

POINTWISE DISPERSIVE ESTIMATES FOR SCHRÖDINGER OPERATORS ON PRODUCT CONES

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ABSTRACT. We investigate the dispersive properties of solutions to the Schrödinger equation with a weakly decaying radial potential on cones. If the potential has sufficient polynomial decay at infinity, then we show that the Schrödinger flow on each eigenspace of the link manifold satisfies a weighted $L^1 \rightarrow L^\infty$ dispersive estimate. In odd dimensions, the decay rate we compute is consistent with that of the Schrödinger equation in a Euclidean space of the same dimension, but the spatial weights reflect the more complicated regularity issues in frequency that we face in the form of the spectral measure. In even dimensions, we prove a similar estimate, but with a loss of $t^{1/2}$ compared to the sharp Euclidean estimate.

1. INTRODUCTION

Let (X, h) be a smooth, compact Riemannian manifold of dimension $n - 1$, and consider the cone on X , denoted $C(X)$ and defined as $\mathbb{R}^+ \times X$ with metric g given by $g = dr^2 + r^2h$. The corresponding Laplace operator on $C(X)$ is given by

$$\Delta_{C(X)} = \partial_r^2 + \frac{n-1}{r} \partial_r + \frac{1}{r^2} \Delta_h,$$

where Δ_h is the Laplacian on X , taken with the negative semidefinite sign convention. We take $\Delta_{C(X)}$ with the Friedrich's extension for simplicity. We are interested in dispersive estimates for the Schrödinger flow

$$e^{itH} P_c, \quad H = -\Delta_{C(X)} + V, \tag{1.1}$$

where P_c denotes projection onto the continuous spectrum of H . Here, we assume that V is a real-valued radial potential satisfying certain decay assumptions at infinity.

Besides giving direct insight into the behavior of waves, dispersive bounds also have interesting applications in nonlinear problems. For example, stability questions around static solutions in nonlinear models such as wave maps have been studied using dispersive decay estimates. See the work of Krieger-Schlag [44] and more recently Krieger-Miao-Schlag [43] for instance. See also the many works of Lawrie-Oh-Shahshahani [45, 46, 47, 48, 49, 50] for treatment of geometric wave and Schrödinger equations in hyperbolic space. Pointwise decay estimates also play a role in obtaining enhanced existence times using normal form methods, see for instance recent works of Ifrim-Tataru [40] and Germain-Pusateri-Rousset [22]. It is also an intrinsically interesting question to understand the interaction between a

background potential and diffraction in order to better characterize the dynamics of waves on manifolds with conic singularities. Conic manifolds have arisen naturally in the work of Hintz-Vasy and Hafner-Hintz-Vasy on general relativity, see [34, 38, 39] and in particular the recent discussion in the work of Hintz [37].

Dispersive behavior of Schrödinger flows has been studied in a tremendous variety of geometric settings and under many different conditions on the asymptotic decay and regularity properties of the potential V . In \mathbb{R}^n , some of the first ideas arose in the seminal paper of Journé-Soffer-Sogge [41], who proved dispersive decay for $n \geq 3$ with potentials that had no zero energy eigenvalues or resonances and were somewhat strongly decaying and regular. Since then, decay estimates have been improved in a variety of settings. Early works by Goldberg and collaborators carefully addressed the regularity required of the potential in higher dimensions and decay rates in 3 dimensions in the absence of embedded resonance and eigenvalues, see [4, 23, 24, 28, 29].

Further works for perturbations of the Euclidean Laplacian have extended dispersive decay results to the setting where $-\Delta + V$ has an embedded resonance at zero energy, which results in a weaker decay estimate in time, see for instance especially the works of Erdogan-Schlag in 3 dimensions [16, 18], Erdogan-Green in two dimensions [14, 17], Green in 5 dimensions [31], as well as Goldberg-Green and Erdogan-Goldberg-Green in odd and even dimensions ≥ 4 [15, 25, 26, 27]. Recent progress by Blair-Sire-Sogge [7] has pushed the construction of the spectral measure for $-\Delta + V$ to cases where the regularity of the potential V is at very critical levels, though the authors have not explored dispersive decay directly. This is by no means an exhaustive list, but these results are representative of the techniques involved, namely careful control of the free resolvent, the use of resolvent expansions, the role of the regularity of the potential V , and the spectral structure of the operator $-\Delta + V$. The survey article by Wilhelm Schlag [61] contains an excellent overview of the key ideas involved.

Dispersive decay estimates have also been studied in several other geometries. For example, Schrödinger operators with potential were studied on hyperbolic space by David Borthwick and the second author in [8]. See the recent article of Bouclet [9] for a broad overview of results on the asymptotically Euclidean setting, the article by Hassell-Zhang [36] and references therein for results on asymptotically conic manifolds, as well as the articles of Schlag-Soffer-Staubach [62, 63] for manifolds with conical ends. Analysis of the Laplacian on product cones is related to the analysis of Schrödinger operators on \mathbb{R}^n with an inverse square potential which have been studied in various settings, e.g. the works [42, 55, 56, 57, 68] by various authors.

The study of the Laplacian on product cones has a rich history. See the classical results of Cheeger-Taylor, [10, 11], where the spectral measure was first described. As a result, there have been several works that studied evolution equations and their decay estimates on product cones, especially wave equations [3, 5, 6, 19, 20, 51, 71]. See also [1] for information about scattering resonances on hyperbolic cones.

Analysis of dispersive estimates for Schrödinger equations using the resolvent and spectral measure on a product cone has been studied in the recent results of Zhang-Zheng [69, 70]. These are the most closely related results to ours, but only study specific types of potentials that can be treated more perturbatively, hence they need not fully explore the regularity and decay of the spectral measure in the same fashion undertaken here. See also the very recent work of Chen [12] that studies local dispersive behavior on manifolds with non-product conic singularities.

On pure product cones, we prove pointwise decay estimates for the mode-by-mode decomposition of the Schrödinger flow (1.1). By this, we mean that if $\{\varphi_j\}_{j=0}^\infty$ is a basis of $L^2(X)$ consisting of eigenfunctions of Δ_h , then the Schrödinger flow on $C(X)$ can be formally decomposed as

$$e^{itH} P_c = \sum_{j=0}^{\infty} e^{itH} P_c E_j, \quad (1.2)$$

where $E_j : L^2(X) \rightarrow L^2(X)$ denotes projection on to the linear span of φ_j . We show that if $V \in \rho^{-2\sigma} L^\infty(\mathbb{R}^+)$ for σ sufficiently large, where $\rho(r) = 1 + r$ is a weight function, and if the perturbed resolvent

$$R_V(z) := (-\Delta_{C(X)} + V - z^2)^{-1}$$

does not have a pole at zero, then each component of (1.2) satisfies a weighted pointwise estimate. In odd dimensions, we prove this with the same $t^{-\frac{n}{2}}$ decay rate as in the Euclidean case, while in even dimensions, there is a loss of $t^{\frac{1}{2}}$ which we do not expect to be sharp.

Theorem 1. *Suppose $C(X)$ is of odd dimension $n \geq 3$. Let $V \in \rho^{-2\sigma} L^\infty(\mathbb{R}^+)$ with*

$$\sigma > 2n \left\lceil \frac{n}{4} \right\rceil.$$

If $R_V(z)$ does not have a pole at $z = 0$, then for any $\alpha \geq 2 \left\lceil \frac{n}{4} \right\rceil (n - 2) - \frac{n-1}{2} + 2$, we have

$$\|\rho^{-\alpha} e^{itH} P_c E_j f\|_{L^\infty(\mathbb{R}^+)} \leq C_{j,\alpha,\sigma} t^{-\frac{n}{2}} \|\rho^\alpha E_j f\|_{L^1(\mathbb{R}^+, r^{n-1} dr)}, \quad (1.3)$$

for some $C_{j,\alpha,\sigma} > 0$.

Remark 1.1. In the case where n is even, the techniques of this article give a slightly weaker estimate of the form

$$\|\rho^{-\alpha} e^{itH} P_c E_j f\|_{L^\infty(\mathbb{R}^+)} \leq C_{j,\alpha,\sigma} t^{-\frac{n-1}{2}} \|\rho^\alpha E_j f\|_{L^1(\mathbb{R}^+, r^{n-1} dr)}, \quad (1.4)$$

for analogous conditions on V and α , where the loss of the $\frac{1}{2}$ power of decay in t arises as a result of regularity issues encountered in the analysis of the spectral measure near zero energy. We expect that with more sophisticated techniques it may be possible to improve this estimate to give the full $t^{-\frac{n}{2}}$ decay rate exhibited in \mathbb{R}^n .

The strategy for proving Theorem 1 is to use the representation of the Schrödinger flow as an oscillatory integral with respect to the continuous part of the spectral measure of $-\Delta_{C(X)} + V$, denoted $d\Pi_V$, which gives

$$e^{itH} P_c = \int_{\mathbb{R}} e^{it\lambda^2} d\Pi_V(\lambda).$$

By Stone's formula, the spectral measure can be expressed in terms of the boundary values of the perturbed resolvent near the continuous spectrum as

$$\begin{aligned} d\Pi_V(\lambda; x, y) &= \frac{1}{2\pi i} [R_V(\lambda + i0; x, y) - R_V(\lambda - i0; x, y)] \lambda d\lambda \\ &= \frac{1}{\pi} \operatorname{Im} R_V(\lambda + i0; x, y) \lambda d\lambda. \end{aligned}$$

Through the use of integration by parts, the $L^1 \rightarrow L^\infty$ mapping properties of the linear Schrödinger flow can be inferred from pointwise estimates on $\operatorname{Im} R_V(\lambda + i0; x, y)$. We accomplish this by decomposing R_V with respect to the basis of eigenfunctions $\{\varphi_j\}$ as

$$R_V(z) = \sum_{j=0}^{\infty} R_{V,j}(z) E_j,$$

for an associated collection of resolvent operators $R_{V,j}$ acting in the radial variable. By analyzing these “radial resolvents,” we are able to demonstrate that $-\Delta_{C(X)} + V$ has no eigenvalues or resonances embedded in the continuous spectrum, and thus we establish weighted L^2 mapping properties for each $R_{V,j}$. To obtain pointwise bounds, we then expand each $R_{V,j}$ in a Birman-Schwinger series to reduce the proof to establishing estimates on the free resolvent, which is easier to analyze due to its explicit nature. In doing this, we find that these estimates are not necessarily summable in j , which is why the projections E_j are required in Theorem 1. Similar weighted mode-by-mode estimates are obtained in the works of Schlag-Soffer-Staubach [62, 63] in the case of surfaces of revolution and related mode by mode decay rates were established for the wave equation on the Schwarzschild space-time in Donniger-Schlag-Soffer [13].

1.1. Outline of the Paper. In Section 2, we discuss the form of the free resolvent on product cones. Then, in Section 3, we prove operator bounds on the free resolvent using properties of Bessel and Hankel functions. Then, we prove weighted L^2 operator bounds on the perturbed resolvent in Section 4. Then, in Section 5, we prove the full pointwise bounds necessary on the perturbed resolvent using a Birman-Schwinger expansion. Finally, in Section 6, we use the representation of the spectral measure in terms of the resolvent combined with the estimates from the previous sections to prove Theorem 1. We also give two appendices at the end of the paper. The first, Appendix A, uses ideas from [19] to give a modified full dispersive estimate on all frequencies provided we measure in L^2 on the cone link. Then, in Appendix B, we discuss absence of embedded eigenvalues for general

Schrödinger operators on product cones, which is a critical component of the proofs in Section 4.

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2. THE FREE RESOLVENT

In this section, we construct the integral kernel for the free resolvent operator

$$R_0(z) = (-\Delta_{C(X)} - z^2)^{-1} : L^2(C(X)) \rightarrow L^2(C(X)), \quad (2.1)$$

for $\text{Im } z \neq 0$, closely following the exposition of [3]. This is equivalent to analyzing solutions of the equation

$$(-\Delta_{C(X)} - z^2)u = f \quad (2.2)$$

for $f \in L^2(C(X))$. To proceed, we decompose u and f into the basis $\{\varphi_j\}$ of eigenfunctions on X as

$$f(r, \theta) = \sum_{j=1}^{\infty} f_j(r) \varphi_j(\theta), \quad u(r, \theta) = \sum_{j=1}^{\infty} u_j(r) \varphi_j(\theta).$$

Denote by $-\mu_j^2$ the eigenvalues of Δ_h associated to each φ_j . Then, we obtain that (2.2) is equivalent to the collection of equations

$$\left(\partial_r^2 + \frac{n-1}{r} \partial_r + z^2 - \frac{\mu_j^2}{r^2} \right) u_j(r) = -f_j(r), \quad j = 0, 1, 2, \dots \quad (2.3)$$

Therefore, we can express the resolvent $R_0(z)$ as

$$R_0(z)f(r, \theta) = \sum_{j=0}^{\infty} u_j(r) \varphi_j(\theta),$$

with u_j as above. If we define the j th *radial resolvent* $R_{0,j}(z)$ by

$$R_{0,j}(z) = \left(\partial_r^2 + \frac{n-1}{r} \partial_r + z^2 - \frac{\mu_j^2}{r^2} \right)^{-1} \quad (2.4)$$

as an operator on $L^2(\mathbb{R}^+, r^{n-1} dr)$, then the full resolvent is given by

$$R_0(z)f(r, \theta) = \sum_{j=0}^{\infty} R_{0,j}(z)f_j(r) \varphi_j(\theta).$$

For each j , the defining equation (2.3) for $R_{0,j}(z)f_j$ is an ODE with a regular singular point at zero, and so by applying the Frobenius method we find that the indicial roots of the equation are $-\frac{n-2}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + \mu_j^2}$. For this reason, we introduce the notations $\delta = -\frac{n-2}{2}$ and $\nu_j = \sqrt{\left(\frac{n-2}{2}\right)^2 + \mu_j^2}$. The structure of the indicial roots suggests that we rescale by r^δ , and so we define ω_j by $u_j(r) = r^\delta \omega_j(r)$ so that ω_j is analytic near $r = 0$. Then, (2.3) becomes

$$\partial_r^2 \omega_j + \frac{1}{r} \partial_r \omega_j + \left(z^2 - \frac{\nu_j^2}{r^2} \right) \omega_j = -r^{-\delta} f_j(r), \quad j = 0, 1, 2, \dots$$

If $z \neq 0$, we can change variables via $\zeta = zr$ to obtain the following inhomogeneous Bessel equation of order ν_j :

$$\tilde{\omega}_j'' + \frac{1}{\zeta} \tilde{\omega}_j' + \left(1 - \frac{\nu_j^2}{\zeta^2} \right) \tilde{\omega}_j = -\frac{\zeta^{-\delta}}{z^2} f_j(\zeta/z), \quad (2.5)$$

where $\tilde{\omega}_j(\zeta) = \omega_j(\zeta/z)$, and the ‘‘prime’’ notation denotes the complex derivative with respect to ζ . For notational convenience, we define $f_{j,z}(\zeta) := -\frac{\zeta^{-\delta}}{z^2} f_j(\zeta/z)$.

The solutions to the homogeneous Bessel equation of order ν are the well-known Bessel functions of the first and second kind, denoted J_ν and Y_ν , respectively. Closely related to these are the Hankel functions $H_\nu^{(1)}$ and $H_\nu^{(2)}$, given by

$$H^{(1)} = J_\nu + iY_\nu, \quad H^{(2)} = J_\nu - iY_\nu.$$

Any two of these Bessel and/or Hankel functions can be used to form a fundamental solution set for the homogeneous equation. Given an appropriate choice of fundamental solution set, we use the method of variation of parameters to construct solutions to the inhomogeneous problem. So let y_1, y_2 be a fundamental solution set for the homogeneous problem associated to (2.5) for some fixed j . We then construct our solution $\tilde{\omega}_j$ as

$$\tilde{\omega}_j = v_1 y_1 + v_2 y_2$$

where all objects above are functions of ζ . Straightforward calculations show that if

$$v_1'(\zeta) = -\frac{y_2(\zeta) f_{j,z}(\zeta)}{\mathscr{W}(y_1, y_2)(\zeta)}, \quad \text{and} \quad v_2'(\zeta) = \frac{y_1(\zeta) f_{j,z}(\zeta)}{\mathscr{W}(y_1, y_2)(\zeta)},$$

then $\tilde{\omega}_j$ as given above solves the inhomogeneous equation (2.5), where $\mathscr{W}(y_1, y_2)(\zeta)$ denotes the Wronskian determinant of y_1 and y_2 evaluated at ζ . Therefore, we may compute v_1 and v_2 by taking path integrals in the complex plane, which yields

$$\tilde{\omega}_j(\zeta) = \left(\int_{\mathcal{C}_1(\zeta)} -\frac{y_2(\xi) f_{j,z}(\xi)}{\mathscr{W}(y_1, y_2)(\xi)} d\xi \right) y_1(\zeta) + \left(\int_{\mathcal{C}_2(\zeta)} \frac{y_1(\xi) f_{j,z}(\xi)}{\mathscr{W}(y_1, y_2)(\xi)} d\xi \right) y_2(\zeta)$$

where $\mathcal{C}_1(\zeta)$, $\mathcal{C}_2(\zeta)$ are any complex contours connecting fixed points $c_1, c_2 \in \mathbb{C}$ to ζ , respectively. In fact, it suffices to take c_1, c_2 on the real line. We then choose our contours

to be the piecewise linear paths defined by

$$\mathcal{C}_1(\zeta) = \{(1-t)c_1 + t \operatorname{Re} \zeta : t \in [0, 1]\} \cup \{\operatorname{Re} \zeta + it \operatorname{Im} \zeta : t \in [0, 1]\}$$

and

$$\mathcal{C}_2(\zeta) = \{(1-t)c_2 + t \operatorname{Re} \zeta : t \in [0, 1]\} \cup \{\operatorname{Re} \zeta + it \operatorname{Im} \zeta : t \in [0, 1]\}.$$

Of particular interest are the boundary values of the resolvent near the continuous spectrum of $-\Delta_{C(X)} + V$. Therefore, if we consider $z = \lambda \pm i\varepsilon$, we have

$$\begin{aligned} \tilde{\omega}_j(zr) = & y_1(zr) \left(\int_{c_1}^{\lambda r} \frac{-y_2(t)f_{j,z}(t)}{\mathcal{W}(y_1, y_2)(t)} dt + i \int_0^{\pm\varepsilon r} \frac{-y_2(\lambda r + it)f_{j,z}(\lambda r + it)}{\mathcal{W}(y_1, y_2)(\lambda r + it)} dt \right) \\ & + y_2(zr) \left(\int_{c_2}^{\lambda r} \frac{y_1(t)f_{j,z}(t)}{\mathcal{W}(y_1, y_2)(t)} dt + i \int_0^{\pm\varepsilon r} \frac{y_1(\lambda r + it)f_{j,z}(\lambda r + it)}{\mathcal{W}(y_1, y_2)(\lambda r + it)} dt \right) \end{aligned}$$

