NODAL LINE ESTIMATES FOR THE SECOND DIRICHLET EIGENFUNCTION

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Abstract. We study the nodal curves of low energy Dirichlet eigenfunctions in generalized curvilinear quadrilaterals. The techniques can be seen as a generalization of the tools developed by Grieser-Jerison in a series of works on convex planar domains and rectangles with one curved edge and a large aspect ratio. Here, we study the structure of the nodal curve in greater detail, in that we find precise bounds on its curvature, with uniform estimates up to the two points where it meets the domain at right angles, and show that many of our results hold for relatively small aspect ratios of the side lengths. We also discuss applications of our results to Courant-sharp eigenfunctions and spectral partitioning.

1. Introduction and statement of results

Understanding the fundamental modes of vibration of a compact domain is a longstanding problem. The original motivation was to describe how a metal sheet with a given shape would vibrate when struck at some fundamental frequency. The main goal being to understand the structure of the set of points in the sheet that are stable, i.e. that are not vibrating. These non-vibrating regions are the zero sets of the Laplace eigenfunctions corresponding to solving the Helmholtz equation on the domain that represents the metal sheet. In the 17th century R. Hook observed these patterns by spilling sand on a glass sheet, and striking the sheet with a violin bow. When the sheet starts vibrating, the sand rearranges itself across the sheet until it is placed on the non-vibrating areas, thus exhibiting the zero sets for the corresponding eigenfunction. This experiment was later reproduced by E. Chladni, who was the first to record an extensive list of zero set configurations. It is nowadays known as the Chladni plates experiment.

We dedicate this article to giving a precise description of the structure of the zero set of the second eigenfunction for a planar domain whose shape is obtained after perturbing a rectangle. While we focus on the second Dirichlet eigenfunction, the techniques developed here can be applied to the low-lying eigenfunctions in general up to a frequency depending upon the length of the domain. Also, there are natural generalizations to Neumann (or more generally Robin) boundary conditions, but for the sake of clarity and presentation we will focus on Dirichlet domains at present.

We note that low-lying eigenvalues and eigenfunctions of the Laplacian on a compact domain also play a role in understanding random walks ([KP89]), heat conductivity ([S$^+$96]) and more. See for instance the recent works [Zel17, Ste17] and references therein for a nice overview of applications and modern topics in the theory of eigenfunctions and nodal sets.

In this work, we study the second Dirichlet eigenfunction of the Laplacian on a planar domain Ω, so that

$$\begin{cases} \Delta v(x,y) &= -\mu v(x,y) \quad \text{in } \Omega \\ v(x,y) &= 0 \quad\quad\quad\ \ \text{on } \partial\Omega \end{cases}$$

where μ is the corresponding eigenvalue. The domain Ω is a curvilinear rectangle that is very nearly rectangular in a Gromov-Hausdorff sense to be made precise below in (1). For convenience, we normalize v so that $\|v\|_{L^\infty} = 1$. We are interested in studying the nodal set of v, which we denote by

$$\Gamma = \{(x,y) \in \mathring{\Omega} : v(x,y) = 0\}.$$

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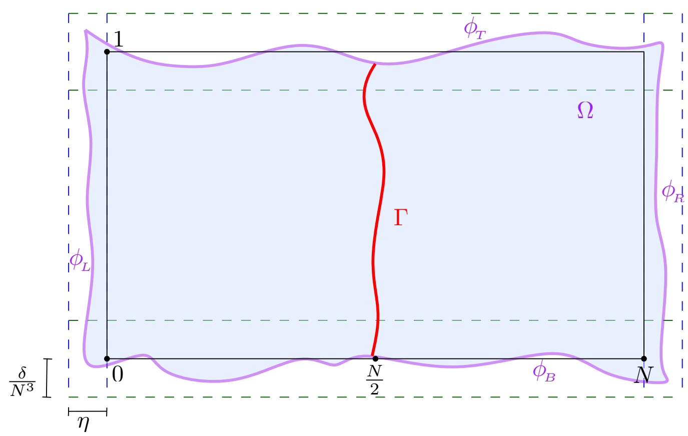
To do this we build off the pioneering works of Jerison [Jer95] and Grieser-Jerison [GJ96, GJ98, GJ09], who studied the low energy eigenfunctions in convex domains and rectangles of high aspect ratio with one curved edge. For convex domains they studied the location of the maximum and nodal line of the first and second Dirichlet eigenfunction respectively, giving estimates that are uniform as the eccentricity of the domain increases. They also derived a method to do a very detailed asymptotic analysis of the location of the nodal line or the location of the maximum for low energy Dirichlet eigenfunction in a rectangle with one curved side. This method is the starting point for our work on curvilinear rectangles.

On a rectangle, the zero set of the second eigenfunction for the Laplacian is a straight line, perpendicular to the long sides, that divides the rectangle in two equal pieces. Here, we study the zero set of the second eigenfunction v on a region Ω that is a perturbation of a rectangle. Indeed, we extend the results in [GJ09] to explore the precise dependence of the nodal set on the properties of the bounding curves and the aspect ratio of the underlying region Ω.

We obtain estimates on the width and regularity of the nodal line of v, with explicit bounds, tracking how the slope and curvature of the nodal line depend upon the top and bottom curve. In particular, we show that there are distinct differences in the nodal line depending upon if some of the bounding curves are flat or if they are curved. As we are imposing Dirichlet boundary conditions, the eigenfunction v vanishes on the boundary of the domain, and analyzing the behavior of the nodal line becomes increasingly delicate as it approaches the boundary. Our techniques allow us to obtain estimates that are uniform up to the boundary and show that the nodal line meets the boundary of Ω orthogonally at two points.

To state our results precisely, we first define the class of domains Ω under consideration. Let $\phi_B, \phi_T, \phi_L, \phi_R$ be functions defining a region Ω in $\mathbb{R}^2$ that is a perturbation of the rectangle $[0,N]\times[0,1]$, for $N>0$, of the form

$$\Omega = \{(x,y)\in\mathbb{R}^2 : \ x\in[-\tfrac{1}{2}, N+\tfrac{1}{2}], \ y\in[-\tfrac{1}{2},\tfrac{3}{2}], \ \phi_B(x)\le y\le\phi_T(x), \ \phi_L(y)\le x\le\phi_R(y)\}. \qquad (1)$$



The functions ϕ_L, ϕ_R defining the sides of the domain are in $C^2([-\frac{1}{2},\frac{3}{2}];\mathbb{R})$, with

$$-\eta\le\phi_L\le 0, \qquad N\le\phi_R\le\eta+N, \qquad \left|\frac{d^j}{dy^j}\phi_L\right|\le\eta, \qquad \left|\frac{d^j}{dy^j}\phi_R\right|\le\eta, \qquad (2)$$

for $j=1,2$ and some $\eta>0$. The functions ϕ_B, ϕ_T defining the top and bottom are in $C^\infty([-\frac{1}{2},N+\frac{1}{2}];\mathbb{R})$, with

$$|\phi_B|\le\frac{\delta}{N^3}, \qquad |\phi_T-1|\le\frac{\delta}{N^3}, \qquad \left|\frac{d^j}{dx^j}\phi_B\right|\le\tilde{C}_j\frac{\delta}{N^3}, \qquad \left|\frac{d^j}{dx^j}\phi_T\right|\le\tilde{C}_j\frac{\delta}{N^3}, \qquad (3)$$

for $j\ge1$ and constants $\tilde{C}_j>0$. Here $0<\delta<\frac{1}{5}$, and $N\ge5$. Note that as η, δ tend to 0, the domain Ω becomes rectangular.

In the case of the rectangle $[0,N]\times[0,1]$ with $N>1$, the second Dirichlet eigenfunction is given by $v(x,y)=\sin\left(\frac{2\pi x}{N}\right)\sin(\pi y)$ and the nodal line Γ is precisely the straight line $\frac{N}{2}\times(0,1)$. The theorem below shows how Γ changes under the above perturbations of the rectangle. Let $\pi_x : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the x-axis.

Theorem 1.1. *There exist $c > 0$, $C > 0$, such that $\pi_x(\Gamma) \subset [\frac{N}{2} - C(\eta + \delta), \frac{N}{2} + C(\eta + \delta)]$ and has diameter bounded by*

$$\operatorname{diam}(\pi_x(\Gamma)) \leq C\left(\eta e^{-cN} + \frac{\delta}{N^2}\right).$$

Moreover, there exists a function $g(y)$ such that $\Gamma \cap \mathring{\Omega} = \{(x, y) \in \mathring{\Omega} : x = g(y)\}$, with

$$|g'(y)| + |g''(y)| \leq C\left(\eta e^{-cN} + \frac{\delta}{N^2}\right).$$

The nodal line $\Gamma \cap \mathring{\Omega}$ touches the boundary of Ω at precisely 2 points, and it meets the boundary orthogonally at these points.

Here and throughout, constants denoted by c, C, C_1, etc, depend on the constants $\tilde{C}_j$, but are independent of η, δ, and N. (In fact we will only require control on derivatives up to $j = 5$.)

Increasing the length of the rectangle by a perturbation of size η decreases the eigenvalue by $O(\eta N^{-3})$, while increasing the width of the rectangle by a perturbation of size δN^{-3} decreases the eigenvalue by $O(\delta N^{-3})$. Thus, by our choice of perturbations to the rectangle, when η and δ are of comparable size, each perturbation leads to the same change in the eigenvalue. Increasing the length of the rectangle by a perturbation of size η should move the nodal set by an amount η (see, for example, Theorem 1 in [GJ96]). Analogous to this, we will see that a global y-perturbation of size δN^{-3} leads to the nodal line moving by an amount at most δ from the unperturbed case. Near the nodal set we will show that the x-derivative of the eigenfunction is of size N^{-1} and the y-perturbation gives an error of size δN^{-3} to the eigenfunction from a sinusoidal function. This is the reason for the error term δN^{-2} appearing in the width and curvature of the nodal line.

An immediate feature to note is that in the special case of flat upper and lower boundaries ($\phi_B(x) \equiv 0$, $\phi_T(x) \equiv 1$), we can set $\delta = 0$ and the factor of $\frac{\delta}{N^2}$ does not appear in the estimates of Theorem 1.1. Therefore, in this flat case the diameter of the nodal line, $\operatorname{diam}(\pi_x(\Gamma))$, is exponentially small in N (rather than the polynomial decay in N when $\delta \neq 0$).

