcal{B}(\varsigma_0 + N, \min(s - 0, 1/2 - \varsigma_0 - N - 0)) \\ &\quad + \sum_{\substack{(\sigma_j, m_j)\in\mathcal{E}_0 \\ \operatorname{Im}\sigma_j > -\varsigma_0 - N}} (\sigma - \sigma_j)^{-m_j} a_j,\end{aligned}$$

where

$$a_j \in \mathcal{B}(\varsigma_0 + N, \operatorname{Im}\sigma_j + 1/2 - 0).$$

Here $\mathcal{E}_0$ is the set of poles of P_σ^{-1}. By an argument identical to the one in the asymptotically Minkowski setting [BVW15, Section 7], these can be identified with the resonances on the hyperbolic cone bounded by S_+, identified above in equation (2).

After inverting the Mellin transform, we can conclude that w enjoys a partial asymptotic expansion. In fact, on M away from the singular locus cf, we have

$$w = \sum_{\substack{(\sigma_j, k)\in\mathcal{E}_0 \\ \operatorname{Im}\sigma_j > -\ell}} \rho^{i\sigma_j}(\log\rho)^k b_{jk} + w',$$

where, for some $C = s + \varsigma_0$ (with s as in Proposition 8.3),

$$w' \in \rho^\ell H_{\mathrm{b}}^{\min(C-\ell-0, 1/2-\varsigma_0-\ell-0)}(M).$$

The coefficients b_{jk} are smooth functions of ρ taking values in $I^{(1/2-\Re(i\sigma_j)-0)}$ and are supported in $\overline{C_+}$. As we look further into the asymptotic expansion of w, the coefficients (and the remainder term) are growing more and more singular; this is because the radiation field is "hiding" at S_+.

In fact, after blowing up S_+, Proposition 2.1 implies that the same arguments in the preceding discussion provide one step toward establishing the polyhomogeneity of w. Indeed, w enjoys an asymptotic expansion at C_+ (away from the singularity at $C_+ \cap \mathrm{cf}$) uniformly up to the corner in $[M; S_+]$.

By Proposition 2.1, the other step needed to establish the polyhomogeneity of w concerns the estimates at $\mathcal{I}^+$. This argument relies on the observation that on M, L and $4D_v(\rho D_\rho + vD_v)$ agree up to terms with additional vanishing near S_+. The vector field $\rho D_\rho + vD_v$ is precisely the vector field that lifts to the b-normal vector field to $\mathcal{I}^+$ on $[M; S_+]$. As it lifts to the radial vector field, we set

$$R = \rho D_\rho + vD_v.$$

The other step establishing the polyhomogeneity of w requires that w enjoys additional vanishing (on $[M; S_+]$) after the application of $(R + ik)\dots(R + i)R$.

We ignore for now these additional terms[14] and suppose $L = 4D_v(\rho D_\rho + vD_v) = 4D_vR$. As Lw is smooth and rapidly decaying, and $w \in H_{\mathrm{b}}^{s,\gamma}$, we know that $D_vRw \in H_{\mathrm{b}}^{s,\gamma}$. Because D_v is elliptic on $\mathrm{WF}_{\mathrm{b}}(w)$, it is microlocally invertible and so $Rw \in H_{\mathrm{b}}^{s+1,\gamma}$, i.e., Rw is one order better than w.

[14]Of course, these additional terms are always there. Managing these terms forms a sizeable part of Section 9.2 of the previous paper [BVW18] and we refer the reader there for a thorough discussion.

To continue this iterative process, we observe that $RD_v = D_v(R+i)$, so that

$$\left(\prod_{j=0}^{k-1}(R+ij)\right) L = \left(\prod_{j=0}^{k-1}(R+ij)\right) D_v R = D_v \left(\prod_{j=0}^{k}(R+ij)\right).$$

An inductive argument then shows that

$$\left(\prod_{j=0}^{k}(R+ij)\right) w \in H_{\mathrm{b}}^{s+k+1,\gamma}(M),$$

so that $(R+ik)\dots(R+i)Rw$ enjoys additional regularity near S_+.