All that remains is to determine that appropriate fundamental solution set y_1, y_2 and constants c_1, c_2 so that our solution is a well defined element of $L^2(C(X))$. If we take $y_2 = J_{\nu_j}$ and $c_1 = 0$, then $\tilde{\omega}_j$ is bounded as $r \rightarrow 0$, provided that the coefficient integrals converge. We then choose y_1 to be either $H_{\nu_j}^{(1)}$ or $H_{\nu_j}^{(2)}$, depending on the sign of $\operatorname{Im} z$. By the asymptotic forms of the Hankel functions, we have

$$H_{\nu_j}^{(1)}(\zeta) \sim \sqrt{\frac{2}{\pi\zeta}} e^{i(\zeta - \frac{\nu_j\pi}{2} - \frac{\pi}{4})}$$

and

$$H_{\nu_j}^{(2)}(\zeta) \sim \sqrt{\frac{2}{\pi\zeta}} e^{-i(\zeta - \frac{\nu_j\pi}{2} - \frac{\pi}{4})}$$

for $-\pi < \arg \zeta < \pi$, and the branch of the square root is defined by $\zeta^{1/2} = e^{\frac{1}{2}(\ln|\zeta| + i\arg \zeta)}$ for such ζ . We can now see that if $z = \lambda + i\varepsilon$, then $\zeta = zr$ also has positive imaginary part, and so $H_{\nu_j}^{(1)}(zr)$ decays exponentially as $r \rightarrow \infty$, while $H_{\nu_j}^{(2)}$ exhibits exponential growth. Hence, when $z = \lambda + i\varepsilon$ we take $y_1 = H_{\nu_j}^{(1)}$ and $c_2 = \infty$, which yields

$$\begin{aligned} \tilde{\omega}_j(zr) = & H_{\nu_j}^{(1)}(zr) \left(\int_0^{\lambda r} \frac{J_{\nu_j}(t)f_{j,z}(t)}{2i/(\pi t)} dt + i \int_0^{\pm\varepsilon r} \frac{J_{\nu_j}(\lambda r + it)f_{j,z}(\lambda r + it)}{2i/[\pi(\lambda r + it)]} dt \right) \\ & + J_{\nu_j}(zr) \left(\int_{\lambda r}^{\infty} \frac{H_{\nu_j}^{(1)}(t)f_{j,z}(t)}{2i/(\pi t)} dt - i \int_0^{\pm\varepsilon r} \frac{H_{\nu_j}^{(1)}(\lambda r + it)f_{j,z}(\lambda r + it)}{2i/[\pi(\lambda r + it)]} dt \right), \end{aligned}$$

since $\mathcal{W}(H_{\nu_j}^{(1)}, J_{\nu_j})(\xi) = -\frac{2i}{\pi\xi}$. We can then take the limit as $\varepsilon \rightarrow 0$ to obtain

$$\tilde{\omega}_j(\lambda r) = \frac{\pi}{2i} H_{\nu_j}^{(1)}(zr) \int_0^{\lambda r} t J_{\nu_j}(t) f_{j,z}(t) dt + \frac{\pi}{2i} J_{\nu_j}(zr) \int_{\lambda r}^{\infty} t H_{\nu_j}^{(1)}(t) f_{j,z}(t) dt.$$

Recalling that $u_j(r) = (zr)^\delta \tilde{\omega}_j(zr)$ and $f_{j,z}(t) = -\frac{t^{\frac{n-2}{2}}}{z^2} f_j(t/z)$, we get that the outgoing solution corresponding to the j th resolvent is

$$\begin{aligned} u_j(r) &= \frac{\pi i}{2} (\lambda r)^{-\frac{n-2}{2}} H_{\nu_j}^{(1)}(\lambda r) \int_0^{\lambda r} \frac{t^{\frac{n}{2}} J_{\nu_j}(t) f_j(t/\lambda)}{\lambda^2} dt \\ &\quad + \frac{\pi i}{2} (\lambda r)^{-\frac{n-2}{2}} J_{\nu_j}(\lambda r) \int_{\lambda r}^{\infty} \frac{t^{\frac{n}{2}} H_{\nu_j}^{(1)}(t) f_j(t/\lambda)}{\lambda^2} dt. \end{aligned}$$

If we then change variables via $t = \lambda s$, we can rewrite the above as

$$u_j(r) = \frac{\pi i}{2} r^{-\frac{n-2}{2}} H_{\nu_j}^{(1)}(\lambda r) \int_0^r s^{\frac{n}{2}} J_{\nu_j}(\lambda s) f_j(s) ds + \frac{\pi i}{2} r^{-\frac{n-2}{2}} J_{\nu_j}(\lambda r) \int_r^{\infty} s^{\frac{n}{2}} H_{\nu_j}^{(1)}(\lambda s) f_j(s) ds.$$

The integral kernel of $R_{0,j}(\lambda + i0)$ with respect to the measure $s^{n-1} ds$ is therefore given by

$$R_{0,j}(\lambda + i0; r, s) = \begin{cases} \frac{\pi i}{2} (rs)^{-\frac{n-2}{2}} J_{\nu_j}(\lambda s) H_{\nu_j}^{(1)}(\lambda r), & s < r \\ \frac{\pi i}{2} (rs)^{-\frac{n-2}{2}} J_{\nu_j}(\lambda r) H_{\nu_j}^{(1)}(\lambda s), & s > r, \end{cases} \quad (2.6)$$

since $s^{\frac{n}{2}} = s^{n-1} s^{-\frac{n-2}{2}}$.

We can repeat this analysis for $z = \lambda - i\varepsilon$, and we find that we must take use $H_{\nu_j}^{(2)}$ instead of $H_{\nu_j}^{(1)}$ due to the asymptotic behavior at infinity, which also causes the Wronskian to change sign, but otherwise the calculations are identical. We therefore obtain

$$R_{0,j}(\lambda - i0; r, s) = \begin{cases} \frac{\pi}{2i} (rs)^{-\frac{n-2}{2}} J_{\nu_j}(\lambda s) H_{\nu_j}^{(2)}(\lambda r), & s < r \\ \frac{\pi}{2i} (rs)^{-\frac{n-2}{2}} J_{\nu_j}(\lambda r) H_{\nu_j}^{(2)}(\lambda s), & s > r. \end{cases} \quad (2.7)$$

We can therefore express the imaginary part of the resolvent kernels $R_{0,j}$ as follows.

Lemma 2.1. *For λ real, we have*

$$\text{Im } R_{0,j}(\lambda + i0; r, s) = \frac{\pi}{2} (rs)^{-\frac{n-2}{2}} J_{\nu_j}(\lambda r) J_{\nu_j}(\lambda s)$$

as an integral kernel with respect to the measure $s^{n-1} ds$.

Proof. This follows immediately from the fact that

$$H_{\nu_j}^{(1)} + H_{\nu_j}^{(2)} = (J_{\nu_j} + iY_{\nu_j}) + (J_{\nu_j} - iY_{\nu_j}) = 2J_{\nu_j}.$$

□

We can now write down an expression for the spectral measure of $-\Delta_{C(X)}$ as in [10], which follows from Stone's formula.

Lemma 2.2. *For λ real,*

$$\operatorname{Im} R_0(\lambda + i0; x, y) = \frac{\pi}{2} (rs)^{-\frac{n-2}{2}} \sum_{j=0}^{\infty} J_{\nu_j}(\lambda r) J_{\nu_j}(\lambda s) \varphi_j(\theta_1) \overline{\varphi_j(\theta_2)}$$

where $x = (r, \theta_1)$ and $y = (s, \theta_2)$ are points in $C(X)$. Moreover, the absolutely continuous part of the spectral measure of $-\Delta_{C(X)}$, with the convention that λ^2 is the spectral parameter, is given by

$$\begin{aligned} d\Pi_0(\lambda; x, y) &= \frac{1}{2\pi i} [R_0(\lambda + i0; x, y) - R_0(\lambda - i0; x, y)] 2\lambda d\lambda \\ &= \sum_{j=0}^{\infty} (rs)^{-\frac{n-2}{2}} J_{\nu_j}(\lambda r) J_{\nu_j}(\lambda s) \varphi_j(\theta_1) \overline{\varphi_j(\theta_2)} \lambda d\lambda \end{aligned}$$

Remark 2.3. We note that by using the convention that λ^2 is the spectral parameter, as opposed to λ , we have the following relations between the boundary values of the resolvent:

$$R_{0,j}(\lambda + i0) = R_{0,j}(-\lambda - i0), \quad \text{and} \quad R_{0,j}(\lambda - i0) = R_{0,j}(-\lambda + i0)$$

for $\lambda \in \mathbb{R}$. This allows us to reduce many of the proofs in the following sections to the case where we consider only $R_{0,j}(\lambda + i0)$ with $\lambda > 0$.

3. ESTIMATES ON THE FREE RESOLVENT

Here, we prove a variety of weighted estimates on the unperturbed radial resolvents $R_{0,j}$. These estimates heavily rely on the asymptotic formulae for the Bessel and Hankel functions near zero and infinity. Of particular interest is the behavior of $R_{0,j}$ measured in the weighted L^q spaces defined by

$$L^{q,\sigma}(\mathbb{R}^+, r^{n-1} dr) = \left\{ f : \mathbb{R}^+ \rightarrow \mathbb{C} : \int_0^\infty |f(r)|^q \rho^{q\sigma}(r) r^{n-1} dr < \infty \right\},$$

where $\rho(r) = 1 + r$. For ease of notation, we simply write $L^{q,\sigma}$ to denote the space $L^{q,\sigma}(\mathbb{R}^+, r^{n-1} dr)$ where there can be no confusion. The estimates for the free resolvent on these spaces will prove useful in Sections 4 and 5 for establishing the mapping properties of the perturbed resolvent.

We begin with a version of the Limiting Absorption Principle for the radial resolvents on a product cone.

Proposition 3.1. *Let $k \geq 0$ be an integer. Then for any $\sigma > \frac{1}{2} + k$,*

$$\|\partial_\lambda^k R_{0,j}(\lambda \pm i0)\|_{L^{2,\sigma} \rightarrow L^{2,-\sigma}} \leq \frac{C_{j,k,\sigma}}{|\lambda|} \quad (3.1)$$

for all $|\lambda| \geq 1$.

Remark 3.2. A noteworthy observation here is that the constant $C_{j,k,\sigma}$ in Proposition 3.1 can exhibit rather dramatic growth as a function of j . This is due to the fact that the constants in the asymptotic formulae for the Bessel and Hankel functions are poorly behaved with respect to order, and is the reason why we must restrict to the mode-by-mode analysis.

Proof. Note that it suffices to prove the proposition for $R_{0,j}(\lambda + i0)$ and for λ positive, since the choice of sign makes no difference in the proof. If $f \in L^{2,\sigma}$, we have

$$\|\partial_\lambda^k R_{0,j}(\lambda + i0)f\|_{L^{2,-\sigma}}^2 = \int_0^\infty \left| \int_0^\infty \partial_\lambda^k R_{0,j}(\lambda + i0; r, s) f(s) s^{n-1} ds \right|^2 (1+r)^{-2\sigma} r^{n-1} dr.$$

Artificially inserting a factor of $(1+s)^{-\sigma}(1+s)^\sigma$ and applying Cauchy-Schwartz, we see that

$$\begin{aligned} \|\partial_\lambda^k R_{0,j}(\lambda + i0)f\|_{L^{2,-\sigma}}^2 &\leq \int_0^\infty \|\partial_\lambda^k R_{0,j}(\lambda + i0; r, \cdot)\|_{L_s^{2,-\sigma}}^2 \|f\|_{L_s^{2,\sigma}}^2 (1+r)^{-2\sigma} r^{n-1} dr \\ &= \|\partial_\lambda^k R_{0,j}(\lambda + i0; \cdot, \cdot)\|_{L_s^{2,-\sigma} L_r^{2,-\sigma}}^2 \|f\|_{L_s^{2,\sigma}}^2. \end{aligned}$$

Hence, it suffices to show that the kernel satisfies

$$\|\partial_\lambda^k R_{0,j}(\lambda + i0; \cdot, \cdot)\|_{L_s^{2,-\sigma} L_r^{2,-\sigma}}^2 \leq \frac{C}{\lambda^2}.$$

By definition,

$$\|\partial_\lambda^k R_{0,j}(\lambda + i0; \cdot, \cdot)\|_{L_s^{2,-\sigma} L_r^{2,-\sigma}}^2 = \int_0^\infty \int_0^\infty \partial_\lambda^k R_{0,j}(\lambda + i0; r, s) (1+s)^{-2\sigma} (1+r)^{-2\sigma} (rs)^{n-1} ds dr, \quad (3.2)$$

Recalling the piecewise formula (2.6) for the resolvent kernel, we have that

$$\begin{aligned} & \|\partial_\lambda^k R_{0,j}(\lambda + i0; \cdot, \cdot)\|_{L_s^{2,-\sigma} L_r^{2,-\sigma}}^2 \\ &= \frac{\pi^2}{4} \int_0^\infty \int_0^r \left[\partial_\lambda^k \left(H_{\nu_j}^{(1)}(\lambda r) J_{\nu_j}(\lambda s) \right) \right]^2 (rs)(1+r)^{-2\sigma}(1+s)^{-2\sigma} ds dr \\ &+ \frac{\pi^2}{4} \int_0^\infty \int_r^\infty \left[\partial_\lambda^k \left(H_{\nu_j}^{(1)}(\lambda s) J_{\nu_j}(\lambda r) \right) \right]^2 (rs)(1+r)^{-2\sigma}(1+s)^{-2\sigma} ds dr. \end{aligned} \quad (3.3)$$

By changing the order of integration, we get that the first term on the right-hand side of (3.3) can be rewritten as

$$\frac{\pi^2}{4} \int_0^\infty \int_s^\infty \left[\partial_\lambda^k \left(H_{\nu_j}^{(1)}(\lambda r) J_{\nu_j}(\lambda s) \right) \right]^2 (rs)(1+r)^{-2\sigma}(1+s)^{-2\sigma} dr ds. \quad (3.4)$$

We note that up to a relabeling of r, s , this is exactly equal to the second term in (3.3), and hence

$$\begin{aligned} & \|\partial_\lambda^k R_{0,j}(\lambda + i0; \cdot, \cdot)\|_{L_s^{2,-\sigma} L_r^{2,-\sigma}}^2 = \\ & \frac{\pi^2}{2} \int_0^\infty \int_r^\infty \left[\partial_\lambda^k \left(H_{\nu_j}^{(1)}(\lambda s) J_{\nu_j}(\lambda r) \right) \right]^2 (rs)(1+r)^{-2\sigma}(1+s)^{-2\sigma} ds dr. \end{aligned} \quad (3.5)$$

Note that if $\mathcal{C}_\nu(x)$ is either a Bessel or Hankel function of order ν , we have

$$\mathcal{C}'_\nu(x) = \frac{1}{2} (\mathcal{C}_{\nu+1}(x) - \mathcal{C}_{\nu-1}(x)), \quad (3.6)$$

and so the triangle inequality reduces the proof of Proposition 3.1 to showing that the following lemma holds.

□

Lemma 3.3. *Let ℓ, m, k be nonnegative integers with $\ell + m = k$, and suppose $\alpha, \beta \in \mathbb{Z}$ are such that $|\alpha| \leq \ell$ and $|\beta| \leq m$. Then for any $\nu \geq \frac{n-2}{2}$, there exists a $C > 0$ depending only on k, ν such that*

$$\int_0^\infty \int_r^\infty |J_{\nu+\alpha}(\lambda r)|^2 |H_{\nu+\beta}(\lambda s)|^2 r^{1+2\ell} s^{1+2m} (1+s)^{-2\sigma} (1+r)^{-2\sigma} ds dr \leq \frac{C}{\lambda^2}, \quad \lambda \geq 1, \quad (3.7)$$

provided that $\sigma > \frac{1}{2} + k$.

Proof. This proof, and others which follow it, make extensive use of asymptotic estimates for the Bessel and Hankel functions, which we record here for later use. For any $\nu \in \mathbb{R}$,

there exist constants $C_\nu, C'_\nu > 0$ such that when $0 < |\tau| \leq 1$,

$$|J_\nu(\tau)| \leq C_\nu \tau^\nu, \quad |H_\nu^{(1)}(\tau)| \leq C_\nu \tau^{-\nu} \quad (3.8)$$

and when $|\tau| \geq 1$,

$$|J_\nu(\tau)| \leq C'_\nu \tau^{-\frac{1}{2}}, \quad |H_\nu^{(1)}(\tau)| \leq C'_\nu \tau^{-\frac{1}{2}}. \quad (3.9)$$

To prove Lemma 3.3, let us first write the left-hand side of (3.7) as $I(\lambda) + \mathbb{I}(\lambda)$, where each term is obtained by restricting the integral in the r variable to $0 < r < \frac{1}{\lambda}$ and $\frac{1}{\lambda} < r < \infty$, respectively. To estimate $I(\lambda)$, note that by (3.9), we have

$$\int_{\frac{1}{\lambda}}^{\infty} s^{1+2m} (1+s)^{-2\sigma} |H_{\nu+\beta}(\lambda s)|^2 ds \leq \frac{C}{\lambda} \int_{\frac{1}{\lambda}}^{\infty} s^{2m} (1+s)^{-2\sigma} ds \leq \frac{C'}{\lambda},$$

as long as $\sigma > m + \frac{1}{2}$. Combining this with (3.8), we have

$$\begin{aligned} \int_0^{\frac{1}{\lambda}} \int_{\frac{1}{\lambda}}^{\infty} |J_{\nu+\alpha}(\lambda r)|^2 |H_{\nu+\beta}(\lambda s)|^2 r^{1+2\ell} s^{1+2m} (1+s)^{-2\sigma} (1+r)^{-2\sigma} ds dr &\leq \frac{C'}{\lambda} \int_0^{\frac{1}{\lambda}} r^{1+2\ell} (\lambda r)^{2(\nu+\alpha)} dr \\ &\leq \frac{C''}{\lambda^2} \end{aligned} \quad (3.10)$$

for any $\ell \geq 0$, since $\lambda \geq 1$. Furthermore, if $0 < r \leq \frac{1}{\lambda}$, we have

$$\begin{aligned} \int_r^{\frac{1}{\lambda}} s^{1+2m} (1+s)^{-2\sigma} |H_{\nu+\beta}(\lambda s)|^2 ds &\leq \frac{C}{\lambda^{2(\nu+\beta)}} \int_r^{\frac{1}{\lambda}} s^{1+2(m-\nu-\beta)} ds \\ &\leq \frac{C}{\lambda^{2(\nu+\beta)}} r^{1+2(m-\nu-\beta)} \left(\frac{1}{\lambda} - r \right) \leq \frac{C'}{\lambda} (\lambda r)^{-2(\nu+\beta)} r^{2m} \end{aligned}$$

since $1 + 2(m - \beta - \nu) \leq 0$. Hence, if we recall that $k = \ell + m \geq |\alpha| + |\beta|$ and apply (3.8), we have

$$\begin{aligned}
& \int_0^{\frac{1}{\lambda}} \int_r^{\frac{1}{\lambda}} |J_{\nu+\alpha}(\lambda r)|^2 |H_{\nu+\beta}(\lambda s)|^2 r^{1+2\ell} s^{1+2m} (1+s)^{-2\sigma} (1+r)^{-2\sigma} ds dr \\
& \leq \frac{C'}{\lambda} \int_0^{\frac{1}{\lambda}} r^{1+2k} (\lambda r)^{2(\alpha-\beta)} dr = \frac{C'}{\lambda^{2k+2}} \int_0^{\frac{1}{\lambda}} (\lambda r)^{1+2(k+\alpha-\beta)} dr \\
& \leq \frac{C'}{\lambda^{2k+3}} \leq \frac{C'}{\lambda^2}
\end{aligned}$$

for all $k \geq 0$ when $\lambda \geq 1$. Combining this with (3.10) proves that $I(\lambda) \leq \frac{C}{\lambda^2}$ for some $C > 0$ and all $\lambda \geq 1$. Since for any fixed k there are only finitely many possibilities for ℓ, m, α, β , we can choose C to depend only on k and ν .