From Theorem 1.1 we see that for N sufficiently large (and $\delta \leq C\eta$), the perturbation of the nodal line from straight is smaller than that of the side perturbations $\phi_L(x)$, $\phi_R(x)$. In the flat case, $\phi_B(x) \equiv 0$, $\phi_T(x) \equiv 1$, we can track the constants in the proof of Theorem 1.1 (see Section 5) to obtain an explicit lower bound on the size of N required for this to occur:

Corollary 1.1. *There exists a constant $N_0 > 0$ such that for $N \geq N_0$ and $\delta \leq \eta$,*

$$\operatorname{diam}(\pi_x(\Gamma)) \leq \tfrac{\eta}{2} \qquad \text{and} \qquad |g'(y)| + |g''(y)| \leq \tfrac{\eta}{2}.$$

In the flat case, $\phi_B(x) \equiv 0$, $\phi_T(x) \equiv 1$, for each $N \geq 8$, we can take $\eta < \eta(N_0)$ and the above estimates hold.

By controlling the behavior of the nodal line up to the boundary, we are able to show for the class of domains under consideration that the nodal line is not closed, but meets the boundary (orthogonally) at two points. More generally, Payne [Pay67] conjectured that the nodal line of the second eigenfunction of a bounded planar domain touches the boundary at 2 points. This was proved for smooth, convex domains by Melas [Mel92], but a counterexample (for a non-simply connected planar domain) was given by Hoffmann-Ostenhof, Hoffmann-Ostenhof, and Nadirashvili [HOHON97].

In [FK08], Freitas and Krejčiřík study the Dirichlet Laplacian for a class of thin curved tubes. As the volume of the cross-section tends to 0 they establish the convergence of the eigenvalues and eigenfunctions in terms of an ordinary differential operator on the base curve of the tube. In particular, they locate the nodal set to sufficient precision to also deduce that the nodal set must intersect the boundary. Krejčiřík and Tušek also prove an analogous result for domains consisting of a thin tubular neighborhood of a hypersurface, [KT15]. The idea of reducing to an associated ordinary differential operator has also been used extensively by Friedlander-Solomyak [FS08], [FS09] and Borisov-Freitas [BF09] to obtain asymptotics of the eigenvalues, eigenfunctions, and resolvent of the Dirichlet Laplacian in thin domains.

Applications to partitioning algorithms. Recently, in work on graph and data partitioning algorithms, Szlam et al in [SMCB05, Szl09] observed the following. If one studies a dense graph that is properly embedded in a domain in $\mathbb{R}^2$ using iterated cuts along nodal sets of the first eigenvector of the graph Laplacian, then the regions tend towards rectangles of bounded aspect ratios. Here, the iteration must be stopped after a finite number of cuts depending upon the graph density in order for the sets to have a geometric interpretation (and hence "rectangles" being recognizable) rather than just a combinatorial set of vertices of the graph. The underlying idea of graph partitions are for instance to cluster data points or to provide a good foundation for a wavelet basis to name just two. There is also the continuum limit version of this, in which one could ask to partition a planar domain using the first non-trivial Neumann or second Dirichlet eigenfunction respectively. Again, for a very general boundary, one expects that such partitions would converge rapidly to a set of near rectangles.

We can use Corollary 1.1 to illustrate such a convergence is possible at least at the outset. Let us focus on the flat top and bottom case for simplicity, with $N \geq 8$, and $\eta = \eta(N)$ sufficiently small. Given such a domain Ω, we can form 2 new domains by *cutting* Ω along the nodal line Γ. Using the estimates above, we can ensure that these two new domains are of the form of Ω but with roughly $N_1 \approx N/2$ length scale in the x direction, and that one side of the new domain satisfies a stronger curvature bound than that of Ω. We can continue to iterate this procedure j times until $N/2^j \approx N_j \leq 8$, with the curvature of one side decreasing in each iteration by Corollary 1.1. However, at some point in the process we once again arrive at a curvilinear quadrilateral with aspect ratio small enough such that the constants fall outside the scope of our strong quantitative estimates. The fluctuations do not decrease as clearly using our methods beyond that point as we would need stronger control over the constants c, C in Theorem 1.1, and hence stronger control on the constituent curves of the domain. Cutting along the nodal line will thus result in a dynamical system of domains with bounded aspect ratios such that the spectral cuts rely on the structure of the iterated component curves. Analogously, for the general top and bottom boundaries considered here, using the estimates in Corollary 1.1, given η, δ, with $\delta \leq \eta$, and for N sufficiently large, we can repeat the above scenario, to again give a possible sequence of domains converging to a rectangle up to a point where we saturate the aspect ratios for which we can prove strong decreasing bounds on the curvature of the component curves. We leave this as a conjecture and focus here on proving quantitative estimates in nearly rectangular domains.

Another partitioning related to the nodal set of Dirichlet eigenfunctions of the Laplacian is the following: Given a domain Ω and integer $k \geq 2$, a spectral minimal k-partition of Ω is a partition of Ω into k disjoint sets Ω_i that minimizes $\max_i \lambda(\Omega_i)$. Here $\lambda(\Omega_i)$ is the first Dirichlet eigenvalue of Ω_i. If $k = 2$, then the spectral minimal partition is given by the nodal domains of a second Dirichlet eigenfunction of Ω. More generally, if a k-th Dirichlet eigenfunction has exactly k-nodal domains (and so gives equality in the Courant nodal domain Theorem), then these nodal domains form a minimal k-partition. See the survey paper of Helffer [Hel10] for greater discussion of spectral minimal partitions and references. It is therefore important to classify examples where the Courant nodal domain Theorem is sharp. For instance, it is a direct computation to observe that the third Dirichlet eigenfunction of the rectangle has three nodal domains whenever the aspect ratio is greater than $\sqrt{8/3}$. This result and many others on spectral partitions and Courant sharp eigenfunctions can be found for instance in the seminal work [HHOT09]. Using the techniques presented here, for any fixed $j \geq 2$, by taking N sufficiently large, for all $2 \leq k \leq j$, the k-th Dirichlet eigenfunction of the perturbed rectangle has exactly k nodal domains (with nodal set approximately equal to the union of the $k-1$ lines $\{\frac{l}{k}N\} \times [0, 1]$ for $1 \leq l \leq k-1$). Thus, in this case, the nodal domains will provide a spectral minimal k-partition.

Outline of the paper. The structure of the rest of the paper as follows: In Section 2 we describe an adiabatic approximation of the eigenfunction that is a key ingredient in the proof of Theorem 1.1. This type of approximation, which can be viewed as an approximate separation of variables for our approximately rectangular domain, has been used in the work of Grieser and Jerison [GJ96], [GJ09]. The approximation has also been used in [BSS97] for numerical analysis of eigenfunctions in partially rectangular billiards, and in [HM12] to analyze non-concentration of eigenfunctions in partially rectangular billiards. In Section 3, we establish the desired properties of the width and regularity of the nodal line using the adiabatic approximation. Then, in Section 4 we demonstrate how in the flat case, we have simple ODE estimates to establish the approximation, and following this, we prove the error estimates for the approximation for our general class of domains. Lastly, in Section 5 we compute an explicit Hadamard variation formula to evaluate the effect the side perturbations have on the eigenfunction. This will in particular allow us to track the constants appearing in the proof of Theorem 1.1 in the flat case and prove Corollary 1.1.

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2. The Adiabatic Ansatz

A key ingredient in the proof of Theorem 1.1 is to establish properties of a Fourier decomposition of the eigenfunction $v(x,y)$. For convenience, we introduce the height function

$$h(x) := \phi_{T}(x) - \phi_{B}(x),$$

and note that by (3) we have $1 - \frac{2\delta}{N^3} \leq h(x) \leq 1 + \frac{2\delta}{N^3}$ for all $x \in [0,N]$. For $(x,y) \in \Omega$ with $0 \leq x \leq N$, we write v as

$$v(x,y) = v_1(x)\sin(\beta(x,y)) + E(x,y), \tag{4}$$

where

$$\beta(x,y) := \frac{\pi(y - \phi_{B}(x))}{h(x)}$$

and where the function $v_1(x)$ is given by

$$v_1(x) = \frac{2}{h(x)} \int_{\phi_{B}(x)}^{\phi_{T}(x)} v(x,y)\sin(\beta(x,y))\,\mathrm{d}y.$$

We will view the first term in the right hand side of (4) as the main term, with $E(x,y)$ an error term when N is large, and δ, η are small. The function $v_1(x)$ is the first Fourier mode in the y-direction. To prove Theorem 1.1 we will use this decomposition of v, and will require a lower bound on $|v_1'(x)|$, together with upper bounds on $v_1(x)$, $E(x,y)$ and their derivatives. In fact, to prove the estimates on regularity of the nodal line near $\partial\Omega$, we need to consider a larger class of decompositions of v: Given $(x_0,y_0) \in \Gamma$, suppose that $(x_1,y_1) \in \partial\Omega$, with $y_1 = \phi_{B}(x_1)$ and $d((x_0,y_0),\partial\Omega) = d((x_0,y_0),(x_1,y_1))$. We now rotate the domain so that (x_1,y_1) is vertically below (x_0,y_0). More precisely, we intoduce the new coordinates $(\tilde{x},\tilde{y}) = F^{-1}(x,y)$, where F is the linear isometry obtained by rotating around (x_0,y_0) followed by a vertical shift so that $(x_1,y_1) = F(\tilde{x}_0,0)$. We then define $w(\tilde{x},\tilde{y})$ to be equal to the eigenfunction v in these rotated coordinates,

$$w = v \circ F.$$

Remark 2.1. *By the bounds on $\phi_{B}'(x)$ and $\phi_{T}'(x)$ from (3), there exists $C > 0$ such that the angle of rotation is bounded by $C\frac{\delta}{N^3}$.*

