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The essential spectrum of periodically-stationary solutions of the complex Ginzburg-Landau equation

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To Matthias Hieber with best wishes

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Abstract We establish the existence and regularity properties of a monodromy operator for the linearization of the cubic-quintic complex Ginzburg-Landau equation about a periodically-stationary (breather) solution. We derive a formula for the essential spectrum of the monodromy operator in terms of that of the associated asymptotic linear differential operator. This result is obtained using the theory of analytic semigroups under the assumption that the Ginzburg-Landau equation includes a spectral filtering (diffusion) term. We discuss applications to the stability of periodically-stationary pulses in ultrafast fiber lasers.

Keywords Nonlinear waves · Breather solutions · Essential spectrum · Analytic semigroups · Fiber lasers

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1 Introduction

The cubic-quintic complex Ginzburg-Landau equation (CQ-CGLE) is a fundamental model for nonlinear waves and coherent structures that arise in fields such as nonlinear optics and condensed matter physics [1, 7]. The CQ-CGLE supports a wide variety of solutions, including stationary pulses, periodically-stationary pulses, fronts, exploding solitons, and chaotic solutions [1]. While stationary pulses (solitons) maintain their shape, periodically-stationary pulses (breathers) change shape as they propagate, returning to the same shape periodically.

Periodically-stationary pulses are the solutions of primary interest to engineers designing ultrafast fiber lasers. These lasers generate pulses with widths in the picosecond to femtosecond range and have applications to time and frequency metrology [9, 27, 33]. Since the advent of the soliton laser [13, 37], researchers have invented several generations of short-pulse, high-energy fiber lasers for a variety of applications. In the mid 1990's stretched-pulse (dispersion-managed) lasers were devised to generate pulses with higher energy and shorter duration than can be acheived with soliton lasers [47, 48]. Dissipative soliton lasers, in which effects such as spectral filtering play a significant role, were introduced in about 2005 and are suitable for high energy applications [8, 21]. Similariton lasers were introduced in 2010 to create femto-second pulses with a high tolerance to noise [9, 17, 23] by exploiting the theoretical discovery of exponentially growing, self-similar pulses in optical fiber amplifiers. The most recent invention is the Mamyshev oscillator which can produce pulses with a peak power in the Megawatt range [41, 46].

The key issue for mathematical modeling of ultrafast lasers is to determine those regions in the design parameter space in which stable pulses exist, and within that space to optimize the pulse parameters. A significant challenge is that from one generation of lasers to the next there has been an increase in the amount by which the pulse changes within each round trip (period). Consequently, soliton perturbation theory [20, 29], which was developed to analyze the stability of stationary pulses, is no longer applicable. Although low-dimensional reduced ODE models [49, 50] and Monte Carlo simulations that use full PDE models [52] have both been employed to assess pulse stability, to date there is no mathematical theory to determine the stability of periodically-stationary laser pulses. In ultrafast lasers, the different physical effects (dispersion, nonlinearity, spectral filtering, saturable gain and loss) occur in different devices within the laser. For a quantitative model it is therefore necessary to use am equation such as the CQ-CGLE in which the coefficients are piecewise constant in the evolution variable. However, constant coefficient models are also often used to gain qualitative insight into the system behavior [42].

In this paper, we take a first step in the development of a stability theory for ultrafast lasers by calculating the essential spectrum of the monodromy operator of the linearization of the CQ-CGLE about a periodically-stationary pulse solution. Because variable coefficient equations are more challenging to analyze, we restrict attention to the constant coefficient CQ-CGLE, which is a phenomenological, distributed model for short-pulse fiber lasers [21,33]. This equation has two important classes of solutions that are periodic in the temporal variable.

The first class is the family of Kuznetsov-Ma (KM) breathers [34,35], which are analytical solutions of the focusing nonlinear Schrödinger equation (FNLSE), a special case of the CQ-CGLE. These solutions, which were discovered using integrable systems techniques, have a non-zero background at spatial infinity. A numerical Floquet spectrum computation by Cuevas-Maraver *et al.* [11] suggests that the KM breather is linearly unstable. Muñoz [38] recently used a Lyapunov functional to prove that because of their non-zero background, these breathers are unstable under small H^s ($s > \frac{1}{2}$) perturbations. Integrable systems techniques have also been used to find breather solutions of the modified and higher-order KdV equations and the Gardner hierarchy [10,4]. In a recent series of papers [3, 5,6], Alejo exploited the integrability structure of these PDE's and Lyapunov functional techniques to establish the nonlinear stability of several such breathers.

The second class consists of the periodically-stationary pulses discovered numerically by Akhmediev and his collaborators [2,49,50]. These solutions were found in the case that the CQ-CGLE includes a spectral filtering term. Although Akhmediev *et al.* provided strong numerical evidence for the existence of these solutions, there are no known analytic formulae for these solutions and no mathematical proof of their existence. However, using numerical simulations and reduced ODE models, they provided numerical evidence for the existence of both stable and unstable periodically-stationary pulses.

In Floquet theory, the stability of periodic solutions of a system of ODE's is characterized by the spectrum of the monodromy matrix of the linearization of the system about the solution. Although Floquet methods have been developed for solutions of PDE's that are periodic in the spatial variables (see for example [19, 24,32,40]), we are only aware of a few results for solutions that are periodic in the temporal (evolution) variable. Wilkening [51] developed a numerical method to study the stability of standing water waves and other time-periodic solutions of the free-surface Euler equations. Motivated by problems from quantum mechanics, Korotyaev [30,31] studied Schrödinger operators that have a real scalar potential which is periodic in time and rapidly decaying in space. He showed that the monodromy operator has no singular continuous spectrum. However, this result relies heavily on fact that the evolution operator is unitary, which is not the case when the CQ-CGLE includes a spectral filtering term. Similar results are discussed in the book of Kuchment [32]. Finally, Sandstede and Scheel [12,44,43] have an extensive body of theoretical and numerical results on the stability of time-periodic perturbations of spatially-periodic traveling waves. However, because of the underlying spatial periodicity in their formulation, these results are not applicable to laser systems.

The results in this paper can be summarized as follows. In Section 2, we review the periodically-stationary solutions of Kuznetsov-Ma and Akhmediev and define the time-periodic operator, $\mathcal{L}(t)$, which is obtained by linearization of the CQ-CGLE about a periodically-stationary solution. We also discuss the asymptotic operator, $\mathcal{L}_\infty$, associated with $\mathcal{L}(t)$. In Section 3, we calculate the essential spectrum of $\mathcal{L}_\infty$ with the aid of some results from the text of Kapitula and Promislow [26], and we show that $\mathcal{L}(t)$ is a relatively compact perturbation of $\mathcal{L}_\infty$. In Section 4, we establish the existence of an evolution family for $\mathcal{L}(t)$ by applying classical results on solutions of initial-value problems for non-autonomous linear differential equations in Banach spaces [39]. In Section 5, we use the results obtained in Section 4 to define the monodromy operator, $\mathcal{M}(s)$, and establish the main result of the paper, which is a formula for the essential spectrum of $\mathcal{M}(s)$ in terms of the essential spectrum of the asymptotic operator, $\mathcal{L}_\infty$. To obtain this result, we assumed that the CQ-CGLE includes a spectral filtering term, which ensures that the semigroup of the asymptotic operator is analytic. Since all fiber lasers have bandlimited gain, this assumption holds in applications. Based on the numerical results of Cuevas-Maraver *et al.* [11], we conjecture that the formula for the essential spectrum of $\mathcal{M}(s)$ also holds for the KM breather. However a different approach will be required to prove such a result, since in this case the asymptotic operator is not analytic.