Now, to estimate $II(\lambda)$, we apply (3.9) to both the Bessel and Hankel functions to obtain

$$\begin{aligned}
II(\lambda) & \leq C \int_{\frac{1}{\lambda}}^{\infty} \int_{\frac{1}{\lambda}}^{\infty} r^{1+2\ell} (1+r)^{-2\sigma} (\lambda r)^{-1} s^{1+2m} (1+s)^{-2\sigma} (\lambda s)^{-1} ds dr \\
& = \frac{C}{\lambda^2} \int_0^{\infty} \int_0^{\infty} (1+r)^{2(\ell-\sigma)} (1+s)^{2(m-\sigma)} ds dr \\
& \leq \frac{C}{\lambda^2},
\end{aligned}$$

provided that $\sigma > k + \frac{1}{2}$, which completes the proof of Lemma 3.3. \square

It will also prove useful to have a bound on the $L^{2,\sigma} \rightarrow L^{2,-\sigma}$ mapping properties of the imaginary part of each $R_{0,j}$ when λ is small. In particular, we are able to show that this operator norm has a precise polynomial rate of vanishing as $\lambda \rightarrow 0$.

Proposition 3.4. *For any integer $k \geq 0$ and any $\sigma > \frac{n}{2} + k$, we have that*

$$\|\partial_\lambda^k \operatorname{Im} R_{0,j}(\lambda \pm i0)\|_{L^{2,\sigma} \rightarrow L^{2,-\sigma}} \leq C_{j,k,\sigma} \lambda^{n-2-k}$$

when $0 < |\lambda| \leq 1$.

Proof. By the discussion at the beginning of the proof of Proposition 3.1, it is sufficient to show that

$$\|\partial_\lambda^k \operatorname{Im} R_{0,j}(\lambda + i0; r, s)\|_{L_r^{2,-\sigma} L_s^{2,-\sigma}} \leq C \lambda^{n-2-k} \quad (3.11)$$

for $0 < \lambda \leq 1$. By (3.5) we have

$$\begin{aligned} & \|\partial_\lambda^k \operatorname{Im} R_{0,j}(\lambda + i0; r, s)\|_{L_r^{2,-\sigma} L_s^{2,-\sigma}}^2 \\ &= C \int_0^\infty \int_0^\infty [\partial_\lambda^k (J_{\nu_j}(\lambda r) J_{\nu_j}(\lambda s))]^2 (1+r)^{-2\sigma} (1+s)^{-2\sigma} (rs) \, dr \, ds. \end{aligned}$$

Using the recursive formula for derivatives of the Bessel functions as before, we can reduce the proof to showing that

$$\int_0^\infty \int_0^\infty r^{1+2\ell} s^{1+2m} |J_{\nu_j+\alpha}(\lambda r)|^2 |J_{\nu_j+\beta}(\lambda s)|^2 (1+r)^{-2\sigma} (1+s)^{-2\sigma} \, dr \, ds \leq C \lambda^{2(n-2-k)} \quad (3.12)$$

for any integers $\ell, m \geq 0$ with $\ell + m = k$ and integers α, β with $|\alpha| \leq \ell$ and $|\beta| \leq m$. Since the above integral is separable, it is in fact enough to show

$$\int_0^\infty r^{1+2\ell} |J_{\nu_j+\alpha}(\lambda r)|^2 (1+r)^{-2\sigma} \, dr \leq C \lambda^{n-2-2\alpha} \quad (3.13)$$

for any $\ell \leq k$ and $|\alpha| \leq \ell$, since analogous estimates will apply to the integral in the s variable. First, notice that (3.8) implies

$$\begin{aligned} & \int_0^{\frac{1}{\lambda}} r^{1+2\ell} |J_{\nu_j+\alpha}(\lambda r)|^2 (1+r)^{-2\sigma} \, dr \leq C \int_0^{\frac{1}{\lambda}} r^{1+2\ell} (\lambda r)^{2(\nu_j+\alpha)} (1+r)^{-2\sigma} \, dr \\ &= C \lambda^{2(\nu_j+\alpha)} \int_0^{\frac{1}{\lambda}} r^{1+2(\ell+\alpha+\nu_j)} (1+r)^{-2\sigma} \, dr \leq C' \lambda^{2(\nu_j+\alpha)} \int_0^{\frac{1}{\lambda}} (1+r)^{1+2(\ell+\alpha+\nu_j-\sigma)} \, dr \\ &\leq C'' \lambda^{2(\nu_j+\alpha)} \left| \left(1 + \frac{1}{\lambda}\right)^{2+2(\ell+\alpha+\nu_j-\sigma)} - 1 \right| \leq C_1 \lambda^{2(\sigma-\ell)-2} (\lambda+1)^{2+2(\ell+\alpha+\nu_j-\sigma)} + C_2 \lambda^{2(\nu_j+\alpha)} \\ &\leq C_1 \lambda^{2(\sigma-\ell)-2} + C_2 \lambda^{2(\nu_j+\alpha)}. \end{aligned}$$

Recalling that $\sigma > \frac{n}{2} + k$ and $\ell \leq k$, we have that $2(\sigma - \ell) > n$. Also, we have $2(\nu_j + \alpha) \geq n - 2 + 2\alpha$, and since $|\alpha| \leq \ell \leq k$, we have that the above is bounded by a constant times $\lambda^{n-2-2\alpha}$ for $0 < \lambda \leq 1$ as claimed.

Next, we consider the integral over the region where $\frac{1}{\lambda} \leq r < \infty$. For this, we use (3.9) to obtain

$$\begin{aligned} \int_{\frac{1}{\lambda}}^{\infty} r^{1+2\ell} |J_{\nu_j+\alpha}(\lambda r)|^2 (1+r)^{-2\sigma} dr &\leq C \int_{\frac{1}{\lambda}}^{\infty} r^{1+2\ell} (\lambda r)^{-1} (1+r)^{-2\sigma} dr \\ &\leq \frac{C}{\lambda} \int_{\frac{1}{\lambda}}^{\infty} (1+r)^{2(\ell-\sigma)} dr \leq \frac{C'}{\lambda} \left(1 + \frac{1}{\lambda}\right)^{1+2(\ell-\sigma)} \\ &= C' \lambda^{2(\sigma-\ell)-2} (\lambda+1)^{1+2(\ell-\sigma)} \leq C'' \lambda^{2(\sigma-\ell)-2}. \end{aligned}$$

The restrictions on σ guarantee that the above is bounded by a constant times λ^{n-2} for $0 < \lambda \leq 1$. Therefore, (3.13) holds, and the proof is complete. \square

Next, we aim to prove weighted L^q estimates on the free radial resolvent kernels $R_{0,j}$, which enables use to control the terms in the Birman-Schwinger series for $R_{V,j}$ when applied iteratively. First, we make note of a technical lemma.

Lemma 3.5. *Let $\nu \geq \frac{n-2}{2}$, and $\lambda > 0$. Suppose that $\beta, m \in \mathbb{Z}$ are such that $|\beta| \leq m$ and $\nu + \beta \geq 0$. Assume also that $1 \leq q < \infty$ and that $\sigma > \frac{n}{q} + m$. Then there exist $C_1, C_2 > 0$ such that*

$$\int_0^{\infty} (\lambda s)^{q(m-\frac{n-2}{2})} |J_{\nu+\beta}(\lambda s)|^q (1+s)^{-q\sigma} s^{n-1} ds \leq \begin{cases} C_1 \lambda^{-n} + C_2 \lambda^{q(m-\frac{n-1}{2})}, & 1 \leq \lambda < \infty \\ C \lambda^{q\sigma-n}, & 0 < \lambda \leq 1. \end{cases} \quad (3.14)$$

Proof. Let us denote by $I(\lambda)$ the integral in the statement above, and observe that $I(\lambda)$ is clearly nonnegative for all $\lambda > 0$. If we split the integral into the the regions where $0 < s < \frac{1}{\lambda}$ and $\frac{1}{\lambda} < s < \infty$, we can apply (3.8) and (3.9) to $J_{\nu+\beta}$ to obtain that

$$I(\lambda) \leq C \int_0^{\frac{1}{\lambda}} (\lambda s)^{q(\nu-\frac{n-2}{2}+\beta+m)} (1+s)^{-q\sigma} s^{n-1} ds + C \int_{\frac{1}{\lambda}}^{\infty} (\lambda s)^{q(m-\frac{n-1}{2})} (1+s)^{-q\sigma} s^{n-1} ds$$

for some constant $C > 0$. To estimate these integrals, we treat the cases $\lambda \geq 1$ and $\lambda \leq 1$ separately. First suppose that $\lambda \geq 1$. Then we see that

$$\int_0^{\frac{1}{\lambda}} (\lambda s)^{q(\nu-\frac{n-2}{2}+\beta+m)} (1+s)^{-q\sigma} s^{n-1} ds \leq \lambda^{1-n} \int_0^{\frac{1}{\lambda}} (1+s)^{-q\sigma} ds \leq C \lambda^{-n}, \quad (3.15)$$

since $\nu - \frac{n-2}{2} + \beta + m \geq 0$ and $q\sigma > 0$. For the integral over $\frac{1}{\lambda} < s < \infty$, we have

$$\int_{\frac{1}{\lambda}}^{\infty} (\lambda s)^{q(m-\frac{n-1}{2})} (1+s)^{-q\sigma} s^{n-1} ds = \lambda^{q(m-\frac{n-1}{2})} \int_{\frac{1}{\lambda}}^{\infty} s^{n-1+q(m-\frac{n-1}{2})} (1+s)^{-q\sigma} ds.$$

Under the hypothesis that $\sigma > \frac{n}{q} + m$, the integral

$$\int_1^{\infty} s^{n-1+q(m-\frac{n-1}{2})} (1+s)^{-q\sigma} ds$$

converges and is bounded by constant which is independent of λ . For the region where $\frac{1}{\lambda} < s < 1$, we have

$$\int_{\frac{1}{\lambda}}^1 s^{n-1+q(m-\frac{n-1}{2})} (1+s)^{-q\sigma} ds \leq C \lambda^{-n-q(m-\frac{n-1}{2})}.$$

Thus,

$$\lambda^{q(m-\frac{n-1}{2})} \int_{\frac{1}{\lambda}}^{\infty} s^{q(m-\frac{n-1}{2})} (1+s)^{-q\sigma} s^{n-1} ds \leq \max\{\lambda^{-n}, \lambda^{q(m-\frac{n-1}{2})}\}$$

when $\lambda \geq 1$.

Now take the case where $0 < \lambda \leq 1$. Then, since $|\beta| \leq m$ and $\nu \geq \frac{n-2}{2}$, we have

$$\begin{aligned} \int_0^{\frac{1}{\lambda}} (\lambda s)^{q(\nu-\frac{n-2}{2}+\beta+m)} (1+s)^{-q\sigma} s^{n-1} ds &\leq C \int_0^{\frac{1}{\lambda}} (1+s)^{n-1-q\sigma} ds \\ &= C' \left(1 + \frac{1}{\lambda}\right)^{n-q\sigma} \leq C'' \lambda^{q\sigma-n}. \end{aligned}$$

For the integral over $\frac{1}{\lambda} \leq s < \infty$, we notice that

$$\int_{\frac{1}{\lambda}}^{\infty} (\lambda s)^{q(m-\frac{n-1}{2})} (1+s)^{-q\sigma} s^{n-1} ds \leq \lambda^{q(m-\frac{n-1}{2})} \int_{\frac{1}{\lambda}}^{\infty} (1+s)^{n-1+q(m-\frac{n-1}{2}-\sigma)} ds$$

since $1 \leq \frac{1}{\lambda} \leq s$. Recalling the assumption that $\sigma > \frac{n}{q} + m$, we can see that

$$n-1+q\left(m-\frac{n-1}{2}-\sigma\right) < -1,$$

and therefore,

$$\begin{aligned} \lambda^{q(m-\frac{n-1}{2})} \int_{\frac{1}{\lambda}}^{\infty} (1+s)^{n-1+q(m-\frac{n-1}{2}-\sigma)} ds &\leq C' \lambda^{q(m-\frac{n-1}{2})} \left(1 + \frac{1}{\lambda}\right)^{n+q(m-\frac{n-1}{2}-\sigma)} \\ &\leq C'' \lambda^{q\sigma-n}. \end{aligned}$$

Therefore, $I(\lambda) \leq C\lambda^{q\sigma-n}$ for $0 < \lambda \leq 1$. \square

Next, we establish some estimates on the $L^{q,\sigma}$ norms of $R_{0,j}(\lambda \pm i0)(r, s)$ when the norm is only taken with respect to one variable.

Proposition 3.6. *Let $k \geq 0$ be an integer. Also assume that $1 \leq q < \infty$ and*

$$\sigma > \frac{n}{q} + k. \quad (3.16)$$

Then for $|\lambda| \geq 1$, we have

$$\begin{aligned} &\|\partial_\lambda^k \operatorname{Im} R_{0,j}(\lambda \pm i0; r, \cdot)\|_{L^{q,\sigma}} \\ &\leq C_{j,q,\sigma,k} |\lambda|^{n-2-k} \sum_{\ell+m=k} \left[(1+|\lambda|r)^{\ell-\frac{n-1}{2}} \left(C_1 |\lambda|^{-\frac{n}{q}} + C_2 |\lambda|^{m-\frac{n-1}{2}} \right) \right] \end{aligned} \quad (3.17)$$

for some $C_{j,q,\sigma,k} > 0$. Furthermore, when $|\lambda| \leq 1$, we have

$$\|\partial_\lambda^k \operatorname{Im} R_{0,j}(\lambda \pm i0; r, \cdot)\|_{L^{q,\sigma}} \leq C_{j,q,\sigma,k} |\lambda|^{n-2} (1+|\lambda|r)^{k-\frac{n-1}{2}}. \quad (3.18)$$

By symmetry, we also have the analogous estimates

$$\begin{aligned} &\|\partial_\lambda^k \operatorname{Im} R_{0,j}(\lambda \pm i0; \cdot, s)\|_{L^{q,\sigma}} \\ &\leq C_{j,q,\sigma,k} |\lambda|^{n-2-k} \sum_{\ell+m=k} \left[(1+|\lambda|s)^{\ell-\frac{n-1}{2}} \left(C_1 |\lambda|^{-\frac{n}{q}} + C_2 |\lambda|^{m-\frac{n-1}{2}} \right) \right] \end{aligned} \quad (3.19)$$

for $|\lambda| \geq 1$, and

$$\|\partial_\lambda^k \operatorname{Im} R_{0,j}(\lambda \pm i0; \cdot, s)\|_{L^{q,\sigma}} \leq C_{j,q,\sigma,k} |\lambda|^{n-2} (1+|\lambda|r)^{k-\frac{n-1}{2}} \quad (3.20)$$

when $|\lambda| \leq 1$.

Remark 3.7. Note that in the special case where the order of differentiation is less than or equal to $\frac{n-1}{2}$, these formulas can be simplified to show simple polynomial behavior in λ . However, if the number of derivatives exceeds this threshold value, we begin to see a non-uniformity with respect to the secondary radial variable. This phenomenon is why the spatial weights are necessary in the statement of Theorem 1.