The function w satisfies

$$(\Delta + \mu)w = 0$$

in the domain

$$\tilde{\Omega} = \{(\tilde{x}, \tilde{y}) : \rho_{L}(\tilde{y}) \le \tilde{x} \le \rho_{R}(\tilde{y}),\ \rho_{B}(\tilde{x}) \le \tilde{y} \le \rho_{T}(\tilde{x})\},$$

with $w|_{\partial\tilde{\Omega}} = 0$. In particular, for $0 \le \tilde{x} \le N$, we have $w(\tilde{x}, \rho_{B}(\tilde{x})) = w(\tilde{x}, \rho_{T}(\tilde{x})) = 0$. Here ρ_{B}, ρ_{T} satisfy the bounds

$$|\rho_{B}(\tilde{x})| \le \frac{2\delta}{N^3}(1 + |\tilde{x} - \tilde{x}_0|), \qquad |\rho_{T}(\tilde{x}) - 1| \le \frac{2\delta}{N^3}(1 + |\tilde{x} - \tilde{x}_0|),$$

$$\left|\frac{d^j}{d\tilde{x}^j}\rho_{B}(\tilde{x})\right| \le 2\tilde{C}_j\frac{\delta}{N^3}, \qquad \left|\frac{d^j}{d\tilde{x}^j}\rho_{T}(\tilde{x})\right| \le 2\tilde{C}_j\frac{\delta}{N^3}, \tag{5}$$

$j \ge 1$. Up to the factor of 2, the derivative bounds are the same as for ϕ_{B}, ϕ_{T}. Moreover, by the construction of F, we have $\rho_{B}(\tilde{x}_0) = \rho_{B}{}'(\tilde{x}_0) = 0$. The functions $\rho_L(\tilde{y})$, $\rho_R(\tilde{y})$ satisfy

$$-\eta - \frac{\delta}{N^3} \le \rho_{L}(\tilde{y}) \le \frac{\delta}{N^3}, \qquad -\frac{\delta}{N^3} \le \rho_{R}(\tilde{y}) - N \le \eta + \frac{\delta}{N^3},$$

$$\left|\frac{d^j}{d\tilde{y}^j}\rho_{L}(\tilde{y})\right| \le \eta + \frac{\delta}{N^3}, \qquad \left|\frac{d^j}{d\tilde{y}^j}\rho_{R}(\tilde{y})\right| \le \eta + \frac{\delta}{N^3}, \tag{6}$$

for $j = 1, 2$. We can make the analogous definition if the closest point to (x_0, y_0) lies on the upper boundary of Ω. For ease of notation, we now drop the tildes, and for each function $w(x, y)$ coming from such a rotation, for $x \in [0, N]$ we write

$$w(x, y) = w_1(x)\sin(\tilde{\beta}(x, y)) + \tilde{E}(x, y), \tag{7}$$

where

$$\tilde{\beta}(x, y) := \frac{\pi(y - \rho_{B}(x))}{\tilde{h}(x)},$$

for the new height function $\tilde{h}(x) = \rho_{T}(x) - \rho_{B}(x)$. To prove Theorem 1.1, we will use the proposition below which gives properties of these decompositions.

Proposition 2.1. *There exist positive constants c, C such that the following properties hold: For each decomposition, there exists a unique point $x_0 \in [\frac{1}{4}N, \frac{3}{4}N]$ such that $w_1(x_0) = 0$, and this point lies in the interval $[\frac{N}{2} - C(\eta + N\delta), \frac{N}{2} + C(\eta + N\delta)]$. Moreover, for $x \in [\frac{1}{4}N, \frac{3}{4}N]$,*

$$|w_1'(x)| \ge C^{-1}N^{-1},$$

and for $x \in [1, N-1]$, $0 \le j \le 3$, we have

$$|w_1^{(j)}(x)| \le CN^{-j}, \qquad \sup_{y \in [\rho_B(x), \rho_T(x)]}\left|\nabla^j\tilde{E}(x, y)\right| \le C\left(\eta e^{-cN} + \frac{\delta}{N^3}\right).$$

Proposition 2.1 is proved in Section 4.

Remark 2.2. *When the rotation is trivial, w is equal to v and the decomposition reduces to the one for v given in (4). Therefore, the properties in this proposition also hold for $v_1(x)$ and $E(x, y)$. In fact, in this case, the unique point x_0 where $v_1(x_0) = 0$ lies in the interval*

$$[\tfrac{N}{2} - C(\eta + \delta)\ ,\ \tfrac{N}{2} + C(\eta + \delta)].$$

3. Estimates on the nodal line

In this section we will prove Theorem 1.1 assuming that Proposition 2.1 holds. We first establish an upper bound on the width of the projection to the x-axis of the nodal line in terms of the error E and its derivatives. We will require a different argument to control the behavior of the nodal line near the boundary, and so we set

$$S(x) = S^B(x) \cup S^T(x) := [\phi_B(x), \phi_B(x) + \tfrac{1}{4}] \cup [\phi_T(x) - \tfrac{1}{4}\,,\, \phi_T(x)].$$

We continue to write $h(x) = \phi_T(x) - \phi_B(x)$ and $\beta(x,y) = \pi(y - \phi_B(x))/h(x)$. Since $h(x) \geq \frac{1}{2}$ for all x, we have that $0 \leq \beta(x,y) \leq \frac{\pi}{2}$ on S^B and $\frac{3\pi}{2} \leq \beta(x,y) \leq \pi$ on S^T. Therefore, this choice yields

$$\sin(\beta(x,y)) \geq \frac{2}{\pi}\beta(x,y) = \frac{2(y - \phi_B(x))}{h(x)} \quad y \in S^B(x),$$

and similarly

$$\sin(\beta(x,y)) \geq \frac{2(\phi_T(x) - y)}{h(x)} \quad y \in S^T(x).$$

Applying Proposition 2.1, we let x_0 be the unique point in the interval $[\frac{1}{4}N, \frac{3}{4}N]$ where $v_1(x_0) = 0$. Using the decomposition of v from (4), define $\tilde{I}$ to be the smallest interval with $x_0 \in \tilde{I}$ and such that

A) $\displaystyle \sup_{\substack{x\in[0,N]\\ y\in S(x)^c}} |E(x,y)| < \frac{1}{2} \inf_{x\in \tilde{I}^c} \frac{|v_1(x)|}{h(x)}$

B) $\displaystyle \sup_{\substack{x\in[0,N]\\ y\in S(x)}} |\partial_y E(x,y)| < 2 \inf_{x\in \tilde{I}^c} \frac{|v_1(x)|}{h(x)}$

Lemma 3.1. *If $x \in \tilde{I}^c$, then $v(x,y) \neq 0$ for all $y \in (\phi_B(x), \phi_T(x))$.*

Proof of Lemma 3.1: Let $y \in S(x)^c$. Then, $\sin(\beta(x,y)) \geq \sin(\beta(x, \phi_B(x) + \frac{1}{4})) \geq \frac{1}{2h(x)}$ and so

$$|v(x,y)| = |v_1(x)\sin(\beta(x,y)) + E(x,y)| \geq |v_1(x)|\tfrac{1}{2h(x)} - |E(x,y)| > 0. \tag{8}$$

By assumption (A), this is strictly positive. Now let $y \in S^B(x)$. Then, since $E(x,\phi_B(x)) = 0$, we have $|E(x,y)| \leq (y - \phi_B(x)) \sup_{y\in S^B(x)} |\partial_y E(x,y)|$. Also, using that $\sin(\beta(x,y)) \geq \frac{2(y-\phi_B(x))}{h(x)}$ we obtain

$$|v(x,y)| = |v_1(x)\sin(\beta(x,y)) + E(x,y)| \geq (y - \phi_B(x))\Big(|v_1(x)|\tfrac{2}{h(x)} - \sup_{y\in S^B(x)} |\partial_y E(x,y)|\Big) > 0, \tag{9}$$

and by assumption (B) this is strictly positive. The case $y \in S^T(x)$ is treated in the same way. □

We now define the interval I,

$$I := [x_0 - \tau, x_0 + \tau]$$

where

$$\tau := \frac{\sup_{x\in\tilde{I}} h(x)}{2\inf_{x\in\tilde{I}} |v_1'(x)|} \max\Big\{4 \sup_{\substack{(x,y)\in\Omega\\ x\in\tilde{I}}} |E(x,y)|\,,\, \sup_{\substack{(x,y)\in\Omega\\ x\in\tilde{I}}} |\partial_y E(x,y)|\Big\}.$$

Lemma 3.2. *If $x \in I^c$, then $v(x,y) \neq 0$ for all $y \in (\phi_B(x), \phi_T(x))$.*

Proof of Lemma 3.2: Let $y \in S(x)^c$. Then, as in (8),

$$|v(x,y)| \geq |v_1(x)|\tfrac{1}{2h(x)} - |E(x,y)|.$$

Also, since $v_1(x_0) = 0$, we have $|v_1(x)| \geq |x - x_0| \inf_{x\in\tilde{I}} |v_1'(x)|$. Therefore, $|v(x,y)| > 0$ provided

$$|x - x_0| > \frac{2h(x)\sup_y |E(x,y)|}{\inf_{x\in\tilde{I}} |v_1'(x)|},$$

and the latter always holds if $|x - x_0| > \tau$.

Now let $y \in S^B(x)$. Then, as in (9)

$$|v(x,y)| \geq (y - \phi_B(x))\Big(|v_1(x)|\tfrac{2}{h(x)} - \sup_{y \in S^B(x)} |\partial_y E(x,y)|\Big)$$

and using $|v_1(x)| \geq |x - x_0| \inf_{x \in \tilde{I}} |v_1'(x)|$ gives

$$|v(x,y)| \geq (y - \phi_B(x))\Big(\tfrac{2}{h(x)}|x - x_0| \inf_{x \in \tilde{I}} |v_1'(x)| - \sup_{y \in S^B(x)} |\partial_y E(x,y)|\Big)$$

Therefore, from the definition of τ, $|v(x,y)| > 0$ for $|x - x_0| > \tau$. $\square$

Using Proposition 2.1 and Remark 2.2, there exist $c > 0$ and $C > 0$ such that $\tau \leq C\left(\eta e^{-cN} + \frac{\delta}{N^2}\right)$ and $I \subset [\frac{N}{2} - CN(\eta + \delta N^{-1}), \frac{N}{2} + CN(\eta + \delta N^{-1})]$, and so the estimate on the width and location of the nodal line in Theorem 1.1 follows immediately from Lemma 3.2.