2 Motivating examples

We consider a class of one-dimensional, constant-coefficient nonlinear Schrödinger equations of the form

$$i\partial_t\psi \;+\; \tfrac{1}{2}\partial_x^2\psi \;+\; f(|\psi|^2)\psi = 0, \tag{2.1}$$

where f is a polynomial with complex coefficients. We call t the *temporal* or *evolution* variable and x the *spatial* variable. We consider solutions, ψ, for which $f(|\psi|^2)\psi \to 0$ at an exponential rate as $x \to \pm\infty$. We say that ψ is a *periodically-stationary solution* of (2.1) if there is a period, T, so that $\psi(t+T,x) = \psi(t,x)$ for all t and x. Our analysis is motivated by two important examples.

Example 2.1 Kuznetsov [34] and Ma [35] independently discovered a family of periodically-stationary solutions of the FNLSE,

$$i\partial_t\psi \;+\; \tfrac{1}{2}\partial_x^2\psi \;+\; (|\psi|^2-\nu_0^2)\psi = 0, \tag{2.2}$$

with the property that

$$\lim_{x\to\pm\infty}\psi(t,x) = \nu_0. \tag{2.3}$$

Here the parameter, $\nu_0 > 0$, is the background amplitude. The Kuznetsov-Ma (KM) breathers are defined in terms of a second parameter, $\nu > \nu_0$, by [18]

$$\psi_{\rm KM}(t,x) \;=\; \nu_0 \;+\; 2\eta\,\frac{\eta\cos(2\nu\eta t) + i\nu\sin(2\nu\eta t)}{\nu_0\cos(2\nu\eta t) - \nu\cosh(2\eta x)}, \tag{2.4}$$

where $\eta = \sqrt{\nu^2-\nu_0^2} > 0$. The period of $\psi_{\rm KM}$ is $T = \pi/\nu\eta$. We observe that $f(|\psi_{\rm KM}|^2)\psi_{\rm KM} = (|\psi_{\rm KM}|^2-\nu_0^2)\psi_{\rm KM} \to 0$ at an exponential rate as $x \to \pm\infty$.

Example 2.2 The CQ-CGLE [1,7] is given by

$$i\partial_t\psi + (\tfrac{D}{2} - i\beta)\partial_x^2\psi - i\delta\psi + \left[\gamma - i\epsilon + (\nu - i\mu)|\psi|^2\right]|\psi|^2\psi \;=\; 0. \tag{2.5}$$

Among other applications, the CQ-CGLE provides a qualitative model for the generation of short-pulses in mode-locked fiber lasers [1,33]. In this context, the parameters in the equation can be interpreted as follows. The coefficient, D, is the fiber dispersion, which is positive in the anomalous or focusing dispersion regime and negative in the normal or defocusing dispersion regime. Spectral filtering is modeling using the term with coefficient $\beta > 0$. The remaining terms model linear gain or loss (δ), saturable nonlinear gain ($\epsilon > 0$ and $\mu < 0$), and the cubic and quintic nonlinear electric susceptibility of the optical fiber ($\gamma > 0$ and $\nu > 0$).

Using a numerical partial differential equation solver, Akhmediev and his collaborators [1,2,49,50] provide strong evidence for the existence of periodically-stationary solutions of (2.5), which they refer to as pulsating solitons. There are no known analytical formulae for these solutions. However, the numerical results show that these solutions decay at an exponential rate as $x \to \pm\infty$.

Equations (2.2) and (2.5) are both special cases of the general nonlinear wave equation

$$i\partial_t \psi + (\tfrac{D}{2} - i\beta)\partial_x^2 \psi + (\alpha - i\delta)\psi + \left[\gamma - i\epsilon + (\nu - i\mu)|\psi|^2\right] |\psi|^2 \psi \;=\; 0. \qquad (2.6)$$

Throughout this paper, we assume that $(D, \beta) \neq (0, 0)$. Since the linearization of (2.6) about a solution involves both the linearized unknown and its complex conjugate, we reformulate (2.6) as the system of equations for $\boldsymbol{\psi} = \left[\Re(\psi)\ \Im(\psi)\right]^T$ given by

$$\partial_t \boldsymbol{\psi} \;=\; \left(\mathbf{B}\,\partial_x^2 + \mathbf{N}_0 + \mathbf{N}_1|\boldsymbol{\psi}|^2 + \mathbf{N}_2|\boldsymbol{\psi}|^4\right)\boldsymbol{\psi}, \qquad (2.7)$$

where

$$\mathbf{B} = \begin{bmatrix} \beta & -\frac{D}{2} \\ \frac{D}{2} & \beta \end{bmatrix}, \qquad (2.8)$$

and

$$\mathbf{N}_0 = \begin{bmatrix} \delta & -\alpha \\ \alpha & \delta \end{bmatrix}, \qquad \mathbf{N}_1 = \begin{bmatrix} \epsilon & -\gamma \\ \gamma & \epsilon \end{bmatrix}, \qquad \mathbf{N}_2 = \begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix}. \qquad (2.9)$$

The linearization of (2.7) about a solution, $\boldsymbol{\psi}$, is of the form

$$\partial_t \mathbf{p} \;=\; \mathcal{L}(t)\mathbf{p}, \qquad (2.10)$$

where $\mathcal{L} = \mathcal{L}(t)$ is a second-order, linear differential operator in x with real, t- and x-dependent, matrix-valued coefficients. If the solution $\boldsymbol{\psi}$ is periodically stationary with period T, then $\mathcal{L}$ is periodic in t with $\mathcal{L}(t+T) = \mathcal{L}(t)$. Substituting $\boldsymbol{\psi}_\varepsilon = \boldsymbol{\psi} + \varepsilon\mathbf{p}$ into (2.7) are keeping only terms of order ε we find that

$$\mathcal{L}(t) \;=\; \mathbf{B}\,\partial_x^2 + \widetilde{\mathbf{M}}(t), \qquad (2.11)$$

where $\widetilde{\mathbf{M}}(t)$ is the operator of multiplication by

$$\widetilde{\mathbf{M}}(t,x) = \mathbf{N}_0 + \mathbf{N}_1|\boldsymbol{\psi}|^2 + \mathbf{N}_2|\boldsymbol{\psi}|^4 + \left(2\mathbf{N}_1 + 4\mathbf{N}_2|\boldsymbol{\psi}|^2\right)\boldsymbol{\psi}\boldsymbol{\psi}^T. \qquad (2.12)$$

3 Essential Spectrum of the Linearized Differential Operator

In this section we introduce the asymptotic operator, $\mathcal{L}_\infty$, associated with the differential operator, $\mathcal{L}(t)$, and determine the essential spectrum of $\mathcal{L}_\infty$. We also prove that $\mathcal{L}(t)$ is a relatively compact perturbation of $\mathcal{L}_\infty$. Our results rely on a general theory summarized by Kapitula and Promislow [26] and on a classical compactness theorem for L^2 due to Kolmogorov and Riesz [22,40]. We begin with the following assumption.