Proof. As before, we consider only $\operatorname{Im} R_{0,j}(\lambda + i0)$ for $\lambda > 0$, since the proof is identical for the other choice of sign. Recall that the kernel of $\operatorname{Im} R_{0,j}(\lambda + i0)$ has the explicit expression

$$\operatorname{Im} R_{0,j}(\lambda + i0; r, s) = \frac{\pi}{2} (rs)^{-\frac{n-2}{2}} J_{\nu_j}(\lambda r) J_{\nu_j}(\lambda s).$$

Since Bessel functions satisfy the recursion relation

$$J'_\nu(x) = \frac{1}{2} (J_{\nu-1}(x) - J_{\nu+1}(x)),$$

we see that $\partial_\lambda^k \operatorname{Im} R_{0,j}(\lambda + i0)$ can be written as a finite linear combination of terms of the form

$$(rs)^{-\frac{n-2}{2}} r^\ell s^m J_{\nu_j+\alpha}(\lambda r) J_{\nu_j+\beta}(\lambda s), \quad (3.21)$$

where ℓ, m, α, β are integers satisfying $\ell + m = k$, $|\alpha| \leq \ell$, and $|\beta| \leq m$. Therefore, by the triangle inequality, it suffices to estimate the weighted L^q norms of such terms. Taking the $L^{q,\sigma}$ norm with respect to the s variable in (3.21) yields

$$\lambda^{q(n-2-k)} (\lambda r)^{q(\ell - \frac{n-2}{2})} |J_{\nu_j+\alpha}(\lambda r)|^q \int_0^\infty (\lambda s)^{q(m - \frac{n-2}{2})} |J_{\nu_j+\beta}(\lambda s)|^q (1+s)^{-q\sigma} s^{n-1} ds.$$

Note that since $|\alpha| \leq \ell$, we have that the product $(\lambda r)^{\ell - \frac{n-2}{2}} |J_{\nu_j+\alpha}(\lambda r)|$ is a continuous function of λr , and thus by (3.9) we obtain

$$(\lambda r)^{q(\ell - \frac{n-2}{2})} |J_{\nu_j+\alpha}(\lambda r)|^q \leq C(1 + \lambda r)^{q(\ell - \frac{n-1}{2})}.$$

Thus, we have that the $L^{q,\sigma}$ norm of (3.21) is bounded by

$$C \lambda^{q(n-2-k)} (1 + \lambda r)^{q(\ell - \frac{n-1}{2})} \int_0^\infty (\lambda s)^{q(m - \frac{n-2}{2})} |J_{\nu_j+\beta}(\lambda s)|^q (1+s)^{-q\sigma} s^{n-1} ds. \quad (3.22)$$

Now, observe that the integral above is in exactly the right form for us to apply Lemma 3.5. Hence, we have that (3.22) is bounded by

$$\begin{cases} C \lambda^{q(n-2-k)} (1 + \lambda r)^{q(\ell - \frac{n-1}{2})} \max\{\lambda^{-n}, \lambda^{q(m - \frac{n-1}{2})}\}, & \lambda \geq 1 \\ C \lambda^{q(n-2-k)+q\sigma-n} (1 + \lambda r)^{q(\ell - \frac{n-1}{2})}, & \lambda \leq 1, \end{cases}$$

for some possibly larger constant C . In the case where $\lambda \geq 1$, simply taking q th roots gives estimate (3.17). When $\lambda \leq 1$, we can use that σ satisfies (3.16) to obtain that $q(n-2-k) + q\sigma - n > q(n-2)$. Once again, taking q th roots gives (3.18). □

Next, we estimate the L^q norm of the resolvent when we do not take the imaginary part.

Proposition 3.8. *Let $k \geq 0$ be an integer and suppose $1 \leq q \leq \frac{n}{n-2}$. Then, if σ satisfies (3.16), we have that when $|\lambda| \geq 1$,*

$$\|\partial_\lambda^k R_{0,j}(\lambda \pm i0; r, \cdot)\|_{L^{q,\sigma}} \leq C |\lambda|^{n-2-k} \sum_{\ell+m=k} \left[(1 + |\lambda|r)^{\ell - \frac{n-1}{2}} \left(C_1 + C_2 |\lambda|^{m - \frac{n-1}{2}} \right) \right] \quad (3.23)$$

for some $C, C_1, C_2 > 0$. If $0 < |\lambda| \leq 1$, then we have

$$\|\partial_\lambda^k R_{0,j}(\lambda \pm i0; r, \cdot)\|_{L^{q,\sigma}} \leq C_1 |\lambda|^{-k} + C_2 |\lambda|^{n-2-k} (1 + |\lambda|r)^{k - \frac{n-1}{2}}. \quad (3.24)$$

Under the same assumptions on σ , we also have

$$\|\partial_\lambda^k R_{0,j}(\lambda \pm i0; \cdot, s)\|_{L^{q,\sigma}} \leq C |\lambda|^{n-2-k} \sum_{\ell+m=k} \left[(1 + |\lambda|s)^{\ell - \frac{n-1}{2}} \left(C_1 + C_2 |\lambda|^{m - \frac{n-1}{2}} \right) \right] \quad (3.25)$$

when $|\lambda| \geq 1$, and

$$\|\partial_\lambda^k R_{0,j}(\lambda \pm i0; \cdot, s)\|_{L^{q,\sigma}} \leq C_1 |\lambda|^{-k} + C_2 |\lambda|^{n-2-k} (1 + |\lambda|r)^{k - \frac{n-1}{2}} \quad (3.26)$$

when $0 < |\lambda| \leq 1$.

Proof. Once again, we consider only the case of $R_{0,j}(\lambda + i0)$ for $\lambda > 0$. Recalling that

$$R_{0,j}(\lambda + i0)(r, s) = \begin{cases} \frac{\pi i}{2} J_{\nu_j}(\lambda s) H_{\nu_j}^{(1)}(\lambda r), & s < r \\ \frac{\pi i}{2} J_{\nu_j}(\lambda r) H_{\nu_j}^{(1)}(\lambda s), & s > r, \end{cases}$$

and (3.6), we see that when $s < r$, $\partial_\lambda^k R_{0,j}(\lambda + i0; r, s)$ can be written as a finite linear combination of terms of the form

$$(rs)^{-\frac{n-2}{2}} r^\ell s^m J_{\nu_j+\alpha}(\lambda s) H_{\nu_j+\beta}^{(1)}(\lambda r),$$

where, as in the proof of Proposition 3.6, ℓ, m are nonnegative integers with $\ell + m = k$ and α, β are any integers with $|\alpha| \leq \ell$ and $|\beta| \leq m$. Similarly, when $r < s$, we can write $\partial_\lambda^k R_{0,j}(\lambda + i0)(r, s)$ as a combination of terms of the same form, but with the roles of r and s reversed. Therefore, it suffices to estimate

$$I(\lambda, r) := \|(rs)^{-\frac{n-2}{2}} r^\ell s^m J_{\nu_j+\beta}(\lambda s) H_{\nu_j+\alpha}^{(1)}(\lambda r) \rho^{-\sigma}(s) \mathbf{1}_{\{s < r\}}\|_{L_s^q}^q \quad (3.27)$$

and

$$II(\lambda, r) := \|(rs)^{-\frac{n-2}{2}} r^\ell s^m J_{\nu_j+\alpha}(\lambda r) H_{\nu_j+\beta}^{(1)}(\lambda s) \rho^{-\sigma}(s) \mathbf{1}_{\{s > r\}}\|_{L_s^q}^q \quad (3.28)$$

for any ℓ, m, α, β as above.

We first estimate $I(\lambda, r)$ in the case where $\lambda r \geq 1$. Under this hypothesis, we can apply (3.9) to obtain

$$I(\lambda, r) \leq C \lambda^{q(n-2-k)} (\lambda r)^{q(\ell - \frac{n-1}{2})} \int_0^r (\lambda s)^{q(m - \frac{n-2}{2})} |J_{\nu_j+\beta}(\lambda s)|^q (1+s)^{-q\sigma} s^{n-1} ds.$$

We now apply Lemma 3.5 to the integral above, which gives

$$I(\lambda, r) \leq \begin{cases} C \lambda^{q(n-2-k)} (\lambda r)^{q(\ell - \frac{n-1}{2})} \max\{\lambda^{-n}, \lambda^{q(m - \frac{n-1}{2})}\}, & \lambda r \geq 1, \lambda \geq 1 \\ C \lambda^{q(n-2-k+\sigma)-n} (\lambda r)^{q(\ell - \frac{n-1}{2})}, & \lambda r \geq 1, 0 < \lambda \leq 1. \end{cases} \quad (3.29)$$

Now let us consider the case where $\lambda r \leq 1$. Here we can apply (3.8), which gives

$$I(\lambda, r) \leq C \lambda^{q(n-2-k)} (\lambda r)^{q(\ell - \frac{n-2}{2} - \nu_j - \alpha)} \int_0^r (\lambda s)^{q(m - \frac{n-2}{2} + \nu_j + \beta)} (1+s)^{-q\sigma} s^{n-1} ds. \quad (3.30)$$

If $r \leq 1$, we can bound the right-hand side of (3.30) by

$$\begin{aligned} C\lambda^{q(n-2-k)}(\lambda r)^{q(k-\alpha+\beta-(n-2))}r^{n-1} \int_0^r (1+s)^{-q\sigma} ds &\leq \tilde{C}\lambda^{q(n-2-k)}(\lambda r)^{-q(n-2)}r^n \\ &= \tilde{C}\lambda^{-qk}r^{n-q(n-2)}, \end{aligned}$$

since $k - \alpha + \beta \geq 0$ and $\int_0^r (1+s)^{-q\sigma} ds \leq C'r$ for some $C' > 0$. Recalling that $q \leq \frac{n}{n-2}$, we obtain

$$I(\lambda, r) \leq C\lambda^{-qk}, \quad \lambda r \leq 1, r \leq 1. \quad (3.31)$$

Now, if $r \geq 1$, we can bound the right-hand side (3.30) by

$$C\lambda^{q(n-2-k)}(\lambda r)^{q(k-\alpha+\beta-(n-2))} \int_0^r (1+s)^{-q\sigma} s^{n-1} ds \leq C\lambda^{q(n-2-k)}(\lambda r)^{-q(n-2)}(1+r)^{n-q\sigma} \quad (3.32)$$

since $k - \alpha + \beta \geq 0$ and $\lambda r \leq 1$. Recalling that $\sigma > \frac{n}{q}$, we have that the right-hand side of (3.31) by

$$C\lambda^{-qk}r^{-q(n-2)} \leq C\lambda^{-qk}. \quad (3.33)$$

Combining (3.31) and (3.33), we have that

$$I(\lambda, r) \leq C\lambda^{-qk}, \quad \lambda r \leq 1. \quad (3.34)$$

Combining (3.34) with (3.29), we have that

$$I(\lambda, r) \leq \begin{cases} C\lambda^{q(n-2-k)}(1+\lambda r)^{q(\ell-\frac{n-1}{2})} \max\{\lambda^{-n}, \lambda^{q(m-\frac{n-1}{2})}\}, & \lambda \geq 1 \\ C_1\lambda^{-qk} + C_2\lambda^{q(n-2)}(1+\lambda r)^{q(\ell-\frac{n-1}{2})}, & 0 < \lambda \leq 1. \end{cases} \quad (3.35)$$

Next, we move on to estimating $II(\lambda, r)$. Again we consider the cases $\lambda r \geq 1$ and $\lambda r \leq 1$ separately. For $\lambda r \geq 1$, we apply (3.8) and (3.9) to obtain

$$II(\lambda, r) \leq C\lambda^{q(n-2-k)}(\lambda r)^{q(\ell-\frac{n-1}{2})} \int_{\frac{1}{\lambda}}^{\infty} (\lambda s)^{q(m-\frac{n-1}{2})} (1+s)^{-q\sigma} s^{n-1} ds.$$

We can then repeat arguments from the proof of Lemma 3.5 to obtain

$$II(\lambda, r) \leq \begin{cases} C\lambda^{q(n-2-k)}(\lambda r)^{q(\ell-\frac{n-1}{2})} \max\{\lambda^{-n}, \lambda^{q(m-\frac{n-1}{2})}\}, & \lambda r \geq 1, \lambda \geq 1 \\ C\lambda^{q(n-2-k)}(\lambda r)^{q(\ell-\frac{n-1}{2})}, & \lambda r \geq 1, \lambda \leq 1. \end{cases} \quad (3.36)$$

Now consider the case where $\lambda r \leq 1$. Here we rewrite $I(\lambda, r)$ as

$$\lambda^{q(n-2-k)}(\lambda r)^{q(\ell - \frac{n-2}{2})} |J_{\nu_j+\alpha}(\lambda r)|^q \left(\int_r^{\frac{1}{\lambda}} + \int_{\frac{1}{\lambda}}^{\infty} \right) (\lambda s)^{q(m - \frac{n-2}{2})} |H_{\nu_j+\beta}^{(1)}(\lambda s)|^q (1+s)^{-q\sigma} s^{n-1} ds.$$

For the integral over $\frac{1}{\lambda} < s < \infty$, we can apply (3.9) to $H_{\nu_j+\beta}^{(1)}$ and (3.8) to $J_{\nu_j+\alpha}$ and repeat previous calculations to show that

$$\begin{aligned} & \lambda^{q(n-2-k)}(\lambda r)^{q(\ell - \frac{n-2}{2})} |J_{\nu_j+\alpha}(\lambda r)|^q \int_{\frac{1}{\lambda}}^{\infty} (\lambda s)^{q(m - \frac{n-2}{2})} |H_{\nu_j+\beta}^{(1)}(\lambda s)|^q (1+s)^{-q\sigma} s^{n-1} ds \\ & \leq \begin{cases} C \lambda^{q(n-2-k)} \max\{\lambda^{-n}, \lambda^{q(m - \frac{n-1}{2})}\}, & \lambda r \leq 1, \lambda \geq 1 \\ C \lambda^{q(n-2-k)} \lambda^{q\sigma-n}, & \lambda r \leq 1, \lambda \leq 1 \end{cases} \quad (3.37) \\ & \leq \begin{cases} C \lambda^{q(n-2-k)} \max\{\lambda^{-n}, \lambda^{q(m - \frac{n-1}{2})}\}, & \lambda r \leq 1, \lambda \geq 1 \\ C \lambda^{q(n-2)}, & \lambda r \leq 1, \lambda \leq 1, \end{cases} \end{aligned}$$

where the last inequality follows since $\sigma > \frac{n}{q} + k$. Now, in the region where $r < s < \frac{1}{\lambda}$, we must apply (3.8), which yields

$$\begin{aligned} \int_r^{\frac{1}{\lambda}} (\lambda s)^{q(m - \frac{n-2}{2})} |H_{\nu_j+\beta}^{(1)}(\lambda s)|^q (1+s)^{-q\sigma} s^{n-1} ds & \leq C \int_r^{\frac{1}{\lambda}} (\lambda s)^{q(m - \frac{n-2}{2} - \nu_j - \beta)} (1+s)^{-q\sigma} s^{n-1} ds \\ & = C \lambda^{q(m - \frac{n-2}{2} - \nu_j - \beta)} \int_r^{\frac{1}{\lambda}} s^{n-1+q(m - \frac{n-2}{2} - \nu_j - \beta)} (1+s)^{-q\sigma} ds. \end{aligned}$$

If $\lambda \geq 1$, then $(1+s)^{-q\sigma}$ is bounded by a uniform constant for all $r < s < \frac{1}{\lambda}$, and so the above is bounded by

$$C \left(\lambda^{-n} - r^n (\lambda r)^{q(m - \frac{n-2}{2} - \nu_j - \beta)} \right)$$

after possibly increasing C . We note that under our assumptions on r and λ , this quantity is still nonnegative. Combining this with (3.8) applied to $J_{\nu_j+\alpha}$, we obtain

$$\begin{aligned}
& \lambda^{q(n-2-k)}(\lambda r)^{q(\ell-\frac{n-2}{2})}|J_{\nu_j+\alpha}(\lambda r)|^q \int_r^{\frac{1}{\lambda}} (\lambda s)^{q(m-\frac{n-2}{2})}|H_{\nu_j+\beta}^{(1)}(\lambda s)|^q (1+s)^{-q\sigma} s^{n-1} ds \\
& \leq C \lambda^{q(n-2-k)}(\lambda r)^{q(\ell-\frac{n-2}{2}+\nu_j+\alpha)} \left(\lambda^{-n} - r^n (\lambda r)^{q(m-\frac{n-2}{2}-\nu_j-\beta)} \right) \\
& \leq \lambda^{q(n-2-k)} \left[C_1 \lambda^{-n} (\lambda r)^{q(\ell-\frac{n-2}{2}+\nu_j+\alpha)} + C_2 r^{n-q(n-2)} (\lambda r)^{q(k+\alpha-\beta)} \right],
\end{aligned} \tag{3.38}$$

for some $C_1, C_2 > 0$. Recalling that $|\alpha| \leq \ell$, $\nu_j \geq \frac{n-2}{2}$, and $|\alpha| + |\beta| \leq k$, we obtain

$$\begin{aligned}
& \lambda^{q(n-2-k)}(\lambda r)^{q(\ell-\frac{n-2}{2})}|J_{\nu_j+\alpha}(\lambda r)|^q \int_r^{\frac{1}{\lambda}} (\lambda s)^{q(m-\frac{n-2}{2})}|H_{\nu_j+\beta}^{(1)}(\lambda s)|^q (1+s)^{-q\sigma} s^{n-1} ds \\
& \leq \lambda^{q(n-2-k)} [C_1 \lambda^{-n} + C_2 r^{n-q(n-2)}] \\
& \leq C \lambda^{q(n-2-k)}
\end{aligned} \tag{3.39}$$

for $\lambda r \leq 1$ and $\lambda \geq 1$, since $q \leq \frac{n}{n-2}$.