To study the regularity of the nodal line, we use the coordinate change described in Section 2. For a given (x_0, y_0) with $v(x_0, y_0) = 0$, $y_0 \leq \frac{1}{2}$, this coordinate change transforms (x_0, y_0) to $(\tilde{x}_0, \tilde{y}_0)$ and the eigenfunction v to $w(\tilde{x}, \tilde{y})$. Dropping the tildes, we have $(\Delta + \mu)w = 0$ in the domain

$$\tilde{\Omega} = \{(x,y) : \rho_L(y) \leq x \leq \rho_R(y), \rho_B(x) \leq y \leq \rho_T(x)\}, \tag{10}$$

with $w(x, \rho_B(x)) = w(x, \rho_T(x)) = 0$. Moreover, $\rho_B(x_0) = {\rho_B}'(x_0) = 0$. Setting $\tilde{h}(x) = \rho_T(x) - \rho_B(x)$, $\tilde{\beta} = \frac{\pi(y - \rho_B(x))}{\tilde{h}(x)}$, we decompose $w(x,y)$ as in (7). Note that in the case of a flat top and bottom boundary, the coordinate change is trivial, and w is identically equal to v. The case $y_0 \geq \frac{1}{2}$ is treated in the analogous way by making a rotation about the top boundary.

We set $\tilde{S}(x) = \tilde{S}^B(x) \cup \tilde{S}^T(x) = [\rho_B(x), \rho_B(x) + \frac{1}{4}] \cup [\rho_T(x) - \frac{1}{4}, \rho_T(x)]$, and to establish the regularity of the nodal line, we first study it away from the boundary of $\tilde{\Omega}$. Define

$$e_1 := \sup_{\substack{(x,y) \in \tilde{\Omega} \\ x \in I}} |\partial_x \tilde{E}(x,y)| + \pi \sup_{x \in I} |w_1(x)| \sup_{x \in I}\Big(\tilde{h}(x)^{-2}\Big(|\tilde{h}'(x)| + |{\rho_B}'(x)\rho_T(x)|\Big)\Big) \tag{11}$$

and

$$\Lambda_1 := \frac{1}{2} \inf_{x \in I} \frac{|w_1'(x)|}{\tilde{h}(x)} - e_1. \tag{12}$$

We will show in the proof of Lemma 3.3 that Λ_1 provides a lower bound for $|\partial_x w(x_0, y_0)|$ for points $(x_0, y_0) \in w^{-1}(0)$ with $y_0 \in \tilde{S}(x_0)^c$.

Lemma 3.3. *Suppose that $\Lambda_1 > 0$. Then, for every $(x_0, y_0) \in w^{-1}(0)$ with $y_0 \in \tilde{S}(x_0)^c$ there exist a smooth real valued function g and a neighborhood U of (x_0, y_0) such that*

$$w^{-1}(0) \cap U = \{(x,y) \in U : \ x = g(y)\},$$

with

$$|g'(y)| \leq \frac{1}{\Lambda_1}\Big(\pi \sup_{x \in I} \frac{|w_1(x)|}{\tilde{h}(x)} + \sup_{\substack{(x,y) \in \tilde{\Omega} \\ x \in I}} |\partial_y \tilde{E}(x,y)|\Big).$$

Proof of Lemma 3.3: Note that for all $(x,y) \in \tilde{\Omega}$

$$\partial_x w(x,y) = w_1'(x) \sin(\tilde{\beta}(x,y)) + w_1(x)\partial_x \tilde{\beta}(x,y) \cos(\tilde{\beta}(x,y)) + \partial_x \tilde{E}(x,y),$$

and

$$\partial_x \tilde{\beta}(x,y) = -\frac{\pi}{\tilde{h}(x)^2}\big(y\tilde{h}'(x) + {\rho_B}'(x)\rho_T(x)\big). \tag{13}$$

Therefore,

$$|\partial_x w(x,y)| \geq |w_1'(x)| \sin(\tilde{\beta}(x,y)) - |w_1(x)\partial_x \tilde{\beta}(x,y) \cos(\tilde{\beta}(x,y)) + \partial_x \tilde{E}(x,y)|.$$

Let $(x_0, y_0) \in w^{-1}(0)$, and suppose that $y_0 \in \tilde{S}(x_0)^c$. Then, using $\sin(\tilde{\beta}(x_0, y_0)) \geq \frac{1}{2\tilde{h}(x_0)}$

$$|\partial_x w(x_0, y_0)| \geq \frac{1}{2} \inf_{x \in I} \frac{|w_1'(x)|}{\tilde{h}(x)} - e_1 = \Lambda_1, \tag{14}$$

with e_1 as defined in (11). Thus, $|\partial_x w(x_0, y_0)| > 0$ since $\Lambda_1 > 0$ by assumption. This implies the existence of the graph function g along a neighborhood of $w^{-1}(0) \cap \{(x,y) \in \Omega : \, y \in \tilde{S}(x)^c\}$. Note that for every y

$$g'(y) = -\frac{\partial_y w(g(y), y)}{\partial_x w(g(y), y)}. \tag{15}$$

We next find an upper bound for $|\partial_y w(g(y), y)|$. Since for all $(x, y) \in \Omega$

$$\partial_y w(x,y) = \pi \frac{w_1(x)}{\tilde{h}(x)} \cos(\tilde{\beta}(x,y)) + \partial_y \tilde{E}(x,y),$$

then

$$|\partial_y w(x,y)| \le \pi \sup_{x \in I} \frac{|w_1(x)|}{\tilde{h}(x)} + \sup_{\substack{(x,y)\in\tilde{\Omega} \\ x \in I}} |\partial_y \tilde{E}(x,y)|. \tag{16}$$

This together with (14) yield the claimed bound on $|g'(y)|$ when $y \in \tilde{S}(x)^c$. □

To study the regularity of the nodal line near $\partial\tilde{\Omega}$, we define

$$e_2 := \sup_{\substack{x \in I \\ y \in \tilde{S}^B(x)}} |\partial_y \partial_x \tilde{E}(x,y)| + \pi \sup_{x\in I} |w_1(x)| \sup_{\substack{(x,y)\in\tilde{\Omega} \\ x \in I}} \frac{|\tilde{h}'(x)|}{\tilde{h}(x)^2} \tag{17}$$

and

$$\Lambda_2 := 2 \inf_{x \in I} \frac{|w_1'(x)|}{\tilde{h}(x)} - e_2. \tag{18}$$

We will show in Lemma 3.4 that Λ_2 provides a lower bound for $|\partial_x w(x_0, y_0)|$ for all points $(x_0, y_0) \in w^{-1}(0)$.

Lemma 3.4. *If $\Lambda_2 > 0$, there exist a neighborhood U of (x_0, y_0), and smooth real valued function g, such that $w^{-1}(0) \cap U = \{(x,y) \in U : \, x = g(y)\}$, and*

$$|g'(y)| \le \frac{|y|}{\Lambda_2} \sup_{\substack{x \in I \\ y \in \tilde{S}^B(x)}} \Big(\tfrac{1}{2} \tilde{h}(x)^{-2} (1 + |\rho_B''(x)|) |\partial_y \tilde{E}(x,y)| + \tfrac{1}{2} |\rho_B''(x)| |\partial_y^2 E(x,y)| + |\partial_y^3 E(x,y)| \Big)$$

Furthermore, $w^{-1}(0)$ meets $\partial\tilde{\Omega}$ orthogonally.

Proof of Lemma 3.4: Since $\partial_x \tilde{E}(x, \rho_B(x)) = 0$, we have $\partial_x \tilde{E}(x_0, 0) = 0$, and so

$$|\partial_x \tilde{E}(x_0, y_0)| \le y_0 \sup_{y \in \tilde{S}^B(x_0)} |\partial_y \partial_x \tilde{E}(x_0, y)|.$$

In addition, using $\rho_B(x_0) = \rho_B'(x_0) = 0$, we know that $\sin(\tilde{\beta}(x_0, y_0)) \ge \frac{2}{\tilde{h}(x_0)} y_0$ and $\partial_x \tilde{\beta}(x_0, y_0) = -\frac{\pi}{\tilde{h}(x_0)^2} y_0 \tilde{h}'(x_0)$. Therefore,

$$|\partial_x w(x_0, y_0)| \ge y_0 \Big(2 \inf_{x\in I} \frac{|w_1'(x)|}{\tilde{h}(x)} - e_2 \Big) = \Lambda_2 \, y_0. \tag{19}$$

This proves the existence of g. For $(x_0, y_0) \in w^{-1}(0)$, with $y_0 \in \tilde{S}^B(x_0)$ we have $w_1(x_0) = -\frac{\tilde{E}(x_0,y_0)}{\sin(\tilde{\beta}(x_0,y_0))}$. Therefore,

$$\partial_y w(x_0, y_0) = -\frac{1}{\tilde{h}(x_0)} \tilde{E}(x_0, y_0) \frac{\pi \cos(\tilde{\beta}(x_0, y_0))}{\sin(\tilde{\beta}(x_0, y_0))} + \partial_y \tilde{E}(x_0, y_0).$$

Note that $\frac{\pi \cos(\pi s)}{\sin(\pi s)} = \frac{1}{s}(1 - r(s))$ with $0 \le r(s) \le \frac{\pi^2}{2} |s|^2$ for $|s| < \frac{1}{2}$. Since $\tilde{\beta}(x_0, y_0) = \frac{\pi y_0}{\tilde{h}(x_0)}$, it follows that

$$\begin{aligned} \partial_y w(x_0, y_0) &= -\tfrac{1}{y_0} \tilde{E}(x_0, y_0)(1 - r(\tfrac{y_0}{\tilde{h}(x_0)})) + \partial_y \tilde{E}(x_0, y_0) \\ &= -\tfrac{1}{y_0} \tilde{E}(x_0, y_0) + \partial_y \tilde{E}(x_0, y_0) + \tfrac{1}{y_0} \tilde{E}(x_0, y_0) r(\tfrac{y_0}{\tilde{h}(x_0)}). \end{aligned} \tag{20}$$