Hypothesis 3.1 *The solution, $\boldsymbol{\psi}$, about which* (2.7) *is linearized is assumed to be periodically stationary with period, T, and has the following properties:*

1. *For each $t \in [0,T]$, the function $\boldsymbol{\psi}(t,\cdot) \in L^\infty(\mathbb{R},\mathbb{C}^2)$;*
2. *For each $t \in [0,T]$, the weak derivative $\boldsymbol{\psi}_x(t,\cdot) \in L^\infty(\mathbb{R},\mathbb{C}^2)$;*
3. *There exist constants $r > 0$ and $\boldsymbol{\psi}_\infty \in \mathbb{C}^2$ so that*

$$\lim_{|x|\to\infty} e^{r|x|} \|\boldsymbol{\psi}(t,x) - \boldsymbol{\psi}_\infty\|_{\mathbb{C}^2} \;=\; 0, \qquad \textit{for all } t \in [0,T].$$

Remark 3.1 The KM breather (2.4) satisfies Hypothesis 3.1. Although we do not have definitive proof, it is reasonable to assume that the numerically computed periodically-stationary solultions discussed in Example 2.2 also satisfy Hypothesis 3.1.

Remark 3.2 Hypothesis 3.1 guarantees that for each t, the multiplication operator, $\widetilde{\mathbf{M}}(t)$, in (2.12) is a bounded operator on $L^2(\mathbb{R},\mathbb{C}^2)$. Since $(D,\beta) \neq (0,0)$, it follows from [26, Lemma 3.1.2] that the operator $\mathcal{L}(t) : H^2(\mathbb{R},\mathbb{C}^2) \subset L^2(\mathbb{R},\mathbb{C}^2) \to L^2(\mathbb{R},\mathbb{C}^2)$ is closed.

Next, we have the following proposition, whose proof is self-evident.

Proposition 3.1 *Assume that Hypothesis 3.1 is met. Then*

$$\mathbf{M}_\infty := \lim_{|x|\to\infty} \widetilde{\mathbf{M}}(t,x) \tag{3.1}$$

exists and is t-independent. Furthermore, the differential operator, $\mathcal{L}$, is exponentially asymptotic in that the leading coefficient, $\mathbf{B}$, is constant and there exists a constant $r > 0$ so that

$$\lim_{|x|\to\infty} e^{r|x|} \left\| \widetilde{\mathbf{M}}(t,x) - \mathbf{M}_\infty \right\|_{\mathbb{C}^{2\times 2}} \;=\; 0 \qquad \textit{for all } t \in [0,T], \tag{3.2}$$

where $\|\cdot\|_{\mathbb{C}^{2\times 2}}$ denotes the matrix norm induced from the Euclidean norm $\|\cdot\|_{\mathbb{C}^2}$.

Definition 3.1 The *asymptotic differential operator*, $\mathcal{L}_\infty$, associated with the exponentially asymptotic operator, $\mathcal{L}(t)$, is the t-independent operator with constant, matrix-valued coefficients given by

$$\mathcal{L}_\infty \;:=\; \mathbf{B}\,\partial_x^2 + \mathbf{M}_\infty. \tag{3.3}$$

Just as for the operator, $\mathcal{L}(t)$, we regard $\mathcal{L}_\infty$ as an operator on $L^2(\mathbb{R},\mathbb{C}^2)$ with domain $H^2(\mathbb{R},\mathbb{C}^2)$. Furthermore, if we define $\mathbf{M}(t,x) := \widetilde{\mathbf{M}}(t,x) - \mathbf{M}_\infty$, then

$$\mathcal{L}(t) = \mathcal{L}_\infty + \mathbf{M}(t), \tag{3.4}$$

where $\mathbf{M}(t,x) \to \mathbf{0}$ as $|x| \to \infty$.

We now review the definition of the essential spectrum we use in this paper.

Definition 3.2 Let X be a Banach space and let $\mathcal{B}(X)$ denote the space of bounded linear operators on X. Let $\mathcal{L} : D(\mathcal{L}) \subset X \to X$ be a closed linear operator with domain $D(\mathcal{L})$ that is dense in X. The *resolvent set* of $\mathcal{L}$ is

$$\rho(\mathcal{L}) \;:=\; \{\lambda \in \mathbb{C} \,|\, \mathcal{L} - \lambda \text{ is invertible and } (\mathcal{L} - \lambda)^{-1} \in \mathcal{B}(X)\}, \tag{3.5}$$

and for each $\lambda \in \rho(\mathcal{L})$, the *resolvent operator* is $R(\lambda : \mathcal{L}) := (\mathcal{L} - \lambda)^{-1}$. The *spectrum* of $\mathcal{L}$ is $\sigma(\mathcal{L}) := \mathbb{C} \setminus \rho(\mathcal{L})$. The *point spectrum* of $\mathcal{L}$ is

$$\sigma_{\rm pt}(\mathcal{L}) \;:=\; \{\lambda \in \mathbb{C} \,|\, \operatorname{Ker}(\mathcal{L} - \lambda) \neq \{0\}\}. \tag{3.6}$$

The *Fredholm point spectrum* of $\mathcal{L}$ is the subset of $\sigma_{\rm pt}(\mathcal{L})$ defined by

$$\sigma_{\rm pt}^{\mathcal{F}}(\mathcal{L}) \;:=\; \{\lambda \in \mathbb{C} \,|\, \mathcal{L} - \lambda \text{ is Fredholm, } \operatorname{Ind}(\mathcal{L} - \lambda) = 0, \text{ and } \operatorname{Ker}(\mathcal{L} - \lambda) \neq \{0\}\}, \tag{3.7}$$

and the *essential spectrum* of $\mathcal{L}$ is $\sigma_{\rm ess}(\mathcal{L}) := \sigma(\mathcal{L}) \setminus \sigma_{\rm pt}^{\mathcal{F}}(\mathcal{L})$. We observe that $\sigma(\mathcal{L}) = \sigma_{\rm pt}(\mathcal{L}) \cup \sigma_{\rm ess}(\mathcal{L})$, but note that this union may not be disjoint.