Finally, we consider the same integral over $r < s < \frac{1}{\lambda}$, once again where $0 < \lambda r \leq 1$ but with $\lambda \leq 1$. For this, we further subdivide into the cases where $r \leq 1$ and $r \geq 1$. If $r \leq 1$, we split the integral into the regions where $r < s < 1$ and $1 < s < \frac{1}{\lambda}$. For the integral over $r < s < 1$, we can repeat the above argument to obtain the same bound as in (3.39). To

bound the integral over $1 < s < \frac{1}{\lambda}$, we use (3.8) to obtain

$$\begin{aligned}
& \lambda^{q(n-2-k)} (\lambda r)^{q(\ell - \frac{n-2}{2})} |J_{\nu+\alpha}(\lambda r)|^q \int_1^{\frac{1}{\lambda}} (\lambda s)^{q(m - \frac{n-2}{2})} |H_{\nu_j+\beta}^{(1)}(\lambda s)|^q (1+s)^{-q\sigma} s^{n-1} ds \\
& \leq \lambda^{q(n-2-k)} (\lambda r)^{q(\ell - \frac{n-2}{2} + \nu_j + \alpha)} \int_1^{\frac{1}{\lambda}} (\lambda s)^{q(m - \frac{n-2}{2} - \nu_j - \beta)} (1+s)^{-q\sigma} s^{n-1} ds \\
& = \lambda^{q(\alpha-\beta)} r^{q(\ell - \frac{n-2}{2} + \nu_j + \alpha)} \int_1^{\frac{1}{\lambda}} s^{q(m - \frac{n-2}{2} - \nu_j - \beta) + n - 1} (1+s)^{-q\sigma} ds \\
& \leq \lambda^{-qk} \int_1^{\infty} s^{qk+n-1} (1+s)^{-q\sigma} ds \\
& \leq C \lambda^{-qk},
\end{aligned}$$

where the last inequality follows from the fact that $\sigma > \frac{n}{q} + k$. Now, if $r \geq 1$, we have

$$\begin{aligned}
& \lambda^{q(n-2-k)} (\lambda r)^{q(\ell - \frac{n-2}{2})} |J_{\nu+\alpha}(\lambda r)|^q \int_r^{\frac{1}{\lambda}} (\lambda s)^{q(m - \frac{n-2}{2})} |H_{\nu_j+\beta}^{(1)}(\lambda s)|^q (1+s)^{-q\sigma} s^{n-1} ds \\
& \leq \lambda^{q(n-2-k)} (\lambda r)^{q(\ell - \frac{n-2}{2} + \nu_j + \alpha)} \int_r^{\frac{1}{\lambda}} (\lambda s)^{q(m - \frac{n-2}{2} - \nu_j - \beta)} (1+s)^{-q\sigma} s^{n-1} ds \\
& \leq C \lambda^{q(\alpha-\beta)} r^{q(\ell - \frac{n-2}{2} + \nu_j + \alpha)} \int_r^{\infty} s^{q(m - \frac{n-2}{2} - \nu_j - \beta) - q\sigma + n - 1} ds \\
& \leq C \lambda^{q(\alpha-\beta)} r^{q(k + \alpha - \beta - (n-2)) - q\sigma + n} \\
& \leq C \lambda^{q(\alpha-\beta)} r^{q(\alpha - \beta - (n-2))},
\end{aligned}$$

where in the last inequality we once again used that $\sigma > \frac{n}{q} + k$. Now, since $r \leq \frac{1}{\lambda}$, we have that the above is bounded by a constant times $\lambda^{q(n-2)}$ for $\lambda \leq 1$. Combining this with (3.36), (3.37), and (3.39), we obtain

$$\mathcal{I}(\lambda, r) \leq \begin{cases} C \lambda^{q(n-2-k)} (1 + \lambda r)^{q(\ell - \frac{n-1}{2})} \max\{1, \lambda^{q(m - \frac{n-1}{2})}\}, & \lambda \geq 1 \\ C_1 \lambda^{-qk} + C_2 \lambda^{q(n-2-k)} (1 + \lambda r)^{q(\ell - \frac{n-1}{2})}, & 0 < \lambda \leq 1. \end{cases} \quad (3.40)$$

In light of (3.35) and (3.40), taking q th roots completes the proof of Proposition 3.8. \square

4. OPERATOR ESTIMATES FOR R_V

In this section we establish some weighted operator norm estimates for the perturbed radial resolvents $R_{V,j}$, defined via the mode-by-mode decomposition of $R_V(z)$:

$$R_V(z) = \sum_{j=0}^{\infty} R_{V,j}(z) E_j.$$

Since V is radial, it follows that we can write

$$R_{V,j}(z) = \left(\partial_r^2 + \frac{n-1}{r} \partial_r + z^2 - \frac{\mu_j^2}{r^2} + V(r) \right)^{-1}, \quad (4.1)$$

wherever this inverse is well defined. Here, we prove that the mapping properties established for $R_{0,j}$ in Proposition 3.1 and Proposition 3.4 extend to $R_{V,j}$. Similar weighted estimates for Schrödinger operators on hyperbolic space are given in Section 4 of [8], and the techniques therein follow an analogous structure.

For a potential $V \in \rho^{-2\sigma} L^\infty(\mathbb{R}^+)$ with $\sigma > \frac{1}{2}$ (recall that we take $\rho(r) = 1+r$), the operator norm $\|VR_0(z)\|_{L^2 \rightarrow L^2}$ is small for $\text{Im } z$ large by the standard resolvent norm estimate on $R_0(z)$, which is computable in a similar fashion to that discussed in Proposition 3.1. Hence, the operator $1 + VR_0(z)$ is invertible by Neumann series for large $\text{Im } z$. For z in this range, we can write

$$R_{V,j}(z) = R_{0,j}(z)(1 + VR_{0,j}(z))^{-1}.$$

We begin our analysis of these perturbed resolvents by proving that each $R_{V,j}$ admits a meromorphic continuation which extends below the real axis. Results of this type are somewhat standard, but we require a more quantitative version, so we provide a careful proof here.

Theorem 2. *For $V \in \rho^{-2\sigma} L^\infty(\mathbb{R}^+)$ with $\sigma > \frac{1}{2}$, the resolvent $R_{V,j}$, chosen with outgoing or incoming boundary conditions admits a meromorphic continuation to $\text{Im } z \geq -\delta$ as a bounded operator*

$$R_{V,j}(z) : L^{2, \frac{1}{2} + \delta}(\mathbb{R}^+, r^{n-1} dr) \rightarrow L^{2, -\frac{1}{2} - \delta}(\mathbb{R}^+, r^{n-1} dr),$$

for $0 < \delta < \sigma - \frac{1}{2}$.

Proof. For the meromorphic extension of an operator of the form

$$\chi R_{0,j} \chi$$

with χ a smooth, compactly supported function, we refer to the work of Melrose-Wunsch [51], Graham-Zworski [30], and the recent discussions in Baskin-Marzuola [1] for a general proof of meromorphic continuation for conic manifolds and Baskin-Yang [3] for the case of product cones and truncated product cones. The meromorphic continuation of $\chi R_{V,j}(\lambda) \chi$ follows from the work of Guillopé-Zworski [33] and the compactness of the resolvent on a

compact manifold with a conic singularity, which can be seen for instance in the treatment of domains for conic operators in the work of Melrose-Wunsch [51]. These results are for the cut-off free resolvent. Some care is required to extend them to the perturbed case.

It follows from a modification of Proposition 3.1 that $\rho(r)^{-\eta}R_{0,j}(z)$ is compact as an operator on $L^{2,\frac{1}{2}+\delta}$ provided that $\text{Im } z > -\delta$ and $\eta > 2\delta + 1$, see Proposition 3.29 of [58] for a very general statement. Since $V \in \rho^{-2\sigma}L^\infty$ and $2\sigma > 2\delta + 1$, we have that $VR_{0,j}(z)$ is compact on $L^{2,\frac{1}{2}+\delta}$ under the same conditions. Therefore, the analytic Fredholm theorem gives a meromorphic continuation of $R_{V,j}(z)$ to the half-plane $\text{Im } z > -\delta$. \square

Next, we rule out the possibility of embedded eigenvalues/resonances in the continuous spectrum of $-\Delta_{C(X)} + V$.

Theorem 3. *For $V \in \rho^{-2\sigma}L^\infty(\mathbb{R}^+)$ with $\sigma > \frac{1}{2}$, the operator $-\Delta_{C(X)} + V$ has continuous spectrum $[0, \infty)$, with no embedded eigenvalues or resonances in the range $(0, \infty)$. Moreover, the continued resolvent $R_{V,j}(z)$ has no poles that stem from eigenvalues on the critical line $\text{Re } z = 0$ except possibly at $z = 0$.*

We postpone the proof of this result to Appendix B, where it is discussed alongside a deeper analysis of the absence of embedded eigenvalues and resonances for the case of more general non-radial potentials.

4.1. The weighted resolvent estimates. Following [28], we observe that mapping properties of $R_{V,j}$ follow from the estimates established for $R_{0,j}$, provided we can rule out embedded resonances. We now show that for the class of potentials considered here, the high-frequency behavior of the resolvent on the critical line is unaffected by the potential.

Proposition 4.1. *For $V \in \rho^{-2\sigma}L^\infty(\mathbb{R}^+)$ with $\sigma > \frac{1}{2} + k$, there exists a constant $M_V > 0$ such that for $\lambda \in \mathbb{R}$ with $|\lambda| \geq M_V$,*

$$\left\| \partial_\lambda^k R_{V,j}(\lambda \pm i0) \right\|_{L^{2,\eta} \rightarrow L^{2,-\eta}} \leq C_{j,k,\sigma} |\lambda|^{-1} \quad (4.2)$$

for $\frac{1}{2} + k < \eta < \sigma$. In particular, there are no resonances on the critical line for $|\lambda| \geq M_V$.

Proof. By the resolvent identity

$$R_{0,j}(z) = R_{V,j}(z) + R_{V,j}(z)V R_{0,j}(z),$$

we can write

$$R_{0,j}(z)\rho^{-\sigma} = R_{V,j}(z)\rho^{-\sigma}(1 + \rho^\sigma V R_{0,j}(z)\rho^{-\sigma})$$

for $\rho(r) = 1 + r$. The factor on the right is meromorphically invertible by the analytic Fredholm theorem, so that

$$R_{V,j}(z)\rho^{-\sigma} = R_{0,j}(z)\rho^{-\sigma}(1 + \rho^\sigma V R_{0,j}(z)\rho^{-\sigma})^{-1}. \quad (4.3)$$

By Proposition 3.1 and the fact that $\rho^\sigma V = \rho^{2\sigma} V \rho^{-\sigma}$, we have

$$\left\| \rho^\sigma V R_{0,j}(\lambda \pm i0)\rho^{-\sigma} \right\|_{L^2 \rightarrow L^2} \leq C \left\| \rho^{2\sigma} V \right\|_{L^\infty} |\lambda|^{-1}.$$

Hence for $V \in \rho^{-2\sigma} L^\infty$, there exists a constant M_V such that for $|\lambda| \geq M_V$,

$$\|\rho^\sigma V R_{0,j}(\lambda \pm i0) \rho^{-\sigma}\|_{L^2 \rightarrow L^2} \leq \frac{1}{2},$$

implying that $(1 + \rho^\sigma V R_{0,j}(\lambda + i0) \rho^{-\sigma})^{-1}$ exists and satisfies

$$\|(1 + \rho^\sigma V R_{0,j}(\lambda \pm i0) \rho^{-\sigma})^{-1}\|_{L^2 \rightarrow L^2} \leq 2.$$

The estimates then follow from (4.3) and Proposition 3.1. \square

Remark 4.2. We note that this quantitative Lemma can be used to establish an absence of embedded resonances for large values of λ . However, to get all the way down to $\lambda = 0$, the more refined techniques in Appendix B are required.

Proposition 4.3. *Let $V \in \rho^{-2\sigma} L^\infty(\mathbb{R}^+)$ with $\sigma > \frac{1}{2} + k$ be such that $-\Delta_{C(X)} + V$ does not have a resonance at 0 energy, then for any $\frac{1}{2} + k < \eta < \sigma$, we have for $|\lambda| \geq 0$,*

$$\|\partial_\lambda^k R_{V,j}(\lambda \pm i0)\|_{L^{2,\eta} \rightarrow L^{2,-\eta}} \leq C_{j,k,\sigma} \langle \lambda \rangle^{-1}. \quad (4.4)$$

Under the stronger hypothesis that $\sigma > \frac{n}{2} + k$, then we have the the low-energy estimate

$$\|\partial_\lambda^k \text{Im } R_{V,j}(\lambda \pm i0)\|_{L^{2,\eta} \rightarrow L^{2,-\eta}} \leq C_{j,k,\sigma} |\lambda|^{n-2-k}, \quad (4.5)$$

for $0 \leq |\lambda| \leq 1$.

Proof. We prove (4.4) only for $R_{V,j}(\lambda + i0)$, since the proof is analogous for $R_{V,j}(\lambda - i0)$. Using a resolvent expansion motivated by [8], we observe that

$$R_{V,j}(\lambda + i0) = R_{0,j}(\lambda + i0)[I + V R_{0,j}(\lambda + i0)]^{-1}.$$

Hence, if we can establish boundedness and regularity of $[I + V R_{0,j}(\lambda + i0)]^{-1}$ through $\lambda = 0$, then (4.4) follows immediately from Proposition 3.1.

We observe that boundedness and regularity of the operator $(I + R_{0,j}(\lambda + i0)V)^{-1}$ follows from the assumption that 0 is not a resonance or an eigenvalue of $-\Delta_{C(X)} + V$ and hence the boundedness of

$$(I + V R_{0,j}(\lambda - i0))^{-1} = [(I + R_{0,j}(\lambda + i0)V)^{-1}]^*$$

follows from analytic Fredholm theory. Thus, we may extend the results of Proposition 4.1 through $\lambda = 0$ to arrive at (4.4).

To prove (4.5), we first consider the case $k = 0$ and establish the pointwise bounds in λ . We require the following resolvent identity

$$\begin{aligned} R_{V,j}(\lambda + i0) - R_{V,j}(\lambda - i0) \\ = (I + R_{0,j}(\lambda + i0)V)^{-1} [R_{0,j}(\lambda + i0) - R_{0,j}(\lambda - i0)] (I + V R_{0,j}(\lambda - i0))^{-1}. \end{aligned} \quad (4.6)$$

This shows that the behavior of $\text{Im } R_{V,j}(\lambda + i0)$ is the same as that of $\text{Im } R_{0,j}(\lambda + i0)$ near $\lambda = 0$, provided the operators

$$(I + R_{0,j}(\lambda + i0)V)^{-1} \quad \text{and} \quad (I + V R_{0,j}(\lambda - i0))^{-1}$$

are bounded for λ in a neighborhood of 0, which we have already observed earlier in the proof. As a result, the $k = 0$ bound in (4.5) clearly follows. The results for $k > 0$ then follow by differentiating term by term and applying Proposition 3.4. \square

5. FULL SPECTRAL RESOLUTION ESTIMATES

Proposition 5.1. *Suppose $V \in \rho^{-2\sigma} L^\infty(\mathbb{R}^+)$ with*

$$\sigma > 4 \left\lceil \frac{n}{4} \right\rceil - 2 + k, \quad (5.1)$$

then for any $\alpha \geq \max\{k - \frac{n-1}{2}, 0\}$,

$$\sup_{r,s>0} \left| \rho^{-\alpha}(r) \partial_\lambda^k \operatorname{Im} R_{V,j}(\lambda \pm i0; r, s) \rho^{-\alpha}(s) \right| \leq C_{j,k,V} \lambda^{2\lceil \frac{n}{4} \rceil (n-2) - 1} \quad (5.2)$$

for all $\lambda \geq 1$ and some $C > 0$. Furthermore, if $0 < \lambda \leq 1$, we have that

$$\sup_{r,s>0} \left| \rho^{-\alpha}(r) \partial_\lambda^k \operatorname{Im} R_{V,j}(\lambda \pm i0; r, s) \rho^{-\alpha}(s) \right| \leq C_{j,k,V} \lambda^{n-2-k}, \quad (5.3)$$

under the same restrictions on α .

The proof proceeds similarly to [8, §6], which utilizes the following modified version of Young's inequality.

Lemma 5.2. *Suppose that on a measure space (Y, μ) the integral kernels $K_j(z, w)$, $j = 1, 2$, satisfy*

$$\|K_1(z, \cdot)\|_{L^{q_1}} \leq A, \quad \|K_1(\cdot, w)\|_{L^{q_1}} \leq A, \quad \|K_2(\cdot, w')\|_{L^{q_2}} \leq B$$

uniformly in z, w, w' for $q_1, q_2 \in [1, \infty]$. Then if $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p} + 1$, we have that

$$\left\| \int K_1(\cdot, w) K_2(w, w') d\mu(w) \right\|_{L^p} \leq AB$$

uniformly in w' . The bound on $\|K_1(\cdot, w)\|_{L^{q_1}}$ is not required if $p = \infty$.

With this lemma in hand, we proceed to the proof of Proposition 5.1.

Proof of Proposition 5.1. We begin by expanding $R_{V,j}$ in a Birman-Schwinger series at all frequencies, as in [28], which gives

$$R_{V,j}(\tau) = \sum_{\ell=0}^{2M-1} R_{0,j}(\tau) (-V R_{0,j}(\tau))^\ell + [R_{0,j}(\tau) V]^M R_{V,j}(\tau) [V R_{0,j}(\tau)]^M. \quad (5.4)$$

As previously discussed, it suffices to consider only the case where we choose $\lambda + i0$ with $\lambda > 0$. For simplicity, we write $R_{0,j}$ for $R_{0,j}(\lambda + i0)$ and $R_{V,j}$ for $R_{V,j}(\lambda + i0)$. We first consider the remainder term $[R_{0,j} V]^M R_{V,j} [V R_{0,j}]^M$. Since $V \in \rho^{-2\sigma} L^\infty$, we may write

$V(r) = \rho^{-2\sigma}(r)f(r)$ for some $f \in L^\infty(\mathbb{R}^+)$. Also, note that for any two operators with Schwartz kernels $A(r, s)$, $B(r, s)$, the kernel of their composition is given by

$$\langle A(r, \cdot), \overline{B(\cdot, s)} \rangle_{L^2(\mathbb{R}^+)} = \langle B(\cdot, s), \overline{A(r, \cdot)} \rangle_{L^2(\mathbb{R}^+)},$$

provided the composition makes sense. Therefore, we can write

$$\rho^{-\alpha}[R_{0,j}V]^M R_{V,j}[VR_{0,j}]^M \rho^{-\alpha}(r, s) = \langle (\rho^{-\sigma} R_{V,j} \rho^{-\sigma})A(\cdot, s), A^*(r, \cdot) \rangle_{L^2}, \quad (5.5)$$

where

$$A(r, s) = (\rho^{-\sigma} f R_{0,j} \rho^{-\sigma})^{M-1} (\rho^{-\sigma} f R_{0,j} \rho^{-\alpha})(r, s),$$

and A^* denotes the adjoint with respect to the L^2 pairing. However, we know that $A^*(r, s) = \overline{A(s, r)}$, and hence we can express the right-hand side of (5.5) as

$$\langle (\rho^{-\sigma} R_{V,j} \rho^{-\sigma})A(\cdot, s), \overline{A(\cdot, r)} \rangle_{L^2}.$$

By (4.4), we have that

$$\left| \partial_\lambda^k \langle (\rho^{-\sigma} R_{V,j} \rho^{-\sigma})A(\cdot, s), \overline{A(\cdot, r)} \rangle_{L^2} \right| \leq \frac{C}{\langle \lambda \rangle} \max_{k_1+k_2 \leq k} (\| \partial_\lambda^{k_1} A(\cdot, s) \|_{L^2} \| \partial_\lambda^{k_2} A(\cdot, r) \|_{L^2}). \quad (5.6)$$

To estimate the norms on the right, we wish to iteratively apply Lemma 5.2 to each factor in the definition of A . For this we consider the high and low frequency cases separately. First, suppose $\lambda \geq 1$. By Proposition 3.8, we have that for $1 \leq q \leq \frac{n}{n-2}$ and any $0 \leq \tilde{k} \leq k_1$,

$$\left\| \rho^{-\sigma}(r) \partial_\lambda^{\tilde{k}} R_{0,j} \rho^{-\sigma}(r, \cdot) \right\|_{L^q} \leq C \rho^{-\sigma}(r) \lambda^{n-2-\tilde{k}} \sum_{\ell+m=\tilde{k}} \left[(1+\lambda r)^{\ell-\frac{n-1}{2}} \left(C_1 + C_2 \lambda^{m-\frac{n-1}{2}} \right) \right]. \quad (5.7)$$

Note that if $\ell \leq \frac{n-1}{2}$, we can see that the corresponding term in (5.7) is bounded by a constant times

$$\lambda^{n-2-\tilde{k}} \max\{1, \lambda^{(\tilde{k}-\ell)-\frac{n-1}{2}}\} \leq \max\{\lambda^{n-2-\tilde{k}}, \lambda^{\frac{n-3}{2}-\ell}\}$$

uniformly for $r \in [0, \infty)$. On the other hand, if $\ell > \frac{n-1}{2}$ then we have that

$$\rho^{-\sigma}(r) (1+\lambda r)^{\ell-\frac{n-1}{2}} \lambda^{n-2-\tilde{k}} \leq (1+\lambda)^{\ell-\frac{n-1}{2}} (1+r)^{\ell-\frac{n-1}{2}-\sigma} \lambda^{n-2-\tilde{k}} \leq C (1+r)^{\tilde{k}-\sigma-\frac{n-1}{2}} \lambda^{\frac{n-3}{2}-(\tilde{k}-\ell)}. \quad (5.8)$$

by Cauchy-Schwarz. Recalling our conditions on σ , we see that $\tilde{k}-\sigma-\frac{n-1}{2} < 0$. Therefore, the corresponding term in (5.7) is bounded by a constant times

$$\lambda^{\frac{n-3}{2}-(\tilde{k}-\ell)} \max\{1, \lambda^{(\tilde{k}-\ell)-\frac{n-1}{2}}\} = \max\{\lambda^{\frac{n-3}{2}-(\tilde{k}-\ell)}, \lambda^{-1}\}$$

uniformly in r . Maximizing over the possible combinations of ℓ, m with $\ell+m=\tilde{k}$, we have that

$$\left\| \rho^{-\sigma} \partial_\lambda^{\tilde{k}} R_{0,j} \rho^{-\sigma}(r, \cdot) \right\|_{L^q} \leq C \max\{\lambda^{n-2-\tilde{k}}, \lambda^{\frac{n-3}{2}}\} \quad (5.9)$$

for some $C > 0$, uniformly in r . A similar argument gives

$$\left\| \rho^{-\sigma} \partial_\lambda^{\tilde{k}} R_{0,j} \rho^{-\sigma}(\cdot, s) \right\|_{L^q} \leq C \max\{\lambda^{n-2-\tilde{k}}, \lambda^{\frac{n-3}{2}}\} \quad (5.10)$$

uniformly in s .