Moreover,

$$\tfrac{1}{y_0}\tilde{E}(x_0,y_0)=\partial_y\tilde{E}(x_0,0)+\tfrac{1}{2}\partial_y^2\tilde{E}(x_0,0)y_0+\tfrac{1}{6}\partial_y^3\tilde{E}(x,y_1)y_0^2,$$

for some $y_1\in\tilde{S}^B(x_0)$, and

$$\partial_y\tilde{E}(x_0,y_0)=\partial_y\tilde{E}(x_0,0)+\partial_y^2\tilde{E}(x_0,0)y_0+\tfrac{1}{2}\partial_y^3\tilde{E}(x_0,y_2)y_0^2,$$

for some $y_2\in\tilde{S}^B(x_0)$. In particular, (20) yields

$$\partial_y w(x_0,y_0)=\tfrac{1}{2}\partial_y^2\tilde{E}(x_0,0)y_0+\tfrac{1}{2}\partial_y^3\tilde{E}(x_0,y_2)y_0^2-\tfrac{1}{6}\partial_y^3\tilde{E}(x,y_1)y_0^2+\tfrac{1}{y_0}\tilde{E}(x_0,y_0)r(\tfrac{y_0}{\tilde{h}(x_0)}). \tag{21}$$

Next, note that since $\tilde{E}(x_0,0)=0$, there exists $y_2\in\tilde{S}^B(x)$ such that $\tilde{E}(x_0,y_0)=\partial_y\tilde{E}(x_0,y_3)y_0$. Since $0\le r(s)\le\frac{\pi^2}{2}|s|^2$ for $|s|<\frac{1}{2}$, it follows that

$$|\partial_y w(x_0,y_0)|\le\tfrac{1}{2}|\partial_y^2\tilde{E}(x_0,0)|y_0+\tfrac{1}{2}|\partial_y^3\tilde{E}(x_0,y_2)|y_0^2+\tfrac{1}{6}|\partial_y^3\tilde{E}(x_0,y_1)|y_0^2+\tfrac{\pi^2}{2}\tilde{h}(x_0)^{-2}|\partial_y\tilde{E}(x_0,y_3)|y_0^2.$$

Therefore,

$$|\partial_y w(x_0,y_0)|\le\tfrac{1}{2}y_0|\partial_y^2\tilde{E}(x_0,0)|+y_0^2\sup_{\substack{x\in I\\ y\in\tilde{S}^B(x)}}\Big[|\partial_y^3\tilde{E}(x,y)|+\tfrac{\pi^2}{2}\tilde{h}(x_0)^{-2}|\partial_y\tilde{E}(x,y)|\Big]. \tag{22}$$

In the same way, we have

$$\partial_y w(x_0,0)=-\frac{\frac{\pi}{2\tilde{h}(x_0)}y_0^2\partial_y^2\tilde{E}(x_0,y_1)}{\sin\left(\frac{\pi y_0}{\tilde{h}(x_0)}\right)}+\partial_y\tilde{E}(x_0,0)r(\tfrac{y_0}{\tilde{h}(x_0)}),$$

and

$$|\partial_y w(x_0,0)|\le y_0\sup_{\substack{x\in I\\ y\in\tilde{S}^B(x)}}|\partial_y^2\tilde{E}(x,y)|+\tfrac{\pi^2}{2}\tilde{h}(x_0)^{-2}y_0^2|\partial_y\tilde{E}(x_0,0)|. \tag{23}$$

To improve (22) and obtain a y_0^2 in the upper bound, we need better control on $\partial_y^2\tilde{E}(x_0,0)$ in (21). To do this we note

$$\partial_y^2\tilde{E}(x_0,0)=\partial_y^2 w(x_0,0)=-\partial_x^2 w(x_0,0)-\mu w(x_0,0)=-\partial_x^2 w(x_0,0)=-\rho_B{}''(x_0)\partial_y w(x_0,0), \tag{24}$$

where the last equality was obtained after differentiating $w(x,\rho_B(x))\equiv 0$ twice and using $\rho_B{}'(x_0)=\rho_B(x_0)=0$. From (22) it follows that for $y\in\tilde{S}^B(x)$ and $(x,y)\in w^{-1}(0)$,

$$|\partial_y w(x_0,y_0)|\le y_0^2\sup_{\substack{x\in I\\ y\in\tilde{S}^B(x)}}\Big(\tfrac{\pi^2}{2}\tilde{h}(x)^{-2}(1+|\rho_B{}''(x)|)|\partial_y\tilde{E}(x,y)|+\tfrac{1}{2}|\rho_B{}''(x)||\partial_y^2 E(x,y)|+|\partial_y^3 E(x,y)|\Big) \tag{25}$$

The bound on g' follows from combining (15) with (19) and (25). In particular $g'(0)=0$, showing that $w^{-1}(0)$ meets $\partial\tilde{\Omega}$ orthogonally. □

Applying Proposition 2.1, the estimate on $|g'(y)|$ given in Theorem 1.1 follows immediately from Lemmas 3.3 and 3.4. The following lemma gives the desired uniform bound on $g''(y)$ and completes the proof of Theorem 1.1:

Lemma 3.5. *There exist constants $c>0$, C such that*

$$|g''(y)|\le C\left(\eta e^{-cN}+\frac{\delta}{N^2}\right).$$

Proof of Lemma 3.5: Let (x_0,y_0) be a point on the nodal line Γ. Then, differentiating $w(g(y),y)=0$ twice gives the expression

$$g''(y_0)=\frac{(\partial_y w(x_0,y_0))^2\partial_x^2 w(x_0,y_0)+(\partial_x w(x_0,y_0))^2\partial_y^2 w(x_0,y_0)-2\partial_y w(x_0,y_0)\partial_x w(x_0,y_0)\partial_x\partial_y w(x_0,y_0)}{(\partial_x w(x_0,y_0))^3}. \tag{26}$$

To bound the denominator, we use the lower bounds on $\partial_x w(x_0, y_0)$ from (14) (in the centre) and (19) (near the boundary). By Proposition 2.1 this implies that

$$|\partial_x w(x_0, y_0)| \geq C^{-1} N^{-1} |y_0|.$$

We also have upper bounds on $\partial_y w(x_0, y_0)$ from (16) (in the centre) and (25) (near the boundary), which again using Proposition 2.1 gives

$$|\partial_y w(x_0, y_0)| \leq C|y_0|^2 \left(\eta e^{-cN} + \frac{\delta}{N^3}\right).$$

Finally, from Proposition 2.1 we have $|\partial_x \partial_y w(x_0, y_0)| \leq CN^{-1}$, $\left|\partial_x^2 w(x_0, y_0)\right| \leq CN^{-2}$, and combining (24) with

$$\left|\partial_y^2 w(x_0, y_0)\right| \leq \left|\partial_y^2 w(x_0, 0)\right| + |y_0| \sup_{y \in [0, y_0]} |\partial_y^3 w(x_0, y_0)|,$$

gives

$$\left|\partial_y^2 w(x_0, y_0)\right| \leq C|y_0| \left(\eta e^{-cN} + \frac{\delta}{N^3}\right).$$

Using these estimates gives the desired bound for the expression for $g''(y_0)$ in (26). □

Remark 3.1. *In the case of flat upper and lower boundaries, we have the following estimates on the quantities appearing in the numerator of* (26)*: First,* $\partial_y v(x_0, y_0)$ *satisfies*

$$|\partial_y v(x_0, y_0)| \leq \pi \sup_{x \in I} |v_1(x)| + \sup_{\substack{(x,y) \in \Omega \\ x \in I}} |\partial_y E(x, y)| \qquad \text{for } \tfrac{1}{4} \leq y_0 \leq \tfrac{3}{4},$$

$$|\partial_y v(x_0, y_0)| \leq y_0^2 \sup_{\substack{(x,y) \in \Omega \\ x \in I}} \left[|\partial_y^3 E(x, y) + \tfrac{1}{2}|\partial_y E(x, y)|\right] \qquad \text{for } y_0 \leq \tfrac{1}{4}, y_0 \geq \tfrac{3}{4},$$

where in the second inequality we have used (22) *(and the fact that* $\partial_y^2 E(x_0, 0) = 0$ *in the flat case). We also immediately have the estimates on the second derivatives of* v *of*

$$|\partial_x \partial_y v(x_0, y_0)| \leq \pi \sup_{x \in I} |v_1'(x)| + \sup_{\substack{(x,y) \in \Omega \\ x \in I}} |\partial_x \partial_y E(x, y)|,$$

$$|\partial_x^2 v(x_0, y_0)| \leq \sup_{x \in I} |v_1''(x)| + \sup_{\substack{(x,y) \in \Omega \\ x \in I}} |\partial_x^2 E(x, y)|.$$

Finally,

$$|\partial_y^2 v(x_0, y_0)| \leq \pi^2 \sup_{x \in I} |v_1(x)| + \sup_{\substack{(x,y) \in \Omega \\ x \in I}} |\partial_y^2 E(x, y)| \qquad \text{for } \tfrac{1}{4} \leq y_0 \leq \tfrac{3}{4}$$

$$|\partial_y^2 v(x_0, y_0)| \leq \pi^3 y_0 \sup_{x \in I} |v_1(x)| + y_0 \sup_{\substack{(x,y) \in \Omega \\ x \in I}} |\partial_y^3 E(x, y)| \qquad \text{for } y_0 \leq \tfrac{1}{4}, y_0 \geq \tfrac{3}{4},$$

where in the second inequality, we have used $|\partial_y^2 v(x_0, y_0)| \leq |\partial_y^2 v(x_0, 0)| + y_0 \sup_{y \in [0, y_0]} |\partial_y^3 v(x_0, y)|$, *and that* $\partial_y^2 v(x_0, 0) = 0$ *in the flat case. We will use these estimates in* Section 5 *when we explicitly track the constants in the flat case.*

4. Proof of Proposition 2.1

In this section we will prove Proposition 2.1 by establishing the required properties of the decompositions of v and w defined in (4) and (7). From the definition of the domain Ω from (1), Ω contains the rectangle $[0, N] \times [\frac{\delta}{N^3}, 1 - \frac{\delta}{N^3}]$, and is contained in the rectangle $[-2\eta, N + 2\eta] \times [-\frac{\delta}{N^3}, 1 + \frac{\delta}{N^3}]$. Therefore, by domain monotonicity for Dirichlet eigenvalues we have the following lemma.