Because w is already conormal to S_+, added regularity near S_+ really measures additional applications of D_v. The vector field vD_v is tangent to S_+ (and so can be applied to w as many times as we like), so we may interpret additional regularity at S_+ as additional vanishing at S_+.[15] This extra vanishing is what is needed for the polyhomogeneity statement.

References

[BM16] Dean Baskin and Jeremy L. Marzuola. Locating resonances on hyperbolic cones. Preprint, arXiv:1608.05278, 2016.

[BVW15] Dean Baskin, András Vasy, and Jared Wunsch. Asymptotics of radiation fields in asymptotically Minkowski space. *American Journal of Mathematics*, 137(5):1293–1364, 2015.

[BVW18] Dean Baskin, András Vasy, and Jared Wunsch. Asymptotics of scalar waves on long-range asymptotically Minkowski spaces. *Adv. Math.*, 328:160–216, 2018.

[CT82a] Jeff Cheeger and Michael Taylor. On the diffraction of waves by conical singularities. I. *Comm. Pure Appl. Math.*, 35(3):275–331, 1982.

[CT82b] Jeff Cheeger and Michael Taylor. On the diffraction of waves by conical singularities. ii. *Communications on Pure and Applied Mathematics*, 35(4):487–529, 1982.

[Eco93] Michael Economakis. *Boundary regularity of the harmonic map problem between asymptotically hyperbolic manifolds*. ProQuest LLC, Ann Arbor, MI, 1993. Thesis (Ph.D.)–University of Washington.

[Fri80] F. G. Friedlander. Radiation fields and hyperbolic scattering theory. *Math. Proc. Cambridge Philos. Soc.*, 88(3):483–515, 1980.

[Fri01] F. G. Friedlander. Notes on the wave equation on asymptotically Euclidean manifolds. *J. Funct. Anal.*, 184(1):1–18, 2001.

[GW18] Oran Gannot and Jared Wunsch. Semiclassical diffraction by conormal potential singularities. Preprint, arXiv:1806.01813, 2018.

[H0̈9] Lars Hörmander. *The analysis of linear partial differential operators. IV*. Classics in Mathematics. Springer-Verlag, Berlin, 2009. Fourier integral operators, Reprint of the 1994 edition.

[Maz91] Rafe Mazzeo. Elliptic theory of differential edge operators i. *Comm. PDE*, 16:1615–1664, 1991.

[Mel93] Richard B. Melrose. *The Atiyah-Patodi-Singer index theorem*, volume 4 of *Research Notes in Mathematics*. A K Peters, Ltd., Wellesley, MA, 1993.

[Mel96] Richard B Melrose. Differential analysis on manifolds with corners, 1996.

[MVW08] Richard Melrose, András Vasy, and Jared Wunsch. Propagation of singularities for the wave equation on edge manifolds. *Duke Mathematical Journal*, 144(1):109–193, 2008.

[MW04a] Richard Melrose and Jared Wunsch. Propagation of singularities for the wave equation on conic manifolds. *Invent. Math.*, 156(2):235–299, 2004.

[MW04b] Richard Melrose and Jared Wunsch. Propagation of singularities for the wave equation on conic manifolds. *Inventiones mathematicae*, 156(2):235–299, 2004.

[Vas08] András Vasy. Propagation of singularities for the wave equation on manifolds with corners. *Ann. of Math. (2)*, 168(3):749–812, 2008.

[15] This interpretation can be formalized by an integration argument and requires keeping track of the factors of the module for which w already possesses regularity; see [BVW18, Section 9.2] for details.

[Vas13] András Vasy. Microlocal analysis of asymptotically hyperbolic and kerr-de sitter spaces (with an appendix by semyon dyatlov). *Inventiones mathematicae*, 194(2):381–513, 2013.

Email address: dbaskin@math.tamu.edu

Department of Mathematics, Texas A&M University, Mailstop 3368, College Station, TX 77843

Email address: marzuola@math.unc.edu

Department of Mathematics, UNC-Chapel Hill, CB#3250 Phillips Hall, Chapel Hill, NC 27599