Remark 3.3 Since the operators we consider are not self-adjoint, there are several non-equivalent definitions of the essential spectrum [14]. An argument that involves the closed graph theorem [28] shows that with the definition we use, $\sigma_{\rm ess}(\mathcal{L})$ consists of those $\lambda \in \mathbb{C}$ so that either (a) $\mathcal{L}-\lambda$ is Fredholm but has $\operatorname{Ind}(\mathcal{L}-\lambda) \neq 0$ or (b) $\mathcal{L}-\lambda$ is not Fredholm. This definition gives the largest subset of the spectrum that is invariant under compact perturbations [14]. However, the operators, $\mathcal{L}$, we consider are given as perturbations of constant coefficient differential operators, $\mathcal{L}_\infty$, by multiplication operators, which are not compact. Nevertheless, we will show below that the operator $\mathcal{L}(t)$ is a *relatively compact perturbation* of $\mathcal{L}_\infty$, by which we mean that $\exists \lambda \in \rho(\mathcal{L}_\infty)$ so that $(\mathcal{L}(t) - \mathcal{L}_\infty)(\mathcal{L}_\infty - \lambda)^{-1} : X \to X$ is compact. Consequently, by Weyl's essential spectrum theorem [26], $\sigma_{\rm ess}(\mathcal{L}(t)) = \sigma_{\rm ess}(\mathcal{L}_\infty)$.

To calculate the spectrum of the asymptotic operator, $\mathcal{L}_\infty$, we convert the equation, $(\mathcal{L}_\infty - \lambda)\mathbf{p} = \mathbf{0}$, to the first-order system,

$$\partial_x \mathbf{Y} \;=\; \mathbf{A}_\infty(\lambda)\mathbf{Y}, \tag{3.8}$$

where $\mathbf{Y} = \begin{bmatrix} \mathbf{p} \\ \mathbf{p}_x \end{bmatrix} \in \mathbb{C}^4$ and $\mathbf{A}_\infty(\lambda) \in \mathbb{C}^{4\times 4}$ is the constant matrix

$$\mathbf{A}_\infty(\lambda) \;:=\; \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{B}^{-1}(\lambda - \mathbf{M}_\infty) & \mathbf{0} \end{bmatrix}. \tag{3.9}$$

Proposition 3.2 *Assume that Hypothesis 3.1 is met. Then,* $\sigma_{\rm pt}(\mathcal{L}_\infty) = \emptyset$ *and*

$$\sigma_{\rm ess}(\mathcal{L}_\infty) \;=\; \{\lambda \in \mathbb{C} \,|\, \exists \mu \in \mathbb{R} : \det[\lambda - \mathbf{M}_\infty + \mu^2 \mathbf{B}] = 0\}. \tag{3.10}$$

Proof Since (3.8) does not have solutions that decay as $x \to +\infty$ and as $x \to -\infty$, $\sigma_{\rm pt}(\mathcal{L}_\infty) = \emptyset$. Arguing as in the proof of Kapitula and Promislow [26, Lemma 3.1.10], we have that $\lambda \in \sigma_{\rm ess}(\mathcal{L}_\infty)$ precisely when the matrix $\mathbf{A}_\infty(\lambda)$ has a pure imaginary eigenvalue, that is

$$\sigma_{\rm ess}(\mathcal{L}_\infty) \;=\; \{\lambda \in \mathbb{C} \,|\, \exists \mu \in \mathbb{R} : \det[\mathbf{A}_\infty(\lambda) - i\mu] = 0\}. \tag{3.11}$$

Premultiplying $\mathbf{A}_\infty(\lambda) - i\mu$ by the invertible matrix $\begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{B} & \mathbf{0} \end{bmatrix}$, and applying the Schur determinental formula [36], we find that $\lambda \in \sigma_{\text{ess}}(\mathcal{L})$ if and only if

$$0 \;=\; \det \begin{bmatrix} \lambda - \mathbf{M}_\infty & -i\mu\mathbf{B} \\ -i\mu\mathbf{B} & \mathbf{B} \end{bmatrix} \;=\; \det \mathbf{B} \det[\lambda - \mathbf{M}_\infty + \mu^2 \mathbf{B}]. \tag{3.12}$$

□

Example 3.1 (KM Breather) For the KM breather discussed in Example 2.1,

$$\mathbf{M}_\infty = \begin{bmatrix} 0 & 0 \\ 2\nu_0^2 & 0 \end{bmatrix}. \tag{3.13}$$

Consequently,

$$\sigma_{\text{ess}}(\mathcal{L}_\infty) \;=\; \{\lambda \in \mathbb{C} \mid \lambda = \pm i\tfrac{\mu}{2}\sqrt{\mu^2 - 4\nu_0^2} \text{ for some } \mu \in \mathbb{R}\}, \tag{3.14}$$

which is the union of the imaginary axis and the interval $[-\nu_0, \nu_0]$ in the real axis. This result is consistent with a result of Cuevas-Maraver *et al.* [11], who performed a modulation instability analysis to show that the frequency, ω, and wave number, k, of a perturbation about a plane wave background, $\psi_\infty(t,x) = \nu_0$, satisfy the dispersion relation $\omega = \pm\frac{k}{2}\sqrt{k^2 - 4\nu_0^2}$.

Example 3.2 (CQ-CGL Breathers) For the CQ-CGL breathers discussed in Example 2.2, $\mathbf{M}_\infty = \delta\mathbf{I}$. Consequently,

$$\sigma_{\text{ess}}(\mathcal{L}_\infty) \;=\; \{\lambda \in \mathbb{C} \mid \lambda = \delta - \mu^2(\beta \pm iD/2) \text{ for some } \mu \in \mathbb{R}\}, \tag{3.15}$$

which is a pair of half-lines in the complex plane that are symmetric about the real axis [25,45]. If the physical system being modeling includes linear loss ($\delta < 0$) and a spectral filter ($\beta > 0$), then the essential spectrum is stable.

We conclude this section by showing that $\mathcal{L}(t)$ is a relatively compact perturbation of $\mathcal{L}_\infty$. This result plays an important role in the proof of our main result, Theorem 5.1, on the essential spectrum of the monodromy operator.

Theorem 3.1 *Assume that Hypothesis 3.1 is met. Then, the differential operator, $\mathcal{L}(t)$, given in* (2.11)*, is a relatively compact perturbation of $\mathcal{L}_\infty$.*

Versions of this result are well known folklore. For example, Kapitula and Promislow [26, Theorem 3.1.11] provide a proof in the special case that $\boldsymbol{\psi} - \boldsymbol{\psi}_\infty$ is compactly supported. For the sake of completeness, and because it may be of independent interest to some readers, we provide a proof of the general case. This proof is based on a characterization of compact subsets of $L^2(\mathbb{R})$ due to Kolmogorov and Riesz [22,40].

In the sequel, for each Banach space, X, we let $\|\cdot\|_X$ denote the standard norm on X.