For the final factor in the definition of A , which has asymmetric weights, we only need an estimate in the left variable in order to apply Lemma 5.2. By Proposition 3.8 we have, for $1 \leq q \leq \frac{n}{n-2}$,

$$\begin{aligned} & \left\| \rho^{-\sigma} g \partial_{\lambda}^{\tilde{k}} R_{0,j} \rho^{-\alpha}(\cdot, s) \right\|_{L^q} \\ & \leq C \rho^{-\alpha}(s) \lambda^{n-2-\tilde{k}} \sum_{\ell+m=\tilde{k}} \left[(1+\lambda s)^{\ell-\frac{n-1}{2}} \left(C_1 + C_2 \lambda^{m-\frac{n-1}{2}} \right) \right]. \end{aligned}$$

We may repeat our previous argument almost exactly in order to bound this quantity. The only difference here is that the analogue of (5.8) has a factor of $\rho^{-\alpha}$ instead of $\rho^{-\sigma}$. So in order to obtain an estimate which is uniform in s , we must enforce the condition that $\alpha \geq \max\{k - \frac{n-1}{2}, 0\}$ and recall that $\tilde{k} \leq k$. Aside from this, the rest of the argument is identical, and so we have

$$\left\| \rho^{-\sigma} f \partial_{\lambda}^{\tilde{k}} R_{0,j} \rho^{-\alpha}(\cdot, s) \right\|_{L^q} \leq C \max\{\lambda^{n-2-\tilde{k}}, \lambda^{\frac{n-3}{2}}\} \quad (5.11)$$

uniformly in s , provided that $\alpha \geq \max\{k - \frac{n-1}{2}, 0\}$.

We can now iteratively apply Lemma 5.2 to $\|\partial_{\lambda}^{k_1} A(\cdot, s)\|_{L^2}$. To do this, we must choose $q = \frac{2M}{2M-1}$ so that $\frac{M}{q} = \frac{1}{2} + (M-1)$. We also require $1 \leq q \leq \frac{n}{n-2}$, which is equivalent to taking $M \geq \frac{n}{4}$. This then implies that we must take $\sigma > \frac{n(2M-1)}{2M} + k_1$ in order for Proposition 3.6 and Proposition 3.8 to apply. In particular, we can take $M = \lceil \frac{n}{4} \rceil$, the smallest integer larger than $\frac{n}{4}$. Using (5.1), we see that

$$\sigma > 4 \left\lceil \frac{n}{4} \right\rceil - 2 + k = n \left(\frac{4M-2}{n} \right) + k \geq n \left(\frac{4M-2}{4M} \right) + k \geq n \left(\frac{2M-1}{2M} \right) + k_1,$$

and so the following argument holds under this condition on σ . Repeatedly applying Lemma 5.2 to $\|\partial_{\lambda}^{k_1} A(\cdot, s)\|_{L^2}$ and using that f is uniformly bounded, we obtain

$$\|\partial_{\lambda}^{k_1} A(\cdot, s)\|_{L^2} \leq C \lambda^{M(n-2)}.$$

The analogous estimate for $\|\partial_{\lambda}^{k_2} A(\cdot, r)\|$ combined with (5.6) gives

$$\left| \partial_{\lambda}^k \langle (\rho^{-\sigma} R_{V,j} \rho^{-\sigma}) A(\cdot, s), \overline{A(\cdot, r)} \rangle_{L^2} \right| \leq C \lambda^{2M(n-2)-1} \quad (5.12)$$

for $\lambda \geq 1$, and this estimate holds uniformly in r and s .

Next, we consider the remainder term in (5.4) when $0 < \lambda \leq 1$. In this case, taking the imaginary part in the left-hand side of (5.6) is essential, so we must estimate

$$\partial_{\lambda}^k \operatorname{Im} \langle \rho^{-\sigma} R_{V,j} \rho^{-\sigma} A(\cdot, s), A(\cdot, r) \rangle_{L^2}. \quad (5.13)$$

First, we note that the above can be written as a finite linear combination of terms where the imaginary part falls on either $R_{V,j}$ or at least one of the factors of A . Thus, by (4.4)

and (4.5), we can write

$$\begin{aligned} \left| \partial_\lambda^k \operatorname{Im} \langle \rho^{-\sigma} R_{V,j} \rho^{-\sigma} A(\cdot, s), A(\cdot, r) \rangle_{L^2} \right| &\leq C \max_{k_1+k_2+k_3 \leq k} \lambda^{n-2-k_3} \|\partial_\lambda^{k_1} A(\cdot, s)\|_{L^2} \|\partial_\lambda^{k_2} A(\cdot, r)\|_{L^2} \\ &\quad + C \max_{k_1+k_2 \leq k} \|\partial_\lambda^{k_1} \operatorname{Im} A(\cdot, s)\|_{L^2} \|\partial_\lambda^{k_2} A(\cdot, r)\|_{L^2} \end{aligned} \quad (5.14)$$

for $0 < \lambda \leq 1$. To estimate the first term on the right-hand side of (5.6), we can argue analogously to the $\lambda \geq 1$ case, but now we use the low-frequency estimates from Proposition 3.8, which give

$$\|\rho^{-\sigma} \partial_\lambda^{\tilde{k}} R_{0,j} \rho^{-\sigma}\|_{L^q} \leq C_1 \lambda^{-\tilde{k}} \rho^{-\sigma}(r) + C_2 \lambda^{n-2-\tilde{k}} (1 + \lambda r)^{\tilde{k} - \frac{n-1}{2}} \rho^{-\sigma}(r) \leq C \lambda^{-\tilde{k}}$$

for any $\tilde{k} \leq k$ and $1 \leq q \leq \frac{n}{n-2}$ as before. Similarly, we have

$$\|\rho^{-\sigma} \partial_\lambda^{\tilde{k}} R_{0,j} \rho^{-\alpha}\|_{L^q} \leq C \lambda^{-\tilde{k}}, \quad (5.15)$$

for $\alpha \geq \max\{k - \frac{n-1}{2}, 0\}$. Therefore, using Lemma 5.2, we have that

$$\|\partial_\lambda^{\tilde{k}} A(\cdot, s)\|_{L^2} \leq C \lambda^{-\tilde{k}}$$

uniformly in s , for $0 < \lambda \leq 1$. Therefore, we have

$$\max_{k_1+k_2+k_3 \leq k} \lambda^{n-2-k_3} \|\partial_\lambda^{k_1} A(\cdot, s)\|_{L^2} \|\partial_\lambda^{k_2} A(\cdot, r)\|_{L^2} \leq C \lambda^{n-2-k}. \quad (5.16)$$

Now, to handle the second term on the right-hand side of (5.14), we note that one may expand $\operatorname{Im} A(\cdot, s)$ into a linear combination of terms in which the imaginary part falls on at least one factor of $R_{0,j}$. Therefore, we can use Proposition 3.6 to obtain that

$$\|\rho^{-\sigma} \partial_\lambda^{\tilde{k}} \operatorname{Im} R_{0,j} \rho^{-\sigma}\|_{L^q} \leq C \lambda^{n-2} (1 + \lambda r)^{k - \frac{n-1}{2}} \rho^{-\sigma} \leq C \lambda^{n-2} \quad (5.17)$$

for any $\tilde{k} \leq k$. Thus, applying Lemma 5.2 in combination with (5.15) and (5.17) gives

$$\|\partial_\lambda^{\tilde{k}} \operatorname{Im} A(\cdot, s)\| \leq C \lambda^{n-2-\tilde{k}}$$

for any $\tilde{k} \leq k$. Hence, we have

$$\max_{k_1+k_2 \leq k} \|\partial_\lambda^{k_1} \operatorname{Im} A(\cdot, s)\|_{L^2} \|\partial_\lambda^{k_2} A(\cdot, r)\|_{L^2} \leq C \lambda^{n-2-k} \quad (5.18)$$

uniformly in r, s . Combining (5.16) and (5.18) with (5.14) yields

$$\partial_\lambda^k \operatorname{Im} \langle \rho^{-\sigma} R_{V,j} \rho^{-\sigma} A(\cdot, s), A(\cdot, r) \rangle_{L^2} \leq C \lambda^{n-2-k}, \quad (5.19)$$

and so the remainder in the Birman-Schwinger expansion of $R_{V,j}$ satisfies the claimed estimate for $0 < \lambda \leq 1$.

Now we consider a generic term in the sum in (5.4) for $1 \leq \ell \leq 2M - 1$. As before, we use the fact that $V = \rho^{-2\sigma}(r)f(r)$ for some $f \in L^\infty(\mathbb{R}^+)$ to write

$$\rho^{-\alpha} \partial_\lambda^k R_{0,j} (V R_{0,j})^\ell \rho^{-\alpha}(r, s) = \partial_\lambda^k (\rho^{-\alpha} R_{0,j} \rho^{-\sigma}) (\rho^{-\sigma} f R_{0,j} \rho^{-\sigma})^{\ell-1} (\rho^{-\sigma} f R_{0,j} \rho^{-\alpha})(r, s). \quad (5.20)$$

Writing the above as an L^2 -pairing, we have

$$\rho^{-\alpha} \partial_\lambda^k R_{0,j} (V R_{0,j})^\ell \rho^{-\alpha}(r, s) = \partial_\lambda^k \langle \rho^{-\alpha} R_{0,j} \rho^{-\sigma}(r, \cdot), (\rho^{-\sigma} f R_{0,j} \rho^{-\sigma})^{\ell-1} (\rho^{-\sigma} f R_{0,j} \rho^{-\alpha})(\cdot, s) \rangle_{L^2}. \quad (5.21)$$

Upon taking the imaginary part, we obtain a finite linear combination of terms of the form (5.21) where at least one factor of $R_{0,j}$ has the imaginary part acting on it. We assume without loss of generality that the leftmost factor on the right-hand side of (5.21) has the imaginary part, and thus we can apply Hölder's inequality to obtain

$$\begin{aligned} & \left| \partial_\lambda^k \langle \rho^{-\alpha} \operatorname{Im} R_{0,j} \rho^{-\sigma}(r, \cdot), (\rho^{-\sigma} f R_{0,j} \rho^{-\sigma})^{\ell-1} (\rho^{-\sigma} f R_{0,j} \rho^{-\alpha})(\cdot, s) \rangle_{L^2} \right| \\ & \leq C \|\rho^{-\alpha} \partial_\lambda^{k_1} \operatorname{Im} R_{0,j} \rho^{-\sigma}(r, \cdot)\|_{L^{q'}} \\ & \quad \times \|(\rho^{-\sigma} f \partial_\lambda^{k_2} R_{0,j} \rho^{-\sigma}) \cdots (\rho^{-\sigma} f \partial_\lambda^{k_{\ell-1}} R_{0,j} \rho^{-\sigma}) (\rho^{-\sigma} f \partial_\lambda^{k_\ell} R_{0,j} \rho^{-\alpha})(\cdot, s)\|_{L^p} \end{aligned} \quad (5.22)$$

for some $1 \leq q' < \infty$ to be determined, and p given by $\frac{1}{q'} + \frac{1}{p} = 1$, where $k_1 + k_2 + \cdots + k_\ell = k$.

If $0 < \lambda \leq 1$, we recall that by Proposition 3.6,

$$\|\rho^{-\alpha} \partial_\lambda^{k_1} \operatorname{Im} R_{0,j} \rho^{-\sigma}(r, \cdot)\|_{L^{q'}} \leq C \lambda^{n-2} \quad (5.23)$$

provided that $\sigma > \frac{n}{q'} + k$ and $\alpha \geq \max\{k_1 - \frac{n-1}{2}, 0\}$. Similarly, for any $\tilde{k} \leq k$, we have by Proposition 3.8 that for any $1 \leq q \leq \frac{n}{n-2}$,

$$\|\rho^{-\sigma} \partial_\lambda^{\tilde{k}} R_{0,j} \rho^{-\sigma}(r, \cdot)\|_{L^q} \leq C \lambda^{-\tilde{k}}, \quad (5.24)$$

if $\sigma > \frac{n}{q} + k$, along with the analogous estimate when the norm is taken with respect to the other variable. Using (5.23), (5.24), and repeated applications of Lemma 5.2 to the right-hand side of (5.22), we obtain

$$\left| \partial_\lambda^k \langle \rho^{-\alpha} \operatorname{Im} R_{0,j} \rho^{-\sigma}(r, \cdot), (\rho^{-\sigma} f R_{0,j} \rho^{-\sigma})^{\ell-1} (\rho^{-\sigma} f R_{0,j} \rho^{-\alpha})(\cdot, s) \rangle_{L^2} \right| \leq C \lambda^{n-2-k} \quad (5.25)$$

when $0 < \lambda \leq 1$, as long as we choose q, q' such that $\frac{1}{q'} + \frac{\ell}{q} = \ell$ and provided that $\sigma > \max\{\frac{n}{q'}, \frac{n}{q}\} + k$. Since $\frac{n}{q} \geq n - 2 - k$, we have by (5.1) that

$$\sigma > 4 \left\lceil \frac{n}{4} \right\rceil - 2 + k \geq n - 2 + k,$$

and so we can ensure that $\sigma > \frac{n}{q} + k$ if q is chosen sufficiently close to, but just below $\frac{n}{n-2}$. Given this choice of q , we also have that q' lies just above $\frac{n}{2\ell}$, and so

$$\sigma > 4 \left\lceil \frac{n}{4} \right\rceil - 2 + k = 2(2M - 1) + k,$$

and since $1 \leq \ell \leq 2M - 1$, we can ensure that $\sigma > \frac{n}{q'} + k$, since $\frac{n}{q'} + k$ can be made arbitrarily close to $2\ell + k$. Therefore, under the claimed conditions on σ and α , we have that

$$\left| \partial_\lambda^k R_{0,j} (V R_{0,j})^\ell \right| \leq C \lambda^{n-2-k}$$

for $0 < \lambda \leq 1$ and any $1 \leq \ell \leq 2M - 1$.

In the case where $\lambda \geq 1$, we have that

$$\|\rho^{-\alpha} \partial_\lambda^{k_1} \operatorname{Im} R_{0,j} \rho^{-\sigma}(r, \cdot)\|_{L^{q'}} \leq C \max\{\lambda^{n-2-k_1-\frac{n}{q'}}, \lambda^{\frac{n-3}{2}}\}$$

uniformly in r as before, provided that $\alpha \geq \max\{k - \frac{n-1}{2}, 0\}$ and $\sigma > \frac{n}{q'} + k$. Next, choose some q which lies just below $\frac{n}{n-2}$ as above. Then, for any $\tilde{k} \leq k$,

$$\|\rho^{-\sigma} \partial_\lambda^{\tilde{k}} R_{0,j} \rho^{-\sigma}(r, \cdot)\|_{L^q} \leq C \max\{\lambda^{n-2-\tilde{k}}, \lambda^{\frac{n-3}{2}}\},$$

uniformly in r , provided that $\sigma > \frac{n}{q} + k$, along with the analogous estimate when the L^q norm is taken over the second variable. We also note that for the rightmost factor in (5.22), we have

$$\|\rho^{-\sigma} \partial_\lambda^{\tilde{k}} R_{0,j} \rho^{-\alpha}(\cdot, s)\|_{L^q} \leq C \max\{\lambda^{n-2-\tilde{k}}, \lambda^{\frac{n-3}{2}}\},$$

uniformly in s , provided $\alpha \geq \max\{k - \frac{n-1}{2}, 0\}$. Given these estimates and Lemma 5.2, we can maximize over the possible combinations of k_1, \dots, k_ℓ to see from (5.22) that

$$\begin{aligned} & \left| \partial_\lambda^k \langle \rho^{-\alpha} \operatorname{Im} R_{0,j} \rho^{-\sigma}(r, \cdot), (\rho^{-\sigma} f R_{0,j} \rho^{-\sigma})^{\ell-1} (\rho^{-\sigma} f R_{0,j} \rho^{-\alpha})(\cdot, s) \rangle_{L^2} \right| \\ & \leq \lambda^{\ell(n-2)-k} \max\{\lambda^{n-2-\frac{n}{q'}}, \lambda^{\frac{n-3}{2}}\} \end{aligned} \quad (5.26)$$

provided that $\frac{1}{q'} + \frac{\ell}{q} = \ell$ and $\sigma > \max\{\frac{n}{q'}, \frac{n}{q}\} + k$. As shown previously, this condition on σ is satisfied under the hypothesis (5.1) if q is chosen close enough to $\frac{n}{n-2}$. Furthermore, for this choice of q , we have that q' lies just above $\frac{n}{2\ell}$. We claim that this implies that the bound (5.26) is smaller than the estimate (5.12). To see this, note that if q' is chosen sufficiently close to $\frac{n}{2\ell}$, then $\frac{n}{q'} = 2\ell + \varepsilon$ for some $\varepsilon > 0$. Then, we have

$$\ell(n-2) + n - 2 - \frac{n}{q'} = \ell(n-2) + (n-2) - 2\ell - \varepsilon \leq \ell(n-4) + (n-2).$$

If $n \leq 4$, then the above is smaller than $n-2$ for all $\ell = 1, \dots, 2M-1$. If $n \geq 4$, then we have

$$\ell(n-4) + (n-2) \leq (2M-1)(n-4) + (n-2) = 2M(n-2) - 2(2M-1) \leq 2M(n-2) - 1.$$

Furthermore, we note that

$$\ell(n-2) + \frac{n-3}{2} \leq (2M-1)(n-2) - \frac{n-3}{2} \leq 2M(n-2) - \frac{n-1}{2} \leq 2M(n-2) - 1.$$

Therefore, the exponent on λ in (5.26) is smaller than that of (5.12) for any ℓ , and hence we have

$$\left| \partial_\lambda^k \langle \rho^{-\alpha} \operatorname{Im} R_{0,j} \rho^{-\sigma}(r, \cdot), (\rho^{-\sigma} f R_{0,j} \rho^{-\sigma})^{\ell-1} (\rho^{-\sigma} f R_{0,j} \rho^{-\alpha})(\cdot, s) \rangle_{L^2} \right| \leq \lambda^{2M(n-2)-1} \quad (5.27)$$

when $\lambda \geq 1$.