Lemma 4.1. *The second Dirichlet eigenvalue* μ *satisfies*

$$\pi^2 \left(1 + 2\tfrac{\delta}{N^3}\right)^{-2} + 4\pi^2 (N + 4\eta)^{-2} \leq \mu \leq \pi^2 \left(1 - 2\tfrac{\delta}{N^3}\right)^{-2} + 4\pi^2 N^{-2}.$$

We can use this eigenvalue bound to obtain control on the growth of the eigenfunction v away from the left and right sides of Ω.

Lemma 4.2. *There exists a constant C (depending only on $\tilde{C}_1$, $\tilde{C}_2$ from (3)) with the following properties: First, $|\nabla v(x,y)| \leq C$ for $(x,y) \in \Omega$, and moreover for all y,*

$$|v(x,y)| \leq CN^{-1}(\eta + x) \quad \text{for } x \leq \tfrac{1}{2}N, \qquad |v(x,y)| \leq CN^{-1}(\eta + N - x) \quad \text{for } x \geq \tfrac{1}{2}N.$$

Proof of Lemma 4.2: The boundary of Ω is C^2-smooth, except at 4 points where the C^2-curves meet at a convex angle. This ensures that the gradient of v is bounded. To obtain the pointwise estimate on v we follow the proof of Lemma 3.12 (a) in [GJ98]: We define a comparison function $R(x,y)$ by

$$R(x,y) = C_1 \sin\left(\pi \frac{x+\eta}{c_1 N}\right) \sin\left(\pi \frac{y - \delta N^{-3}}{1 + 2\delta N^{-3}}\right).$$

Here $c_1 > 0$ is chosen so that $(\Delta + \mu)R(x,y) < 0$ for $(x,y) \in \Omega$ with $x \leq \frac{1}{2}c_1 N - \eta$. This is possible by the eigenvalue upper bound on μ from Lemma 4.1. Since v vanishes on $\partial\Omega$ and its gradient is bounded, we can then choose the constant C_1 so that $|v(x,y)| \leq R(x,y)$ for $(x,y) \in \Omega$ with $x = \frac{1}{2}c_1 N - \eta$. Moreover, $R > 0$ on the part of $\partial\Omega$ with $x \leq \frac{1}{2}c_1 N - \eta$. Therefore, by applying the maximum principle to v and R to the subset of Ω with $x \leq \frac{1}{2}c_1 N - \eta$ we have

$$|v(x,y)| \leq R(x,y) \qquad \text{for } (x,y) \in \Omega \text{ with } x \leq \tfrac{1}{2}c_1 N - \eta.$$

This gives the desired estimate on $|v(x,y)|$ for $x \leq \frac{1}{2}N$ and the case of $x \geq \frac{1}{2}N$ can be handled analogously. □

We recall that the function w satisfies $(\Delta + \mu)w = 0$ in the domain $\tilde{\Omega}$ as in (10), with $w(x, \rho_B(x)) = w(x, \rho_T(x)) = 0$. We recall that ρ_B, ρ_T, ρ_R, ρ_L, satisfy the bounds (5) and (6). The function w is equal to v in the rotated coordinates, and as noted in Remark 2.1, the angle of rotation in the definition of w is bounded by $C\frac{\delta}{N^3}$ for some $C > 0$. Therefore, by Lemma 4.2 we have $|\nabla w(x,y)| \leq C$ and

$$|w(x,y)| \leq CN^{-1}(\eta + \delta N^{-2} + x) \quad \textit{for } x \leq \tfrac{1}{2}N, \quad |w(x,y)| \leq CN^{-1}(\eta + \delta N^{-2} + N - x) \quad \textit{for } x \geq \tfrac{1}{2}N. \tag{27}$$

Defining a height function by $\tilde{h}(x) = \rho_B(x) - \rho_T(x) \geq \frac{3}{5}$, we write w as the Fourier series

$$w(x,y) = \sum_{k \geq 1} w_k(x) \sin(k\tilde{\beta}(x,y)).$$

Here the k-th mode $w_k(x)$ is given by

$$w_k(x) = \frac{2}{\tilde{h}(x)} \int_{\rho_B(x)}^{\rho_T(x)} w(x,y) \sin(k\tilde{\beta}(x,y))\,\mathrm{d}y.$$

To prove Proposition 2.1, we will first bound each mode $w_k(x)$, then sum over k, and finally use elliptic estimates to extend these to derivative bounds. To estimate $w_k(x)$, we use the eigenfunction equation to find the equation that it satisfies, and then use the Duhamel principle to find an implicit expression. To bound this expression we need control on the boundary values $w_k(0)$, $w_k(N)$.

Lemma 4.3. *There exists a constant C such that*

$$|w_k(0)| + |w_k(N)| \leq C\left(\tfrac{\eta}{N} + \tfrac{\delta}{N^3}\right).$$

Proof of Lemma 4.3: By definition

$$w_k(0) = \frac{2}{\tilde{h}(0)} \int_{\rho_B(0)}^{\rho_T(0)} w(0,y) \sin(k\tilde{\beta}(0,y))\,\mathrm{d}y. \tag{28}$$

The estimate on $|w_k(0)|$ therefore follows immediately from (27). The estimate for $w_k(N)$ follows in the same way. □

Proof of Proposition 2.1: Flat case. Let us first consider the case of a flat top and bottom, with $w = v$, and $\rho_B(x) \equiv 0$, $\rho_T(x) \equiv 1$. In this case, we can remove the factor of $\frac{\delta}{N^3}$ in the estimate in Lemma 4.3 above. Using that $(\Delta + \mu)v(x, y) = 0$ for $0 \le x \le N$, the function $v_k(x) = 2\int_0^1 v(x, y)\sin(k\pi y)\,\mathrm{d}y$ satisfies the ODE

$$v_k''(x) + (\mu - \pi^2 k^2)v_k(x) = 0. \tag{29}$$

Writing $\mu_k^2 = \pi^2 k^2 - \mu \ge \pi^2(k^2 - 1) - 4\pi^2 N^{-2} \ge (k^2 - 2)\pi^2$ for $k \ge 2$ (by Lemma 4.1, provided $N \ge 2$), we therefore have, for $k \ge 2$,

$$\begin{aligned} v_k(x) &= \frac{1}{e^{\mu_k N} - e^{-\mu_k N}}\left(v_k(0)\left(e^{\mu_k(N-x)} - e^{\mu_k(x-N)}\right) + v_k(N)\left(e^{\mu_k x} - e^{-\mu_k x}\right)\right) \\ &= \frac{1}{\sinh(\mu_k N)}\left(v_k(0)\sinh(\mu_k(N-x)) + v_k(N)\sinh(\mu_k x)\right). \end{aligned} \tag{30}$$

Writing $v(x, y) = v_1(x)\sin(\pi y) + E(x, y)$ as in (4), this expression gives $|E(x, y)| \le C\eta e^{-cN}$, and likewise for derivatives of $E(x, y)$. For $k = 1$,

$$4\pi^2(N + 4\eta)^{-2} \le \mu - \pi^2 \le 4\pi^2 N^{-2},$$

and we set $\mu_1^2 = \mu - \pi^2$. The function $v_1(x)$ satisfies

$$v_1(x) = v_1(0)\cos(\mu_1 x) + \frac{v_1(N) - v_1(0)\cos(\mu_1 N)}{\sin(\mu_1 N)}\sin(\mu_1 x). \tag{31}$$

Setting $A_1 = \frac{v_1(N) - v_1(0)\cos(\mu_1 N)}{\sin(\mu_1 N)}$ gives $|v_1(x) - A_1\sin(\mu_1 x)| \le C\eta/N$, and since $\|v\|_{L^\infty} = 1$, this implies that $||A_1| - 1| \le C\eta/N$. The estimates from Proposition 2.1 then follow readily from the expressions in (30) and (31).

Proof of Proposition 2.1: General case. In the general case, $w_k(x)$ satisfies an approximate version of the ODE in (29), with an error depending on $\rho_B(x)$, $\rho_T(x)$ and their first two derivatives: Fix $x^* \in [0, N]$, and set

$$e_k(x, y) = \frac{2}{\tilde{h}(x)}\sin(k\tilde{\beta}(x, y)). \tag{32}$$

Lemma 4.4. *The function $w_k(x)$ satisfies the equation*

$$w_k''(x) + \left(\mu - \frac{\pi^2 k^2}{\tilde{h}(x^*)^2}\right)w_k(x) = F_k(x),$$

where $F_k(x)$ has the bound

$$|F_k(x)| \le Ck\left(\left|\tfrac{1}{\tilde{h}(x)^2} - \tfrac{1}{\tilde{h}(x^*)^2}\right| + |\rho_T{}'(x)| + |\rho_B{}'(x)| + |\rho_T{}''(x)| + |\rho_B{}''(x)|\right),$$

for an absolute constant C.

Proof of Lemma 4.4: The function $F_k(x)$ is equal to

$$F_k(x) = \pi^2 k^2\left(\frac{1}{\tilde{h}(x)^2} - \frac{1}{\tilde{h}(x^*)^2}\right)w_k(x) + 2\int_{\rho_B(x)}^{\rho_T(x)} \partial_x w(x, y)\partial_x e_k(x, y)\,\mathrm{d}y + \int_{\rho_B(x)}^{\rho_T(x)} w(x, y)\partial_x^2 e_k(x, y)\,\mathrm{d}y. \tag{33}$$

Applying the bounds we have derived for w in (27) and the definition of e_k in (32), the terms that do not immediately obey the estimate of the lemma are the first term and the term in the final integral given by

$$-\frac{2k^2}{\tilde{h}(x)}\int_{\rho_B(x)}^{\rho_T(x)} w(x, y)\left(\partial_x\tilde{\beta}(x, y)\right)^2\sin\left(k\tilde{\beta}(x, y)\right)\,\mathrm{d}y. \tag{34}$$

This is because this is the only term in the final integral in (33) for which a factor k^2 appears in the expression for $\partial_x^2 e(x, y)$. All of the other terms in the last two integrals in (33) contain at most one derivative of w, two derivatives of ρ_T and ρ_B, and one factor of k. After an integration by parts $w_k(x)$ is equal to

$$\frac{2}{k\pi}\int_{\rho_B(x)}^{\rho_T(x)} \partial_y w(x,y)\cos\left(k\tilde{\beta}(x,y)\right)\,\mathrm{d}y,$$

and (34) can be written as

$$-\frac{2k}{\pi}\int_{\rho_B(x)}^{\rho_T(x)} \partial_y\left(w(x,y)\left(\partial_x\tilde{\beta}(x,y)\right)^2\right)\cos\left(k\tilde{\beta}(x,y)\right)\,\mathrm{d}y.$$

Since $|\partial_y w(x,y)|$ is bounded by a constant, both of these terms are of the desired form. □

The function $w_k(x)$ also satisfies the boundary conditions

$$w_k(0)=\alpha_k^{(1)}, \qquad w_k(N)=\alpha_k^{(2)},$$

where $\alpha_k^{(i)}$ are values coming from the side variation of the domain, wih bounds in Lemma 4.3.