Proof Let $\lambda \in \rho(\mathcal{L}_\infty)$. We must show that the operator $\mathcal{K} := \mathbf{M}(t) \circ (\mathcal{L}_\infty - \lambda)^{-1} \,:\, L^2(\mathbb{R},\mathbb{C}^2) \to H^2(\mathbb{R},\mathbb{C}^2) \to L^2(\mathbb{R},\mathbb{C}^2)$ is compact. It suffices to show that for any bounded family of functions, $\mathcal{H} \subset L^2(\mathbb{R},\mathbb{C}^2)$, the subset $\mathcal{F} = \mathcal{K}(\mathcal{H}) \subset L^2(\mathbb{R},\mathbb{C}^2)$ is precompact, or equivalently is totally bounded. A classical theorem due to Kolmogorov and Riesz [22] states that $\mathcal{F} \subset L^2(\mathbb{R},\mathbb{C}^2)$ is totally bounded if and only if the following three conditions hold:

1. $\mathcal{F}$ is bounded,
2. for all $\epsilon > 0$ there is an $R > 0$ so that for all $f \in \mathcal{F}$,

$$\int_{|x|>R} \|f(x)\|_{\mathbb{C}^2}^2 \, dx \;<\; \epsilon^2, \qquad \text{and} \tag{3.16}$$

3. for all $\epsilon > 0$ there is a $\delta > 0$ so that for all $f \in \mathcal{F}$ and $y \in \mathbb{R}$ with $|y| < \delta$,

$$\int_{\mathbb{R}} \|f(x+y) - f(x)\|_{\mathbb{C}^2}^2 \, dx \;<\; \epsilon^2. \tag{3.17}$$

The first condition holds as the subset $\mathcal{H}$ and the operator $\mathcal{K}$ are bounded. For the second condition, we first observe that there is a $C > 0$ so that for all $h \in \mathcal{H}$,

$$\|(\mathcal{L}_\infty - \lambda)^{-1} h\|_{H^2(\mathbb{R},\mathbb{C}^2)} < C. \tag{3.18}$$

Next, by Hypothesis 3.1, there is an $\widetilde{R} > 0$ so that $\|\mathbf{M}(t,x)\|_{\mathbb{C}^{2\times 2}} < e^{-r|x|}/C$ for all $|x| > \widetilde{R}$. Let $R > \widetilde{R}$. Since every $f \in \mathcal{F} = \mathcal{K}(\mathcal{H})$ is of the form $f = \mathbf{M}(t)g$ for some $g \in (\mathcal{L}_\infty - \lambda)^{-1}(\mathcal{H})$,

$$\int_{|x|>R} \|f(x)\|_{\mathbb{C}^2}^2 \, dx \;<\; \frac{1}{C^2} e^{-2rR} \|g\|_{H^2(\mathbb{R},\mathbb{C}^2)} \;<\; \epsilon^2, \tag{3.19}$$

provided that $R > |\log \epsilon|/r$.[1]

For the third condition, we first observe that $f = \mathbf{M}(t)g \in H^1(\mathbb{R},\mathbb{C}^2)$, by Hypothesis 3.1 and since $g \in H^2(\mathbb{R},\mathbb{C}^2)$. Appealing to a result in Evans [16, §5.8.2] on difference quotients for H^1 functions, we have that

$$\begin{aligned} &\int_{\mathbb{R}} \|f(x+y) - f(x)\|_{\mathbb{C}^2}^2 \, dx \le |y|^2 \, \|f'\|_{L^2(\mathbb{R},\mathbb{C}^2)}^2 \\ &\le |y|^2 \max\{\|\mathbf{M}(t)\|_{L^\infty(\mathbb{R},\mathbb{C}^2)}^2, \|\mathbf{M}_x(t)\|_{L^\infty(\mathbb{R},\mathbb{C}^2)}^2\} \, \|g\|_{H^2(\mathbb{R},\mathbb{C}^2)}^2, \end{aligned}$$

which can be made arbitrarily small, provided y is close enough to zero. □

4 The Evolution Family

In this section, we establish the existence of an evolution family for the linearized equation (2.10) by applying classical results on the existence, uniqueness, and regularity of solutions of initial-value problems for non-autonomous linear differential equations in Banach spaces. In Section 5, the evolution family will be used to define the monodromy operator.

We study solutions, $\mathbf{u} : [s,\infty) \to H^2(\mathbb{R},\mathbb{C}^2)$, of the initial-value problem

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} \;&=\; \mathcal{L}(t)\mathbf{u}, \qquad \text{for } t > s, \\ \mathbf{u}(s) &= \mathbf{v}, \end{aligned} \tag{4.1}$$

[1] We observe that the third condition in Hypothesis 3.1 that ψ decays exponentially can be significantly relaxed. In particular, we do not even require that ψ or ψ_x belong to $L^2(\mathbb{R},\mathbb{C}^2)$.

where $\mathcal{L}(t) = \mathbf{B}\,\partial_x^2 + \widetilde{\mathbf{M}}(t)$ is the t-dependent family of operators on $L^2(\mathbb{R}, \mathbb{C}^2)$ defined in (2.11), and $\mathbf{v} \in H^2(\mathbb{R}, \mathbb{C}^2)$.

The existence of solutions of linear differential equations in Banach spaces is typically established using semigroup theory. Since the operator, $\mathcal{L}(t)$, is t-dependent, we utilize the theory of evolution families [39], in which the solution of (4.1) is represented in the form, $\mathbf{u}(t) = \mathcal{U}(t,s)\mathbf{v}$, where $\mathcal{U}(t,s)$ is an evolution operator. The following theorem establishes conditions on a solution, $\psi = \psi(t,x)$, of the nonlinear wave equation (2.6) that ensure the existence of an evolution operator, $\mathcal{U}(t,s)$, for the linearized equation, (4.1).

Hypothesis 4.1 *The T-periodic solution, $\boldsymbol{\psi}$, about which* (2.7) *is linearized has the property that both $\boldsymbol{\psi}$ and $\boldsymbol{\psi}_t$ are bounded and are continuous on* $[0,T] \times \mathbb{R}$, *uniformly in x.*

Definition 4.1 Let $\mathbf{A} = \mathbf{A}(t,x) : [0,\infty) \times \mathbb{R} \to \mathbb{C}^{2\times 2}$ be a bounded matrix-valued function. We define

$$\|\mathbf{A}\|_\infty \ := \ \sup_{(t,x)} \|\mathbf{A}(t,x)\|_{\mathbb{C}^{2\times 2}}. \tag{4.2}$$

Theorem 4.1 *Assume that Hypothesis 4.1 is met and that $\beta \geq 0$. Then, there exists a unique evolution operator, $\mathcal{U}(t,s) \in \mathcal{B}(L^2(\mathbb{R}, \mathbb{C}^2))$, for $0 \leq s \leq t < \infty$, such that*

1. $\|\mathcal{U}(t,s)\|_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))} \leq \exp[\,\|\widetilde{\mathbf{M}}\|_\infty (t-s)\,]$,
2. $\mathcal{U}(t,s)(H^2(\mathbb{R}, \mathbb{C}^2)) \subset H^2(\mathbb{R}, \mathbb{C}^2)$, *and*
3. *For each s, $\mathcal{U}(\cdot, s)$ is strongly continuous in that for all $\mathbf{v} \in L^2(\mathbb{R}, \mathbb{C}^2)$, the mapping $t \mapsto \mathcal{U}(t,s)\mathbf{v}$ is continuous, and*
4. *For each $\mathbf{v} \in H^2(\mathbb{R}, \mathbb{C}^2)$, the function $\mathbf{u}(t) = \mathcal{U}(t,s)\mathbf{v}$ is the unique solution of the initial value problem* (4.1) *for which $\mathbf{u} \in C([s,\infty), H^2(\mathbb{R}, \mathbb{C}^2))$ and $\mathbf{u} \in C^1((s,\infty), L^2(\mathbb{R}, \mathbb{C}^2))$.*

Remark 4.1 In the case that $\beta > 0$, for any $\mathbf{v} \in L^2(\mathbb{R}, \mathbb{C}^2)$ we have that $\mathbf{u}(t) = \mathcal{U}(t,s)\mathbf{v} \in H^2(\mathbb{R}, \mathbb{C}^2)$. However, when $\beta = 0$, we require that $\mathbf{v} \in H^2(\mathbb{R}, \mathbb{C}^2)$ in order that $\mathbf{u}(t) \in H^2(\mathbb{R}, \mathbb{C}^2)$.