Now, if the imaginary part falls on any factor other than the first on the right-hand side of (5.21), we simply repeat the preceding argument, but with the $L^{q'}$ norm on that factor.

Finally, we consider the case where $\ell = 0$ in (5.4). For this term, we must simply obtain pointwise bounds on $\rho^{-\alpha}(r)\partial_\lambda^k \operatorname{Im} R_{0,j}(r, s)\rho^{-\alpha}(s)$. Recall that by Lemma 2.1, we have

$$\operatorname{Im} R_{0,j}(r, s) = \frac{\pi}{2} \lambda^{n-2} (\lambda r \lambda s)^{-\frac{n-2}{2}} J_{\nu_j}(\lambda r) J_{\nu_j}(\lambda s).$$

Therefore, $\partial_\lambda^k R_{0,j}(r, s)$ can be written as a finite linear combination of terms of the form

$$\lambda^{n-2-k} (\lambda r)^{\ell - \frac{n-2}{2}} (\lambda s)^{m - \frac{n-2}{2}} J_{\nu_j + \alpha}(\lambda r) J_{\nu_j + \beta}(\lambda s) \quad (5.28)$$

for $\ell + m = k$, $|\alpha| \leq \ell$, and $|\beta| \leq m$. Using the standard asymptotics of the Bessel functions, we have that the above is bounded in absolute value by a constant times

$$\lambda^{n-2-k} (1 + \lambda r)^{\ell - \frac{n-1}{2}} (1 + \lambda s)^{m - \frac{n-1}{2}}. \quad (5.29)$$

Next, we note that

$$\rho^{-\alpha}(r) (1 + \lambda r)^{\ell - \frac{n-1}{2}} \leq C(1 + \lambda)^\ell,$$

for all λ , uniformly in r , under the assumption that $\alpha \geq \max\{k - \frac{n-1}{2}, 0\}$. The analogous estimate holds for $\rho^{-\alpha}(s) (1 + \lambda s)^{m - \frac{n-1}{2}}$, and therefore, we have that

$$|\rho^{-\alpha}(r) \partial_\lambda^k R_{0,j}(r, s) \rho^{-\alpha}(s)| \leq C \lambda^{n-2-k} (1 + \lambda)^k. \quad (5.30)$$

Combining (5.30) with (5.19), (5.12), (5.25), and (5.26), the proof of Proposition 5.1 is complete. \square

6. DISPERSIVE ESTIMATES

In this section, we prove the main estimate in Theorem 1. To accomplish this, we write the continuous part of the spectral measure for $-\Delta_{C(X)} + V$ as

$$d\Pi_V(\lambda; x, y) = \frac{1}{\pi i} [R_V(\lambda + i0; x, y) - R_V(\lambda - i0; x, y)] \lambda d\lambda = \frac{1}{\pi} \operatorname{Im} R_V(\lambda + i0; x, y) \lambda d\lambda.$$

Then, we can write

$$[e^{it(-\Delta_{C(X)} + V)} P_c](x, y) = \int_{-\infty}^{\infty} e^{it\lambda^2} d\Pi_V(\lambda; x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{it\lambda^2} \operatorname{Im} R_V(\lambda + i0; x, y) \lambda d\lambda,$$

where P_c denotes projection onto the continuous spectrum of $-\Delta_{C(X)} + V$. Projecting further onto the span of φ_j , we obtain

$$\frac{1}{\pi} \int_{-\infty}^{\infty} e^{it\lambda^2} \operatorname{Im} R_{V,j}(\lambda + i0; r, s) \lambda d\lambda \quad (6.1)$$

since V is radial. Therefore, the proof of Theorem 1 is equivalent to showing that

$$\left| \frac{1}{\pi} \int_{-\infty}^{\infty} e^{it\lambda^2} \rho^{-\alpha}(r) \operatorname{Im} R_{V,j}(\lambda + i0; r, s) \rho^{-\alpha}(s) \lambda d\lambda \right| \leq Ct^{-\frac{n}{2}} \quad (6.2)$$

for $\alpha > 2 \left\lceil \frac{n}{4} \right\rceil (n-2) - \frac{n-1}{2} + 2$.

Proof of Theorem 1. Assume that n is odd, and let $\chi \in C_0^\infty(\mathbb{R})$ be a cutoff function which is identically one on $[-1/2, 1/2]$ and zero outside $[-1, 1]$. We then consider the low-frequency component of the left-hand side of (6.2), given by

$$\frac{1}{\pi} \int_{\mathbb{R}} e^{it\lambda^2} \chi(\lambda) \rho^{-\alpha}(r) \operatorname{Im} R_{V,j}(\lambda + i0; r, s) \rho^{-\alpha}(s) \lambda d\lambda. \quad (6.3)$$

Noting that the operator $\frac{1}{2it\lambda} \partial_\lambda$ preserves $e^{it\lambda^2}$, we may integrate by parts N times in λ to obtain

$$\frac{C_N}{t^N} \int_{-\infty}^{\infty} e^{it\lambda^2} \partial_\lambda \left(\frac{1}{\lambda} \partial_\lambda \right)^{N-1} [\chi(\lambda) \rho^{-\alpha}(r) \operatorname{Im} R_{V,j}(\lambda + i0; r, s) \rho^{-\alpha}(s)] d\lambda \quad (6.4)$$

for some $C_N \in \mathbb{C} \setminus 0$. Now, observe that when expanding the integrand via the product rule, any terms in which a derivative falls on the factor of $\chi(\lambda)$ can be written as

$$C_N t^{-N} \int_{-\infty}^{\infty} e^{it\lambda^2} G(\lambda; r, s) d\lambda$$

for some $G(\lambda; r, s)$ which is smooth and compactly supported away from 0 in λ , and bounded uniformly in r, s by Proposition 5.1. Applying the standard dispersive estimate for the Schrödinger equation on \mathbb{R} , we have

$$t^{-N} \left| \int_{-\infty}^{\infty} e^{it\lambda^2} G(\lambda; r, s) d\lambda \right| \leq Ct^{-N-\frac{1}{2}} \|\widehat{G}(\cdot; \cdot, r, s)\|_{L^1}, \quad (6.5)$$

where \widehat{G} denotes the Fourier transform in λ . Since n is odd, we may choose $N = \frac{n-1}{2}$, so the right-hand side of (6.5) is bounded by $Ct^{-\frac{n}{2}}$ as claimed, after possibly increasing C .

Now, any terms obtained from expanding (6.4) where no derivatives fall on the factor of χ must be of the form

$$\lambda^{1-2N+k} \chi(\lambda) \rho^{-\alpha}(r) \partial_\lambda^k \operatorname{Im} R_{V,j}(\lambda + i0; r, s) \rho^{-\alpha}(s) \quad (6.6)$$

for some $k = 1, 2, \dots, N$ since at least one derivative always falls on the factor of $\operatorname{Im} R_{V,j}$. By Proposition 5.1, we have that each of the above terms is bounded in absolute value by a constant times

$$\lambda^{1-2N+k} \chi(\lambda) \lambda^{n-2-k} = \lambda^{n-1-2N} \chi(\lambda)$$

uniformly for $r, s > 0$. For our choice of $N = \frac{n-1}{2}$, we have that $\lambda^{n-1-2N} = 1$, and hence (6.6) is a smooth function of λ , and so its Fourier transform is bounded in L^1 . Once again, using the standard $L^1 \rightarrow L^\infty$ dispersive estimate for the free one-dimensional Schrödinger equation, we have that

$$t^{-N} \left| \int_{-\infty}^{\infty} e^{it\lambda^2} \lambda^{1-2N+k} \chi(\lambda) \rho^{-\alpha}(r) \partial_\lambda^k \operatorname{Im} R_{V,j}(\lambda + i0)(r, s) \rho^{-\alpha}(s) d\lambda \right| \leq Ct^{-N-\frac{1}{2}} = Ct^{-\frac{n}{2}}, \quad (6.7)$$

uniformly in r, s . We remark that it is in this calculation that the choice of $N = \frac{n-1}{2}$, and hence the power of $t^{-\frac{n}{2}}$, cannot be improved, since any additional derivatives which fall on $\operatorname{Im} R_{V,j}(\lambda + i0)$ would yield an integrand which is not bounded smooth near $\lambda = 0$. We also note that for this portion of the argument, we only require that $\alpha \geq 0$, since we did not differentiate $\operatorname{Im} R_{V,j}$ more than $\frac{n-1}{2}$ times.

Next, we consider the ‘‘high-frequency’’ component of (6.2), which we define by

$$\frac{1}{\pi} \int_{-\infty}^{\infty} e^{it\lambda^2} \chi(\lambda/R)(1 - \chi(\lambda)) \rho^{-\alpha}(r) \operatorname{Im} R_{V,j}(\lambda + i0; r, s) \rho^{-\alpha}(s) \lambda d\lambda \quad (6.8)$$

for any $R \in [1, \infty)$. To control this term, we integrate by parts as before to obtain

$$\begin{aligned} & \frac{1}{\pi} \int_{-\infty}^{\infty} e^{it\lambda^2} \chi(\lambda/R)(1 - \chi(\lambda)) \rho^{-\alpha}(r) \operatorname{Im} R_{V,j}(\lambda + i0; r, s) \rho^{-\alpha}(s) \lambda d\lambda \\ &= C_N t^{-N} \int_{-\infty}^{\infty} e^{it\lambda^2} \partial_\lambda \left(\frac{1}{\lambda} \partial_\lambda \right)^{N-1} [\chi(\lambda/R)(1 - \chi(\lambda)) \rho^{-\alpha}(r) \operatorname{Im} R_{V,j}(\lambda + i0; r, s) \rho^{-\alpha}(s)] d\lambda \end{aligned}$$

for any $N > 0$ and some corresponding constant C_N . We aim to show that the integrand can be bounded uniformly in $L^1(\mathbb{R}, d\lambda)$ as $R \rightarrow \infty$. We also claim that it is sufficient to consider the case where all the derivatives in λ fall on the factor of $\operatorname{Im} R_{V,j}$. To see this, note that $\partial_\lambda(1 - \chi(\lambda))$ is supported in a fixed compact set which is bounded away from $\lambda = 0$, and that $\partial_\lambda \chi(\lambda/R) = \frac{1}{R} \chi'(\lambda/R)$ is supported away from $\lambda = 0$ in a set of size $\mathcal{O}(R)$. Therefore, we need only show that

$$\frac{1}{\lambda^{N-1}} \chi(\lambda/R)(1 - \chi(\lambda)) \rho^{-\alpha}(r) \partial_\lambda^N \operatorname{Im} R_{V,j}(\lambda + i0; r, s) \rho^{-\alpha}(s) \quad (6.9)$$

has bounded L^1 norm, and that the estimate is uniform with respect to r, s , and R . For this, we utilize Proposition 5.1, which implies that if $\alpha \geq \max\{N - \frac{n-1}{2}, 0\}$ and $\sigma > 4 \lceil \frac{n}{4} \rceil - 2 + N$, then (6.9) is bounded by a constant times $\langle \lambda \rangle^{1-N+L}$, uniformly in r, s , and R , where

$L = 2 \left\lceil \frac{n}{4} \right\rceil (n - 2) - 1$. Thus, choosing

$$N = 2 \left\lceil \frac{n}{4} \right\rceil (n - 2) + 2,$$

guarantees that (6.9) is uniformly bounded in L^1 . Noting that $N \geq \frac{n}{2}$ if N is chosen as above, we obtain

$$\lim_{R \rightarrow \infty} \left| \frac{1}{\pi} \int_{-\infty}^{\infty} e^{it\lambda^2} \chi(\lambda/R) (1 - \chi(\lambda)) \lambda \operatorname{Im} R_{V,j}(\lambda + i0; r, s) d\lambda \right| \leq Ct^{-\frac{n}{2}}, \quad (6.10)$$

where $C > 0$ is independent of r, s and R . Our choice of N also determines the maximum number of derivatives of $\operatorname{Im} R_{V,j}(\lambda + i0; r, s)$ that must be taken, which yields

$$\sigma > 4 \left\lceil \frac{n}{4} \right\rceil - 2 + 2 \left\lceil \frac{n}{4} \right\rceil (n - 2) + 2 = 2n \left\lceil \frac{n}{4} \right\rceil$$

as the sufficient condition on the decay rate of V . Also, the condition on α becomes

$$\alpha > N - \frac{n-1}{2} = 2 \left\lceil \frac{n}{4} \right\rceil (n - 2) + 2 - \frac{n-1}{2},$$

as stated in Theorem 1. Under these conditions on the weights, we can combine (6.7) and (6.10) to obtain (6.2), which completes the proof of Theorem 1. \square

Remark 6.1. In the case where n is even, we find that in repeating the argument just prior to (6.7), the largest N we can choose is $\frac{n-2}{2}$, which leads to a decay rate of $t^{-\frac{n-1}{2}}$ in the $L^1 \rightarrow L^\infty$ estimate. The remainder of the argument goes through without modification, yielding (1.4).

APPENDIX A. DISPERSIVE ESTIMATES FOR THE FREE SCHRÖDINGER EQUATION

Here we discuss bounds on solutions of the unperturbed Schrödinger equation, given by

$$\begin{cases} (\frac{1}{i}\partial_t - \Delta_{C(X)}) u = 0, \\ u|_{t=0} = f. \end{cases} \quad (\text{A.1})$$

We prove that the solution to this equation satisfies a dispersive estimate analogous to Theorem 1, but without the need for projection onto the harmonics of the link, provided that the solution is measured in $L^\infty(\mathbb{R}^+; L^2(X))$, rather than simply $L^\infty(C(X))$. We do this by using a modification of the techniques outlined in [19], which handled flat two-dimensional cones, to obtain an explicit asymptotic formula for the kernel of $e^{it\Delta_{C(X)}}$ as a function of a rescaled variable.

Theorem 4. *Let $C(X) = \mathbb{R}^+ \times X$ be the product cone on X , for (X, h) a compact Riemannian manifold of dimension $n - 1$. Then the solution to (A.1) satisfies*

$$\|e^{it\Delta_{C(X)}} f\|_{L^\infty(\mathbb{R}^+; L^2(X))} \leq Ct^{-\frac{n}{2}} \|f\|_{L^1(\mathbb{R}^+; L^2(X))}, \quad t > 0,$$

for some $C > 0$. Here, $L^1(\mathbb{R}^+)$ is defined with respect to the measure $r^{n-1} dr$.

Remark A.1. We note that this result is somewhat weaker than similar estimates obtained in [71], but we include it here because the proof is quite short and requires significantly less machinery.