For $k=1$, set $\mu_1^2=\mu-\frac{\pi^2}{\tilde{h}(x^*)^2}\geq 0$ and for $k\geq 2$, set $\mu_k^2=\mu_k(x^*)^2=\frac{\pi^2k^2}{\tilde{h}(x^*)^2}-\mu\geq 0$.

Lemma 4.5. *Define the functions $W_1(x)$ and $W_k(x)$ (for $k\geq 2$) by*

$$W_1(x)=\frac{1}{\mu_1}\sin(\mu_1 x), \qquad W_k(x)=\frac{1}{2\mu_k}\left(e^{\mu_k x}-e^{-\mu_k x}\right)=\frac{1}{\mu_k}\sinh(\mu_k x).$$

Then,

$$w_1(x)=\int_x^N W_1(t-x)F_1(t)\,\mathrm{d}t+A_1\sin(\mu_1(N-x))+\alpha_1^{(2)}\cos(\mu_1(N-x)),$$

with $\alpha_1^{(1)}=\int_0^N W_1(t)F_1(t)\,\mathrm{d}t+A_1\sin(\mu_1 N)+\alpha_1^{(2)}\cos(\mu_1 N)$. *Also,*

$$w_k(x)=\int_0^x W_k(x-t)F_k(t)\,\mathrm{d}t+A_ke^{\mu_k x}+B_ke^{-\mu_k x},$$

for constants A_k, B_k, with

$$\begin{aligned}\alpha_k^{(1)}&=A_k+B_k\\ \alpha_k^{(2)}&=A_ke^{\mu_k N}+B_ke^{-\mu_k N}+\int_0^N W_k(N-t)F_k(t)\,\mathrm{d}t.\end{aligned}$$

Proof of Lemma 4.5: The functions $W_1(x)$ and $W_k(x)$ satisfy

$$\begin{aligned}W_1''(x)+\mu_1^2W_1(x)&=0, \qquad W_1(0)=0, \quad W_1'(0)=1\\ W_k''(x)-\mu_k^2W_k(x)&=0, \qquad W_k(0)=0, \quad W_k'(0)=1.\end{aligned}$$

The lemma then follows from Lemma 4.4 and the boundary conditions of $w_k(x)$ at $x=0,N$. □

Combining Lemmas 4.4 and 4.5, we can bound $w_k(x)$.

Proposition 4.1. *There exist constants c, C, such that for $x\in[0,N]$, $k\geq 2$,*

$$|w_k(x)|\leq C\left(\frac{\eta}{N}e^{-c\mu_k d(x)}+k^{-1}\frac{\delta}{N^3}\right).$$

Here $d(x)=\min\{x,N-x\}$ is the distance of x from the endpoints of $[0,N]$.

Proof of Proposition 4.1: We fix $x^*\in[0,N]$, and use Lemmas 4.4 and 4.5 to bound w_k at $x=x^*$ (with a bound independent of x^*). The constants A_k, B_k from Lemma 4.5 can be written for $k\geq 2$ as

$$A_k=-\frac{1}{2\mu_k}\int_0^N\frac{W_k(N-t)}{W_k(N)}F_k(t)\,\mathrm{d}t+\frac{-e^{-\mu_k N}\alpha_k^{(1)}+\alpha_k^{(2)}}{e^{\mu_k N}-e^{-\mu_k N}},$$

with $B_k = \alpha_k^{(1)} - A_k$. From Lemma 4.4, we have the bound

$$|F_k(t)| \leq Ck\frac{\delta}{N^3}(1+|t-x^*|), \tag{35}$$

and from Lemma 4.3, $|w_k(0)|, |w_k(N)| \leq C\left(\frac{\eta}{N}+\frac{\delta}{N^3}\right)$. Therefore, since $\mu_k \geq \pi\sqrt{k^2-2}$, the only terms in the expression for $w_k(x^*)$ from Lemma 4.5 that do not immediately satisfy the required estimates are

$$\int_0^{x^*} W_k(x^*-t)F_k(t)\,\mathrm{d}t - \frac{1}{2\mu_k}e^{\mu_k x^*}\int_0^N \frac{e^{\mu_k(N-t)}}{e^{\mu_k N}}F_k(t)\,\mathrm{d}t.$$

However, these integrals can be combined to be written as

$$-\frac{1}{2\mu_k}\int_0^{x^*} e^{-\mu_k(x^*-t)}F_k(t)\,\mathrm{d}t - \frac{1}{2\mu_k}\int_{x^*}^N e^{\mu_k(x^*-t)}F_k(t)\,\mathrm{d}t. \tag{36}$$

Using the bound on $F_k(t)$ from (35) and integrating gives the desired bound. □

We write

$$w(x,y) = w_1(x)\sin\left(\tilde{\beta}(x,y)\right) + \sum_{k\geq 2} w(x,y)\sin\left(k\tilde{\beta}(x,y)\right) = V_1(x,y) + \tilde{E}(x,y).$$

Summing the estimate from Proposition 4.1 over k we can control the L^2-norm of $\tilde{E}$. For the rest of the section, fix $x^* \in [1, N-1]$ and denote the cross-section at x^* by $U(x^*) = \tilde{\Omega} \cap \{(x,y) : x = x^*\}$.

Corollary 4.1. *There exist constants c, C such that*

$$\|\tilde{E}\|_{L^2(U(x^*))} \leq C\left(\frac{\eta}{N}e^{-cd(x^*)} + \frac{\delta}{N^3}\right).$$

We now convert this L^2-estimate into bounds on derivatives of $\tilde{E}$.

Proposition 4.2. *For each $j \geq 0$, and with $c > 0$ as in Corollary 4.1, there exists a constant C_j such that*

$$\|\tilde{E}\|_{H^j(U(x^*))} \leq C_j\left(\frac{\eta}{N}e^{-cd(x^*)} + \frac{\delta}{N^3}\right).$$

Proof of Proposition 4.2: To obtain this estimate on $\tilde{E}$ we find the elliptic equation that it satisfies. For $V_1(x,y) := w_1(x)\sin\left(\frac{\pi(y-\rho_B(x))}{\tilde{h}(x)}\right)$, we have

$$\begin{aligned}\Delta V_1(x,y) &= \frac{2}{\tilde{h}(x)}\left(\int_{\rho_B(x)}^{\rho_T(x)} \partial_x^2 w(x,y')\sin\left(\tilde{\beta}(x,y')\right)\,\mathrm{d}y'\right)\sin\left(\tilde{\beta}(x,y)\right) - \frac{\pi^2}{\tilde{h}(x)^2}V_1(x,y) + G_1(x,y)\\ &= -\mu V_1(x,y) + G_1(x,y).\end{aligned}$$

The function $G_1(x,y)$ consists of terms where at least one derivative in x has been applied to a factor of $\rho_T(x)$ or $\rho_B(x)$, and so for each $j \geq 0$, there exists a constant C_j such that

$$\|G_1\|_{H^j(U(x^*))} \leq C_j\frac{\delta}{N^3}. \tag{37}$$

Using the eigenfunction equation, $\tilde{E}(x,y)$ satisfies

$$\begin{cases} \Delta\tilde{E}(x,y) = -\mu\tilde{E}(x,y) - G_1(x,y) & \text{in } \tilde{\Omega} \\ \tilde{E}(x,y) = 0 & \text{on } \partial\tilde{\Omega}. \end{cases}$$

Applying elliptic estimates to this equation, (37) and the estimate on $\tilde{E}$ from Corollary 4.1 establishes the proposition. □

Using Proposition 4.2, we can obtain more refined information about the first Fourier mode $w_1(x)$ and complete the proof of Proposition 2.1.

Proposition 4.3. *There exists a constant C such that in the interval $[\frac{N}{4}, \frac{3N}{4}]$ the function $w_1(x)$ has a unique zero at $x = x_0$ with $\left|x_0 - \frac{N}{2}\right| \leq C(\eta + N\delta)$. Moreover, $|w_1'(x)| \geq C^{-1}N^{-1}$ for this range of x, and for $x \in [0, N]$, $1 \leq j \leq 3$, we have,*

$$|w_1(x) - A_1 \sin(\mu_1(N-x))| \leq C\left(\eta/N + \delta\right), \qquad |w_1^{(j)}(x)| \leq CN^{-j}$$

Here the constant A_1 is as in Lemma 4.5 *and satisfies $||A_1| - 1| \leq C(\eta/N + \delta)$.*

Proof of Proposition 4.3: By Lemma 4.1, $\left|\mu_1 - \frac{2\pi}{N}\right| \leq CN^{-2}(\delta + \eta)$ for a constant C. Therefore, using the expression for $w_1(x)$ from Lemma 4.5 and the bound $|F_1(t)| \leq C\frac{\delta}{N^3}(1 + |t - x^*|)$ from Lemma 4.4, we have

$$|w_1(x) - A_1 \sin(\mu_1(N-x))| \leq C(\eta/N + \delta). \tag{38}$$

Here C is a constant (changing from line-to-line). Moreover, since $\|w\|_{L^\infty} = 1$, and

$$w(x,y) = w_1(x)\sin\left(\tilde{\beta}(x,y)\right) + \tilde{E}(x,y),$$

combining (38) with Proposition 4.2, we have

$$||A_1| - 1| \leq C(\eta/N + \delta).$$

To complete the proof of the lemma, we need to bound $w_1'(x)$. Differentiating the expression from Lemma 4.5 gives

$$w_1'(x) = -\int_x^N W_1'(t-x)F_1(t)\,\mathrm{d}t - \mu_1 A_1 \cos(\mu_1(N-x)) + \mu_1\alpha_1^{(2)}\sin(\mu_1(N-x)).$$