Proof The theorem follows from a combination of the Hille-Yosida Theorem (see Theorem 3.1 of Pazy [39, Ch. 1]) and Theorems 2.3 and 4.8 of [39, Ch. 5]. The following three lemmas ensure that the hypotheses of these three theorems hold.

Lemma 4.1 *The linear operator, $\mathbf{B}\partial_x^2 : H^2(\mathbb{R}, \mathbb{C}^2) \subset L^2(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$ is closed with domain $H^2(\mathbb{R}, \mathbb{C}^2)$. Furthermore, $\mathbb{R}^+ \subset \rho(\mathbf{B}\partial_x^2)$ and the resolvent operator (see Definition 3.2) satisfies*

$$\|R(\lambda : \mathbf{B}\partial_x^2)\|_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))} \ \leq \ \frac{1}{\lambda} \qquad \textit{for all } \lambda > 0. \tag{4.3}$$

Consequently, by the Hille-Yosida Theorem, $\mathbf{B}\partial_x^2$ is the infinitesimal generator of a C_0-semigroup of contractions on $L^2(\mathbb{R}, \mathbb{C}^2)$.

Proof The closedness of the operator $\mathbf{B}\partial_x^2$ is discussed in Remark 3.2. By Theorem 3.1, $\sigma(\mathbf{B}\partial_x^2) = \{s(\beta \pm iD/2) \,|\, s \leq 0\}$, and so $\mathbb{R}^+ \subset \rho(\mathbf{B}\partial_x^2)$. To establish the bound (4.3) on the norm of the resolvent, we use the Fourier transform, $\mathcal{F} : L^2(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$, defined by

$$\widehat{f}(\xi) \;=\; \mathcal{F}[f](\xi) \;=\; \int_{\mathbb{R}} f(x) \exp(-2\pi i x \xi)\, dx. \tag{4.4}$$

Now, for $\mathbf{v} \in L^2(\mathbb{R}, \mathbb{C}^2)$

$$\mathcal{F}[R(\lambda : \mathbf{B}\partial_x^2)\mathbf{v}](\xi) \;=\; \mathbf{C}(\xi)\widehat{\mathbf{v}}(\xi), \tag{4.5}$$

where

$$\mathbf{C}(\xi) \;=\; (-4\pi\xi^2\mathbf{B} - \lambda)^{-1} \;=\; \frac{1}{d(\xi)} \begin{bmatrix} -4\pi\xi^2\beta - \lambda & -4\pi\xi^2 D/2 \\ 4\pi\xi^2 D/2 & -4\pi\xi^2\beta - \lambda \end{bmatrix}, \tag{4.6}$$

with $d(\xi) = (4\pi\xi^2\beta + \lambda)^2 + (4\pi\xi^2 D/2)^2$. By Parseval's Theorem,

$$\begin{aligned} \|R(\lambda : \mathbf{B}\partial_x^2)\mathbf{v}\|^2_{L^2(\mathbb{R},\mathbb{C}^2)} \;&=\; \|\mathbf{C}\widehat{\mathbf{v}}\|^2_{L^2(\mathbb{R},\mathbb{C}^2)} \\ &\leq\; \int_{\mathbb{R}} \|\mathbf{C}(\xi)\|^2_{\mathbb{C}^{2\times 2}} \|\widehat{\mathbf{v}}(\xi)\|^2_{\mathbb{C}^2}\, d\xi \;\leq\; \|\mathbf{C}\|^2_\infty \|\mathbf{v}\|^2_{L^2(\mathbb{R},\mathbb{C}^2)}, \end{aligned}$$

where $\|\mathbf{C}\|_\infty := \sup_{\xi\in\mathbb{R}} \|\mathbf{C}(\xi)\|_{\mathbb{C}^{2\times 2}}$. Since $\|\mathbf{C}(\xi)\|^2_{\mathbb{C}^{2\times 2}}$ is equal to the largest eigenvalue of $\mathbf{C}(\xi)^T\mathbf{C}(\xi) = d(\xi)^{-1}\mathbf{I}_{2\times 2}$, we conclude that $\|\mathbf{C}(\xi)\|_{\mathbb{C}^{2\times 2}} = d(\xi)^{-1/2} \leq \lambda^{-1}$, for all $\lambda > 0$. Since $\beta > 0$, we conclude that $\|R(\lambda : \mathbf{B}\partial_x^2)\|_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))} \leq \lambda^{-1}$, as required. □

Lemma 4.2 *Assume that Hypothesis 4.1 is met. Then, for all $t > 0$,*

$$\|\widetilde{\mathbf{M}}(t)\|_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))} \;\leq\; \|\widetilde{\mathbf{M}}\|_\infty \;<\; \infty. \tag{4.7}$$

Proof Let $\mathbf{u} \in L^2(\mathbb{R}, \mathbb{C}^2)$. Then,

$$\|\widetilde{\mathbf{M}}(t)\mathbf{u}\|^2_{L^2(\mathbb{R},\mathbb{C}^2)} \;\leq\; \int_{\mathbb{R}} \|\widetilde{\mathbf{M}}(t,x)\|^2_{\mathbb{C}^{2\times 2}} \|\mathbf{u}(x)\|^2_{\mathbb{C}^2}\, dx \;\leq\; \|\widetilde{\mathbf{M}}\|^2_\infty \|\mathbf{u}\|^2_{L^2(\mathbb{R},\mathbb{C}^2)}.$$

To show that $\|\widetilde{\mathbf{M}}\|_\infty < \infty$ we use Hypothesis 4.1 together with the fact that the matrix 2-norm is bounded by the Frobenius norm, $\|\mathbf{A}\|_{\mathbb{C}^{2\times 2}} \leq \|\mathbf{A}\|_F$, where $\|\mathbf{A}\|_F^2 := \sum_{i,j=1}^{2} |A_{ij}|^2$. □

Taken together, Theorem 2.3 of [39, Ch. 5] and Lemmas 4.1 and 4.2 imply that $\{\mathcal{L}(t)\}_{t\geq 0}$ is a stable family of infinitesimal generators of C_0-semigroups on $L^2(\mathbb{R}, \mathbb{C}^2)$, which is one of the assumptions required for the application of Theorem 4.8 of Pazy [39, Ch. 5]. The remaining assumption in that theorem also holds thanks to the following lemma.