In particular $|w_1'(x) + \mu_1 A_1\cos(\mu_1(N-x))| \leq CN^{-1}(\eta+\delta)$, and combining this with the estimate for A_1 gives the required bound for $|w_1'(x)|$. The expression for $w_1''(x)$ from Lemma 4.4 gives $|w_1''(x)| \leq CN^{-2}$, and differentiating we have $|w_1'''(x)| \leq CN^{-3}$. Since $|w_1'(x)|$ is non-zero on $[\frac{N}{4}, \frac{3N}{4}]$, $w_1(x)$ has at most one zero in this interval. The function $\sin(\mu_1(N-x))$ has its unique zero in this interval at $\tilde{x}_0$, with $|\tilde{x}_0 - \frac{N}{2}| \leq C(\delta + \eta)$, and its derivative is bounded below by $C^{-1}N^{-1}$. Therefore, $w_1(x)$ also has a unique zero at $x = x_0$, and by (38) we have $|x_0 - \frac{N}{2}| \leq CN(\eta/N + \delta)$. □

Remark 4.1. *In the case that no rotation has been applied (so that $v_1 = w_1$), the function $F_1(t)$ from Lemma 4.4 satisfies the stronger bound $|F_1(t)| \leq C\frac{\delta}{N^3}$. Inserting this stronger estimate into the argument above, the point x_0 satisfies $|x_0 - \frac{N}{2}| \leq C(\eta + \delta)$.*

5. An explicit Hadamard variation formula and constant tracking

To prove Proposition 2.1, we used the estimate on the boundary values of the Fourier modes, $w_k(0)$ and $w_k(N)$, in Lemma 4.3, which follows directly from a pointwise estimate on the eigenfunction. In order to track the constants appearing in the error estimates in Proposition 2.1 in the flat case, we require a more explicit bound on these boundary values. We do this as follows, using a variant of a calculation given in [GJ09].

Proposition 5.1. *There exists a constant C such that*

$$\left|w_k(0) - \frac{4\pi}{N}\int_{\rho_B(0)}^{\rho_T(0)} \rho_L(y)\sin(\tilde{\beta}(0,y))\sin(k\tilde{\beta}(0,y))\,\mathrm{d}y\right| \leq C(\eta + \delta N^{-3})\left(\eta^{3/4} + \delta + k^2(\eta + \delta N^{-3})^2\right). \tag{39}$$

Proof of Proposition 5.1: We extract the main term in $w_k(0)$ as follows. First, we integrate by parts to write $w_k(0)$ as

$$\begin{aligned} 2\int_{\rho_B(0)}^{\rho_T(0)} w(0,y)\sin(k\tilde{\beta}(0,y))\,\mathrm{d}y &= 2\int_{\partial\tilde{\Omega}_0} w(x,y)\frac{\partial}{\partial\nu}\left(x\sin(k\tilde{\beta}(0,y))\right)\,\mathrm{d}\sigma \\ &= -2\int_{\partial\tilde{\Omega}_0}\frac{\partial w}{\partial\nu}(x,y)x\sin(k\tilde{\beta}(0,y))\,\mathrm{d}\sigma + 2\left(\mu - \frac{k^2\pi^2}{\tilde{h}(0)^2}\right)\int_{\tilde{\Omega}_0} w(x,y)x\sin(k\tilde{\beta}(0,y))\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{40}$$

The domain $\tilde{\Omega}_0$ is the domain $\{(x,y)\in\tilde{\Omega}:\rho_L(y)\le x\le 0\}$. The second integral in (40) is bounded in absolute value by $Ck^2(\eta+\delta N^{-3})^3$. The first integral in (40) consists of three terms. Two of these integrals are over portions of the top and bottom boundaries of $\tilde{\Omega}$ of length bounded by $C(\eta+\delta N^{-3})$, and so since the gradient of w is bounded, these integrals are bounded in absolute value by $C(\eta+\delta N^{-3})^2$. The remaining contribution to (40) is given by

$$-2\int_{\rho_B(0)}^{\rho_T(0)}\left(\partial_x-\rho_L'(y)\partial_y\right)w(\rho_L(y),y)\rho_L(y)\sin(k\tilde{\beta}(0,y))\,\mathrm{d}y. \tag{41}$$

To pick out the main term in (41) we write

$$w(x,y)=\sin(\mu_1(N-x))\sin(\tilde{\beta}(0,y))+B(x,y),$$

with $\mu_1^2=\mu_1^2(0)=\mu-\frac{\pi^2}{\tilde{h}(0)^2}$ and for an error term $B(x,y)$ to be estimated below. Using $|\mu_1-\frac{2\pi}{N}|\le CN^{-2}(\eta+\delta)$, (41) becomes

$$\frac{4\pi}{N}\int_{\rho_B(0)}^{\rho_T(0)}\rho_L(y)\sin(\tilde{\beta}(0,y))\sin(k\tilde{\beta}(0,y))\,\mathrm{d}y-2\int_{\rho_B(0)}^{\rho_T(0)}\left(\partial_x-\rho_L'(y)\partial_y\right)B(\rho_L(y),y)\rho_L(y)\sin(k\tilde{\beta}(0,y))\,\mathrm{d}y \tag{42}$$

up to an error $C(\eta+\delta)(\eta+\delta N^{-3})$. We are left to bound the second integral in (42), and to do this we will use the results of Section 4 to estimate $B(x,y)$. Summing the estimate from Proposition 4.1 over $k\ge 2$, we obtain a bound on $\tilde{E}(x,y)$ for $x\ge 0$ of

$$\left\|\tilde{E}(x,y)\right\|_{L^2(U(x))}\le C\left(\frac{\eta}{N\max\{1,x\}}e^{-cx}+\frac{\delta}{N^3}\right),$$

for constants c, C, where we recall that $U(x)$ is the cross-section of $\tilde{\Omega}$ at x. Combining this with Proposition 4.3 shows that for $0\le x\le 1$, we have

$$\|B(x,y)\|_{L^2(U(x))}\le C\left(\frac{\eta}{N\max\{1,x\}}e^{-cx}+\frac{\delta}{N^3}\right)+C(\eta N^{-1}+\delta) \tag{43}$$

Using Lemma 4.2, we can also bound $B(x,y)$ in a different way for $0\le x\le 1$ via

$$|B(x,y)|\le|w(x,y)|+\left|\sin(\mu_1(N-x))\sin(\tilde{\beta}(0,y))\right|\le CN^{-1}(\eta+\delta N^{-2}+x). \tag{44}$$

In particular, using (43) and (44), we have $\|B\|_{L^2(\tilde{\Omega}_1)}\le C\left(\eta^{3/4}+\delta\right)$. Moreover, B satisfies the equation

$$\Delta B=\Delta w+\left(\mu_1^2+\tfrac{\pi^2}{\tilde{h}(0)^2}\right)(w-B)=-\left(\mu_1^2+\tfrac{\pi^2}{\tilde{h}(0)^2}\right)B.$$

We can use this to bound the second integral in (42). Let $\chi(x)$ be a smooth cut-off function, equal to 1 for $x\le\frac14$ and 0 for $x\ge\frac34$. There exists an extension $H(x,y)$ of $\sin(\mu_1(N-x))\sin(\tilde{\beta}(0,y))\big|_{x=\rho_L(y)}$ to $\tilde{\Omega}_1$, with $H(1,y)\equiv 0$ such that

$$\|H\|_{H^1(\Omega_1)}\le C\eta.$$

The function $\chi(x)B(x,y)-H(x,y)$ therefore vanishes on $\partial\tilde{\Omega}_1$, and satisfies

$$\Delta\left(\chi(x)B(x,y)-H(x,y)\right)=(\Delta\chi)B+2\nabla\chi.\nabla B+\chi\Delta B-\Delta H=:F.$$

Elliptic estimates therefore imply that

$$\|\chi B-H\|_{H^1(\tilde{\Omega}_1)}\le\|F\|_{H^{-1}(\tilde{\Omega}_1)}\le C\left(\|B\|_{L^2(\tilde{\Omega}_1)}+\|H\|_{H^1(\tilde{\Omega}_1)}\right).$$

Using $\|B\|_{L^2(\tilde{\Omega}_1)}\le C(\eta^{3/4}+\delta)$, we have the same bound on $\|\chi B-H\|_{H^1(\Omega_1)}$. Elliptic estimates thus give $\left\|\frac{\partial B}{\partial\nu}\right\|_{L^2(D)}\le C(\eta^{3/4}+\delta)$, where $D=\{(\rho_L(y),y):\rho_B(0)\le y\le\rho_T(0)\}$. Applying this estimate in the second integral in (42), we see that the integral can be bounded by $C(\eta+\delta N^{-3})(\eta^{3/4}+\delta)$, and this therefore concludes the proof of the proposition. □

Now consider the case where the domain Ω has flat top and bottom boundaries (so that $\phi_T(x) = 1$, $\phi_B(x) = 0$). Proposition 5.1 then allows us to track the constants appearing in the error estimates in Proposition 2.1: Given N (not necessarily large) and a small constant $c > 0$, we can choose $\eta = \eta(N)$ sufficiently small so that

$$|v_k(0)|, |v_k(N)| \leq \frac{8\eta}{N} + c\left(\frac{\eta}{N} + \frac{k^2\eta}{N^2}\right).$$

By choosing c small compared to 8, we can use this in (30) and (31) to get explicit estimates on $E(x, y)$ and its derivatives, with the estimates not depending on any unknown constants. Using this in the quantities appearing in Section 3, for any $N \geq 8$ fixed and $\eta = \eta(N)$ sufficiently small, this provides the following bounds and proves Corollary 1.1: We have

$$\tau \leq 10^{-4}\eta,$$

where we recall from Lemma 3.2 that the width of the nodal line is bounded by 2τ. Moreover, we have

$$\Lambda_1 \geq 0.3, \qquad \Lambda_2 \geq 1.2,$$

and from Lemmas 3.3 and 3.4 this gives the upper bound on $|g'(y)|$ of

$$|g'(y)| \leq 10^{-2}\eta.$$

Finally, inserting the bounds from Remark 3.1 on the terms appearing in $g''(y)$ gives

$$|g''(y)| \leq 10^{-2}\eta.$$

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