Lemma 4.3 *Assume that Hypothesis 4.1 is met. Then, for each $\mathbf{v} \in H^2(\mathbb{R}, \mathbb{C}^2)$, we have that $F(\cdot) := \mathcal{L}(\cdot)\mathbf{v} : (0, \infty) \to L^2(\mathbb{R}, \mathbb{C}^2)$ is C^1.*

Proof We show that F is differentiable with $F'(t) = \partial_t \widetilde{\mathbf{M}}(t)\mathbf{v}$. The proof that F' is continuous is similar. First we observe that since $\boldsymbol{\psi}$ and $\boldsymbol{\psi}_t$ are assumed to be bounded, $\mathcal{L}(t)\mathbf{v}$ and $\partial_t \widetilde{\mathbf{M}}(t)\mathbf{v}$ are in $L^2(\mathbb{R}, \mathbb{C}^2)$. Next, we apply the fundamental theorem of calculus and use the equivalences between the matrix 2-norm and the Frobenius norm to obtain the estimate

$$\begin{aligned}
&\|F(t+h) - F(t) - hF'(t)\|^2_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&\leq \int_{\mathbb{R}} \|\widetilde{\mathbf{M}}(t+h,x) - \widetilde{\mathbf{M}}(t,x) - h\partial_t \widetilde{\mathbf{M}}(t,x)\|^2_{\mathbb{C}^{2\times 2}} \|\mathbf{v}(x)\|^2_{\mathbb{C}^2}\, dx \\
&= \int_{\mathbb{R}} \left\| \int_t^{t+h} \left[\partial_t \widetilde{\mathbf{M}}(\tau,x) - \partial_t \widetilde{\mathbf{M}}(t,x)\right] d\tau \right\|^2_{\mathbb{C}^{2\times 2}} \|\mathbf{v}(x)\|^2_{\mathbb{C}^2}\, dx \\
&\leq h^2 \sum_{i,j=1}^{2} \sup_{(\tau,x)\in(t,t+h)\times\mathbb{R}} \left(\left|\partial_t \widetilde{\mathbf{M}}_{i,j}(\tau,x) - \partial_t \widetilde{\mathbf{M}}_{i,j}(t,x)\right|^2 \right) \|\mathbf{v}\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&\leq 8h^2 \sup_{(\tau,x)\in(t,t+h)\times\mathbb{R}} \left\|\partial_t \widetilde{\mathbf{M}}(\tau,x) - \partial_t \widetilde{\mathbf{M}}(t,x)\right\|^2_{\mathbb{C}^{2\times 2}} \|\mathbf{v}\|_{L^2(\mathbb{R},\mathbb{C}^2)}.
\end{aligned}$$

The result now follows, since by Hypothesis 4.1, $\partial_t \widetilde{\mathbf{M}}$ is continuous in t, uniformly in x. □

□

5 The essential spectrum of the monodromy operator [Case: $\beta > 0$]

In this section, we define the monodromy operator that is the main focus of this paper, and derive a formula for its essential spectrum in terms of that of the asymptotic operator, $\mathcal{L}_\infty$. Our proof relies on the fact that the semigroup, $e^{t\mathcal{L}_\infty}$, associated with $\mathcal{L}_\infty$ is analytic, which only holds when $\beta > 0$. Therefore, although the results in this section apply to the CQ-CGL breathers in Example 2.2, they do not apply to the KM breather in Example 2.1.

Definition 5.1 Let ψ be a periodically-stationary solution of the constant coefficient CQ-CGL equation (2.6) and let $s \in \mathbb{R}$. The *monodromy operator*, $\mathcal{M}(s)$, associated with the linearization (2.10) of (2.6) about ψ is the bounded operator on $L^2(\mathbb{R}, \mathbb{C}^2)$ defined by $\mathcal{M}(s) = \mathcal{U}(s+T, s)$, where $\mathcal{U}$ is the evolution operator of Theorem 4.1 and T is the period of ψ.

Theorem 5.1 *Suppose that $\beta > 0$ and that Hypotheses 3.1 and 4.1 are met. Then the C^0-semigroup, $e^{t\mathcal{L}_\infty}$, generated by $\mathcal{L}_\infty$ is analytic. Furthermore, the essential spectrum of the evolution operator, $\mathcal{U}(t,s)$, is given by*

$$\sigma_{\mathrm{ess}}(\mathcal{U}(t,s)) = \sigma_{\mathrm{ess}}(e^{(t-s)\mathcal{L}_\infty}). \tag{5.1}$$

Therefore, the essential spectrum of the monodromy operator, $\mathcal{M}(s)$, is given by

$$\sigma_{\mathrm{ess}}(\mathcal{M}(s)) = \sigma_{\mathrm{ess}}(e^{T\mathcal{L}_\infty}), \tag{5.2}$$

which is independent of s.

Before proving this theorem, we state and prove a Corollary.

Corollary 5.1 *Suppose that $\beta > 0$ and Hypothesis 4.1 is met. Then, the essential spectrum of the monodromy operator, $\mathcal{M}(s)$, is given by*

$$\sigma_{\mathrm{ess}}(\mathcal{M}(s)) \setminus \{0\} \;=\; e^{T\sigma_{\mathrm{ess}}(\mathcal{L}_\infty)}. \tag{5.3}$$

In particular, in the case of a CQ-CGL breather, if $\delta < 0$ then the essential spectrum lies inside a circle of radius $e^{\delta T} < 1$.

Proof Since the spectral mapping theorem holds for the point spectrum of a C_0-semigroup [15], $\sigma_{\mathrm{pt}}(e^{T\mathcal{L}_\infty})\setminus\{0\} = e^{T\sigma_{\mathrm{pt}}(\mathcal{L}_\infty)} = \emptyset$, by Theorem 3.1. Since $\sigma_{\mathrm{pt}}^{\mathcal{F}} \subset \sigma_{\mathrm{pt}}$, $\sigma_{\mathrm{pt}}^{\mathcal{F}}(e^{T\mathcal{L}_\infty}) \setminus \{0\} = \emptyset$. Consequently, $\sigma_{\mathrm{ess}}(e^{T\mathcal{L}_\infty}) \setminus \{0\} = \sigma(e^{T\mathcal{L}_\infty}) \setminus \{0\} = e^{T\sigma(\mathcal{L}_\infty)}$, since the spectral mapping theorem also holds for the spectrum of an analytic semigroup [15]. The result now follows since $\sigma(\mathcal{L}_\infty) = \sigma_{\mathrm{ess}}(\mathcal{L}_\infty)$. □

The proof of Theorem 5.1 relies on the following two lemmas.

Lemma 5.1 *Suppose that Hypothesis 4.1 is met. Then*

$$\mathcal{U}(t,s) \;=\; e^{(t-s)\mathcal{L}_\infty} \;+\; \int_s^t e^{(t-\tau)\mathcal{L}_\infty}\,\mathbf{M}(\tau)\,\mathcal{U}(\tau,s)\,d\tau, \qquad \text{in } \mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2)). \tag{5.4}$$

Proof We refer to Engel and Nagel [15, App. C] for a summary of the theory of Lebesgue integration for functions, $f : J \to X$, from an interval $J \subset \mathbb{R}$ to a Banach space, X. As in the proof of Theorem 4.1, the asymptotic operator, $\mathcal{L}_\infty$, generates a C_0-semigroup on $L^2(\mathbb{R},\mathbb{C}^2)$. For each $\tau \in [s,t]$ and $\mathbf{v} \in H^2(\mathbb{R},\mathbb{C}^2)$, let $f(\tau) := \mathbf{M}(\tau)\mathcal{U}(\tau,s)\mathbf{v} \in L^2(\mathbb{R},\mathbb{C}^2)$. By Hypothesis 4.1, $f \in L^1([s,t], L^2(\mathbb{R},\mathbb{C}^2))$. Since $\mathcal{L}(t) = \mathcal{L}_\infty + \mathbf{M}(t)$, the result follows from the variation of parameters formula (see Corollary 2.2 of [39, Ch. 4], together with Theorem 4.1 above).

Corollary 10.6 of [39, Ch. 1] implies that $(e^{t\mathcal{L}_\infty})^* = e^{t\mathcal{L}_\infty^*}$, where $\mathcal{L}_\infty^* = \mathbf{B}^T\,\partial_x^2 + \mathbf{M}_\infty^T$ is the adjoint of $\mathcal{L}_\infty$.

Lemma 5.2 *Suppose that $\beta > 0$ and Hypothesis 4.1 is met. Then the semigroups $e^{t\mathcal{L}_\infty}$ and $e^{t\mathcal{L}_\infty^*}$ are analytic.*

Proof We will show that for all $\sigma > 0$ and $\tau \neq 0$,

$$\|R(\sigma + i\tau : \mathbf{B}\partial_x^2)\| \;\le\; \frac{\sqrt{1 + (D/2\beta)^2}}{|\tau|}. \tag{5.5}$$

Therefore, by Theorem 5.2 of [39, Ch. 2] (and the discussion preceeding it), $\mathbf{B}\partial_x^2$ is the infinitesimal generator of an analytical semigroup. Since $\mathcal{L}_\infty = \mathbf{B}\partial_x^2 + \mathbf{M}_\infty$, where $\mathbf{M}_\infty$ is a bounded operator, it follows from Corollary 2.2 of [39, Ch. 3] that $\mathcal{L}_\infty$ is the infinitesimal generator of an analytical semigroup. The same argument holds for the adjoint. Note that as $\beta \to 0$, the constant on the right-hand side of (5.5) blows up. Consequently, this proof cannot be extended to the case $\beta = 0$.

To establish (5.5), as in the proof of Lemma 4.1, and with $\lambda = \sigma + i\tau$, we observe that

$$\|R(\sigma + i\tau : \mathbf{B}\partial_x^2)\|^2 \;\le\; \sup_{\xi\in\mathbb{R}} \mu_{\max}[\mathbf{C}(\xi)^*\mathbf{C}(\xi)] \tag{5.6}$$

$$= \left(\inf_{\xi\in\mathbb{R}} \mu_{\min}[(-4\pi\xi^2\mathbf{B} - \lambda)(-4\pi\xi^2\mathbf{B}^T - \overline{\lambda})]\right)^{-1}, \tag{5.7}$$

since the largest eigenvalue of a non-negative definite Hermitian matrix is the inverse of the smallest eigenvalue of its inverse. Let $a = 4\pi\xi^2\beta > 0$ and $b = 2\pi\xi^2 D$. A calculation shows that $\mu_{\min} = |a+\lambda|^2 + b^2 - 2|b\tau|$. If we let $z = -a + i\,\mathrm{sgn}(\tau b)b$, then $\mu_{\min} = |z-\lambda|^2$. The points, z, lie on the half-line in the left-half plane given by $y = -\mathrm{sgn}(\tau)mx$ with $x < 0$, where $m = |D/2\beta|$, while λ is in the right half-plane. We observe that z and λ either both lie above or both lie below the real axis. Consequently, the problem of minimizing $\mu_{\min}$ as a function of ξ is that of finding the minimum of the square of the distance from λ to this half-line, which is greater than the minimum squared distance to the full line. Therefore,

$$\|R(\sigma + i\tau : \mathbf{B}\partial_x^2)\| \;\le\; \frac{\sqrt{1+m^2}}{m\sigma + |\tau|} \;\le\; \frac{\sqrt{1+m^2}}{|\tau|}, \tag{5.8}$$

as required. □

Proof of Theorem 5.1 Since the essential spectrum is invariant under compact perturbations, to prove the theorem, we just need to show that the integral in (5.4) is a compact operator. By Engel and Nagel [15, Theorem C.7], it suffices to show that for each $\tau \in (s,t)$ the integrand $\mathcal{K}(\tau) = e^{(t-\tau)\mathcal{L}_\infty}\mathbf{M}(\tau)\mathcal{U}(\tau,s)$ is compact, and that the function $\mathcal{K} : (s,t) \to \mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))$ is strongly continuous, in that for all $\mathbf{v} \in L^2(\mathbb{R},\mathbb{C}^2)$, $\|\mathcal{K}(\tau)\mathbf{v} - \mathcal{K}(\tau_0)\mathbf{v}\|_{L^2(\mathbb{R},\mathbb{C}^2)} \to 0$ as $\tau \to \tau_0$.

To show that $\mathcal{K}(\tau)$ is compact we show that the adjoint, $\mathcal{K}^*(\tau)$, is compact. As in Theorem 3.1, $\mathcal{L}^*(\tau)$ is a relatively compact perturbation of $\mathcal{L}_\infty^*$. Therefore there is a $\lambda \in \rho(\mathcal{L}_\infty^*)$ so that $\mathbf{M}^*(\tau)(\mathcal{L}_\infty^* - \lambda)^{-1} \in \mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))$ is compact. Since the semigroup $e^{t\mathcal{L}_\infty^*}$ is analytic, the operator $(\mathcal{L}_\infty^* - \lambda)e^{(t-\tau)\mathcal{L}_\infty^*}$ is bounded (see Theorem 5.2 of [39, Ch. 2] together with the discussion preceding that result). Therefore the composition,

$$\mathcal{K}^*(\tau) = \mathcal{U}^*(\tau,s)\,\mathbf{M}^*(\tau)(\mathcal{L}_\infty^* - \lambda)^{-1}(\mathcal{L}_\infty^* - \lambda)e^{(t-\tau)\mathcal{L}_\infty^*}, \tag{5.9}$$

is also compact.

Finally, by Hypothesis 4.1, the function $\mathbf{M} : (s,t) \to \mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))$ is uniformly bounded and strongly continuous. Furthermore, by Theorem 4.1, $\mathcal{U}(\cdot,s)$ is uniformly bounded and strongly continuous for each s. Therefore, $\mathcal{K}(\tau)$ is strongly continuous, since the composition of uniformly bounded, strongly continuous functions is strongly continuous. □

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