Since X is compact, there exists an orthonormal basis $\{\varphi_j\}_{j=0}^\infty$ of $L^2(X)$, satisfying

$$-\Delta_h \varphi_j = \mu_j^2 \varphi_j$$

for $0 = \mu_0^2 < \mu_1^2 \leq \dots$ repeated according to multiplicity. By the functional calculus of Cheeger [10] discussed in Section 2, we can define the shifted eigenvalues

$$\nu_j = \sqrt{\mu_j^2 + \left(\frac{n-2}{2}\right)^2},$$

in order to write the spectral measure of $-\Delta_{C(X)}$ as

$$d\Pi_0(r_1, \theta_1, r_2, \theta_2) = (r_1 r_2)^{-\frac{n-2}{2}} \sum_{j=0}^{\infty} J_{\nu_j}(\lambda r_1) J_{\nu_j}(\lambda r_2) \varphi_j(\theta_1) \overline{\varphi_j(\theta_2)} \lambda d\lambda,$$

where J_ν is the Bessel function of the first kind of order ν . Hence, the fundamental solution to (A.1) has the form

$$K_{e^{it\Delta_{C(X)}}}(r_1, \theta_1, r_2, \theta_2) = (r_1 r_2)^{-\left(\frac{n-2}{2}\right)} \sum_{j=0}^{\infty} \left(\int_0^{\infty} e^{it\lambda^2} J_{\nu_j}(\lambda r_1) J_{\nu_j}(\lambda r_2) \lambda d\lambda \right) \varphi_j(\theta_1) \overline{\varphi_j(\theta_2)}, \quad (\text{A.2})$$

with respect to the standard measure on the cone, $r^{n-1} dr dv_h(\theta)$, where dv_h is the Riemannian volume measure on X . As in [19], we let $t = is$ in the above expression to obtain a formula for the heat kernel $e^{-s\Delta_{C(X)}}$. By Weber's second exponential integral formula, we have that

$$\int_0^{\infty} e^{-s\lambda^2} J_\nu(\lambda r_1) J_\nu(\lambda r_2) \lambda d\lambda = \frac{1}{2s} e^{-\frac{r_1^2 + r_2^2}{4s}} I_\nu\left(\frac{r_1 r_2}{2s}\right),$$

where I_ν is the modified Bessel function of order ν , defined by

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\nu + k + 1)} \left(\frac{x}{2}\right)^{2k + \nu}.$$

Analytic continuation in s and taking $s = -it$ gives us

$$K_{e^{it\Delta_{C(X)}}}(r_1, \theta_1, r_2, \theta_2) = \frac{ie^{\frac{r_1^2 + r_2^2}{4it}}}{2t(r_1 r_2)^{\frac{n-2}{2}}} \sum_{j=0}^{\infty} i^{\nu_j} J_{\nu_j}\left(\frac{r_1 r_2}{2t}\right) \varphi_j(\theta_1) \overline{\varphi_j(\theta_2)},$$

since $I_\nu(ix) = i^\nu J_\nu(x)$. For non-integer values of ν , we choose z^ν to have its branch cut along the negative real axis. For convenience, we define $x = \frac{r_1 r_2}{2t}$ and let

$$S(x, \theta_1, \theta_2) = x^{-\left(\frac{n-2}{2}\right)} \sum_{j=0}^{\infty} i^{\nu_j} J_{\nu_j}(x) \varphi_j(\theta_1) \overline{\varphi_j(\theta_2)},$$

so that

$$K_{e^{it\Delta_{C(X)}}}(r_1, \theta_1, r_2, \theta_2) = \frac{i \exp\left(\frac{r_1^2 + r_2^2}{4it}\right)}{(2t)^{\frac{n}{2}}} S(x, \theta_1, \theta_2). \quad (\text{A.3})$$

Furthermore, we define the family of operators $S(x) : C^\infty(X) \rightarrow \mathcal{D}'(X)$ by

$$S(x)f(\theta_1) = \int_X S(x, \theta_1, \theta_2) f(\theta_2) dv_h(\theta_2) = x^{-\left(\frac{n-2}{2}\right)} \sum_{j=0}^{\infty} i^{\nu_j} J_{\nu_j}(x) \langle f, \varphi_j \rangle \varphi_j(\theta).$$

Next, we make note of an asymptotic expansion for $K_{e^{it\Delta_{C(X)}}}$ in the regime where $x \rightarrow 0$, which is analogous to [19, Prop 4.1].

Proposition A.2. *The free Schrödinger propagator has the asymptotic behavior*

$$K_{e^{it\Delta_{C(X)}}}(r_1, \theta_1, r_2, \theta_2) = \frac{i \exp\left(\frac{r_1^2 + r_2^2}{4it}\right)}{(2t)^{\frac{n}{2}}} \left[\frac{(i/2)^{\frac{n-2}{2}}}{\Gamma\left(\frac{n}{2}\right) \text{vol}(X)} + \mathcal{O}\left(\left(\frac{r_1 r_2}{2t}\right)^\alpha\right) \right], \quad \text{as } \frac{r_1 r_2}{2t} \rightarrow 0,$$

where $\alpha = \min\{2, \nu_1 - \frac{n-2}{2}\}$.

Proof. It suffices to show that

$$S(x, \theta_1, \theta_2) = \frac{(i/2)^{\frac{n-2}{2}}}{\Gamma\left(\frac{n}{2}\right) \text{vol}(X)} + \mathcal{O}(x^\alpha), \quad \text{as } x \rightarrow 0, \quad (\text{A.4})$$

uniformly in θ_1, θ_2 . Since the φ_j are L^2 -normalized, we have that $\varphi_0 = \frac{1}{\sqrt{\text{vol}(X)}}$, and so

$$S(x, \theta_1, \theta_2) = \frac{(i/2)^{\frac{n-2}{2}}}{\text{vol}(X)} \left(\frac{x}{2}\right)^{-\left(\frac{n-2}{2}\right)} J_{\frac{n-2}{2}}(x) + x^{-\left(\frac{n-2}{2}\right)} \sum_{j=1}^{\infty} i^{\nu_j} J_{\nu_j}(x) \varphi_j(\theta_1) \overline{\varphi_j(\theta_2)}.$$

By the standard power series representation for $J_{\frac{n-2}{2}}$, we have

$$\left| S(x, \theta_1, \theta_2) - \frac{(i/2)^{\frac{n-2}{2}}}{\Gamma\left(\frac{n}{2}\right) \text{vol}(X)} \right| = \left| \frac{(i/2)^{\frac{n-2}{2}}}{\text{vol}(X)} \sum_{k=1}^{\infty} \frac{(-1)^k}{k! \Gamma\left(\frac{n}{2} + k\right)} \left(\frac{x}{2}\right)^{2k} + x^{-\left(\frac{n-2}{2}\right)} \sum_{j=1}^{\infty} i^{\nu_j} J_{\nu_j}(x) \varphi_j(\theta_1) \overline{\varphi_j(\theta_2)} \right|.$$

Using the bound $|J_\nu(x)| \leq \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu$ for x real and $\nu > 0$, along with the standard L^∞ eigenfunction estimate $\|\varphi_j\|_{L^\infty} \leq C\mu_j^{\frac{n-2}{2}}$, shows that for $0 \leq x < 2$, we have

$$\begin{aligned} \left| S(x, \theta_1, \theta_2) - \frac{(i/2)^{\frac{n-2}{2}}}{\Gamma(\frac{n}{2})\text{vol}(X)} \right| &\leq \frac{1}{\text{vol}(X)} \sum_{k=1}^{\infty} \left(\frac{x^2}{4}\right)^k + \sum_{j=1}^{\infty} \frac{\mu_j^{2(n-1)}}{2^{\frac{n-2}{2}}\Gamma(\nu_j+1)} \left(\frac{x}{2}\right)^{\nu_j - \frac{n-2}{2}} \\ &\leq \frac{x^2}{\text{vol}(X)(4-x^2)} + \left(\frac{x}{2}\right)^{\nu_1 - \frac{n-2}{2}} \sum_{j=1}^{\infty} \frac{\mu_j^{2(n-1)}}{\Gamma(\nu_j+1)}. \end{aligned} \quad (\text{A.5})$$

Note that $\mu_j \sim Cj^{\frac{1}{n-1}}$ by the Weyl law for the eigenvalues of $-\Delta_h$. Since $\nu_j \geq \mu_j$ for all j , the summation in the last inequality of (A.5) converges, which demonstrates (A.4) with $\alpha = \min\{2, \nu_1 - \frac{n-2}{2}\}$. We observe that by definition, $\nu_1 - \frac{n-2}{2} > 0$. Hence, by (A.3), the proof is complete. \square

Corollary A.3. *The family of operators $S(x)$ satisfies*

$$\|S(x)f\|_{L^2(X)} \leq C\|f\|_{L^2(X)}$$

for all $x \geq 0$ and for some $C > 0$ which is uniform in x .

Proof. For $x < 2 - \varepsilon$ with $\varepsilon > 0$, the estimate follows from the proof of Proposition A.2, which shows that $|S(x, \theta_1, \theta_2)| \leq C$ for some C which is uniform in θ_1, θ_2 . Thus, for such x ,

$$\begin{aligned} \|S(x)f\|_{L^2(X)}^2 &= \int_X \left| \int_X S(x, \theta_1, \theta_2) f(\theta_2) dv_h(\theta_2) \right|^2 dv_h(\theta_1) \\ &\leq \int_X \|S(x, \theta_1, \cdot)\|_{L^2(X)}^2 \|f\|_{L^2(X)}^2 dv_h(\theta_1) \\ &\leq C^2 \text{vol}(X)^2 \|f\|_{L^2(X)}^2. \end{aligned}$$

For $x \geq 1$, we can simply use the fact that $|J_\nu(x)|$ is bounded uniformly in both x and ν to see that

$$\|S(x)f\|_{L^2(X)}^2 = x^{2-n} \sum_{j=0}^{\infty} J_{\nu_j}^2(x) \langle f, \varphi_j \rangle^2 \leq C \sum_{j=0}^{\infty} \langle f, \varphi_j \rangle^2 = C\|f\|_{L^2(X)}^2,$$

for some $C > 0$, since $n \geq 2$. \square

We are now ready to present the proof of the dispersive estimate in Theorem 4.

Proof of Theorem 4. Recalling that $K_{e^{it\Delta_{C(X)}}}(r_1, \theta_1, r_2, \theta_2)$ is the Schwartz kernel of $e^{it\Delta_{C(X)}}$ and applying (A.3), we have that for any $r_1, t > 0$,

$$\begin{aligned}
& \|e^{it\Delta_{C(X)}} f(r_1, \cdot)\|_{L^2(X)}^2 \\
&= \int_X \left| \int_0^\infty \int_X K_{e^{it\Delta_{C(X)}}}(r_1, \theta_1, r_2, \theta_2) f(r_2, \theta_2) r_2^{n-1} dv_h(\theta_2) dr_2 \right|^2 dv_h(\theta_1) \\
&= \int_X \left(\int_0^\infty \frac{ie^{\left(\frac{r_1^2+r_2^2}{4it}\right)}}{(2t)^{\frac{n}{2}}} \left[S\left(\frac{r_1 r_2}{2t}\right) f(r_2, \theta_1) \right] r_2^{n-1} dr_2 \right) \\
&\quad \times \overline{\left(\int_0^\infty \frac{ie^{\left(\frac{r_1^2+r_3^2}{4it}\right)}}{(2t)^{\frac{n}{2}}} \left[S\left(\frac{r_1 r_3}{2t}\right) f(r_3, \theta_1) \right] r_3^{n-1} dr_3 \right)} dv_h(\theta_1) \\
&= (2t)^{-n} \int_0^\infty \int_0^\infty e^{\left(\frac{r_2^2-r_3^2}{4it}\right)} \left(\int_X S\left(\frac{r_1 r_2}{2t}\right) f(r_2, \theta_1) \overline{S\left(\frac{r_2 r_3}{2t}\right) f(r_3, \theta_1)} dv_h(\theta_1) \right) (r_2 r_3)^{n-1} dr_2 dr_3 \\
&\leq (2t)^{-n} \int_0^\infty \int_0^\infty \left\| S\left(\frac{r_1 r_2}{2t}\right) f(r_2, \cdot) \right\|_{L^2(X)} \left\| S\left(\frac{r_1 r_3}{2t}\right) f(r_3, \cdot) \right\|_{L^2(X)} (r_2 r_3)^{n-1} dr_2 dr_3.
\end{aligned}$$

In the last inequality, we are able to omit the complex exponential factor by taking absolute values, since the integral is known to be real-valued and non-negative. By Corollary A.3, the above is bounded by

$$Ct^{-n} \int_0^\infty \int_0^\infty \|f(r_2, \cdot)\|_{L^2(X)} \|f(r_3, \cdot)\|_{L^2(X)} (r_2 r_3)^{n-1} dr_2 dr_3 = Ct^{-n} \|f\|_{L^1(L^2(X), r^{n-1} dr)}^2,$$

for some $C > 0$ which is independent of r_1 . Taking square roots completes the proof of Theorem 4. \square

APPENDIX B. ABSENCE OF EMBEDDED RESONANCES ON PRODUCT CONES

B.1. Radial potentials and ODE methods. In this appendix, we establish the absence of embedded resonances and eigenvalues result claimed in Theorem 3 using the results from Theorem XIII.56 of [59]. To prove this, we need to use the fact that V is decaying to treat Vu as a perturbative term in the limit and prove that if $u \in L^2$, then hence Vu is

perturbative and we can write

$$u_{\pm}(r) = \frac{e^{\pm i\lambda r}}{r^{\frac{n-1}{2}}} \mp \int_r^{\infty} \frac{\sin(\lambda(r-s))}{\lambda(rs)^{\frac{n-1}{2}}} V(s) u_{\pm}(s) ds,$$

where by u_{\pm} we mean the outgoing/incoming functions converging to the Jost solution asymptotic of the form

$$\frac{e^{\pm i\lambda r}}{r^{\frac{n-1}{2}}}.$$

This gives the exact integrability condition in Theorem XIII.56 of [59]. It is also an integral equation that can also be solved using Picard iteration. Using the variation of parameters formula to solve

$$(-\Delta_{C(X)} - \lambda^2)u = V(r)u$$

for any V with $\int_a^{\infty} V(r)dr < \infty$ for some $a > 0$, we can write

$$\begin{aligned} u(r) &= c_+ u_+(r) + c_- u_-(r) \\ &+ \int_r^{\infty} \frac{u_+(r)u_-(s)}{W(s)} V(s)u(s)s^{n-1} ds + \int_r^{\infty} \frac{u_-(r)u_+(s)}{W(s)} V(s)u(s)s^{n-1} ds, \end{aligned}$$

where $W(s) = 2s^{n-1}$. Hence, if we have a resonance $u \in L^{2,\sigma}$ for $\sigma > \frac{1}{2}$, we have that indeed we can see the integral terms on the right converge and hence derive a contradiction to the existence of resonances that are not eigenvalues.

For radial potentials, we have observed that there are no embedded eigenvalues using the ODE based tools of Theorem XIII.56 of [59], but the issue of absence of embedded resonances down to $\lambda = 0$ still must be established. We state the result here.

Proposition B.1. *For $V \in \rho^{-2\sigma}L^{\infty}(\mathbb{R}^+)$ with $\sigma > \frac{1}{2}$, if $R_{V,j}(z)$ has a pole at τ for $\tau \in \mathbb{R} \setminus \{0\}$, then τ^2 is an embedded eigenvalue for $-\Delta_{C(X)} + V$.*

Proof of Theorem 3. By Proposition B.1, it suffices to prove that $R_V(z)$ has no real poles corresponding to eigenvalues of $-\Delta_{C(X)+V}$ embedded in the continuous spectrum. It also suffices to prove this fact for $R_{V,j}(z)$, since the fact that V is radial means that R_V respects the decomposition into harmonics on the link. The method of proof from Theorem XIII.56 of [59] proves that for V sufficiently decaying, a radial operator has no positive eigenvalues. This relies on a formal analysis of Jost solutions, proving that no linear combination of them can be an L^2 function. In particular, if we assume we have an embedded eigenvalue at energy λ^2 with corresponding radial eigenfunction ϕ on the j th harmonic, it satisfies

$$\left(\partial_r^2 + \frac{n-1}{r} \partial_r + \lambda^2 - \frac{\mu_j^2}{r^2} \right) \phi = 0$$

meaning that ∞ is a regular singular point. As a result, as $r \rightarrow \infty$ the Jost solutions are of the form

$$\phi_{\pm}(r) = e^{\pm i\lambda r} r^{-\frac{n-1}{2}} + O(r^{\frac{n-3}{2}}).$$

Hence, linear combinations generically take the form

$$A \sin(\lambda r + \theta) r^{-\frac{n-1}{2}},$$

which is easily seen to not be in $L^2(C(X))$, which yields a contradiction. Further discussion of the Jost solutions can be computed as in [60], Ch. 5. □

B.2. Absence of Embedded Resonances for Non-Radial Potentials. For the purposes of ruling out embedded resonances, a geometrically robust approach is to use a boundary pairing formula on radially compactified space, as in [54, §2.3], to prove that an embedded resonance is an embedded eigenvalue and hence has at least L^2 decay. The boundary pairing formula for Schrödinger operators on Euclidean space is derived from the observation that a pole of the resolvent corresponds to a solution to

$$(-\Delta_{\mathbb{R}^n} + V - \lambda^2)u = f,$$

say for $f \in \mathcal{S}$ a Schwartz class function. The solution to this equation takes the form

$$u = e^{i\frac{\lambda}{x}} x^{\frac{1}{2}(n-1)} w_+ + e^{-i\frac{\lambda}{x}} x^{\frac{1}{2}(n-1)} w_-, \quad w_{\pm} \in C^\infty(\mathbb{S}_+^n)$$

where $x = \frac{1}{r}$ and \mathbb{S}_+^n is the upper hemisphere of the sphere \mathbb{S}^n . The boundary pairing formula states that for solutions

$$(-\Delta + V - \lambda^2)u^{(\ell)} = f^{(\ell)}$$

with $\ell = 1, 2$, we have

$$2i\lambda \int_{\mathbb{S}^{n-1}} (v_+^{(1)} \overline{v_+^{(2)}} - v_-^{(1)} \overline{v_-^{(2)}}) dz = \int_{\mathbb{R}^n} (f^{(1)} \overline{u^{(2)}} + u^{(1)} \overline{f^{(2)}}) dz$$

with $v_{\pm}^{(\ell)} = w_{\pm}^{(\ell)}|_{\partial\mathbb{S}_+^n}$. For a pole of the outgoing resolvent, we observe that v_+ vanishes identically.

Hence, any embedded resonance can be seen to be an embedded eigenvalue. To eliminate embedded eigenvalues, we follow the work of Froese et al [21] to prove that L^2 eigenfunctions must in fact exhibit super-polynomial decay. The arguments there involve constructing a series of positive commutator arguments to obtain this rapid decay. To begin, take $\varepsilon, \gamma > 0$ and define $\rho(|x|) = \langle x \rangle$. Then, the function

$$F(x) = \gamma \ln(\rho(1 + \varepsilon\rho)^{-1}).$$

Then, $\nabla F = xg$ for $g = \gamma\rho^{-2}(1 + \varepsilon\rho)^{-1}$ and

$$(x \cdot \nabla)^2 g - (x \cdot \nabla)(\nabla F)^2 \leq 4\gamma(\gamma + 2)\rho^{-2}.$$

Defining $\psi_R = e^F \psi$, a conjugated form of the equation can be written as a modified quadratic form. Coupling this form with a Mourre estimate (positive commutator using $r\partial r$ (the radial version of $x \cdot \nabla$) we can prove that the set of polynomial weights for which

$\psi \in L^2$ is open and can be extended to ∞ . The Mourre estimate serves as a means to construct the weak limit of the resolvent at the real axis.

Once super-polynomial behavior is established, a similar open set for exponential decay can be established using the assumption that $e^{\alpha_0 \rho} \rho^\lambda \psi \in L^2$ for all λ and showing that this implies then that $e^{(\alpha_0 + \gamma) \rho} \psi \in L^2$. To do this, build the function

$$F(x) = \alpha_0 \rho + \lambda \ln(1 + \gamma \rho / \lambda)$$

and derive a similar contradiction.

Once super-exponential decay is established, the strategy of Vasy-Wunsch [66] can be applied to prove a unique continuation argument by conjugating the operator to

$$P_\alpha = e^{\alpha r'} (-\Delta + V - \lambda) e^{-\alpha r'}$$

for r' some smoothed version of r to be determined. Then,

$$0 = \|P_\alpha \phi_\alpha\|^2 = \|\operatorname{Re} P_\alpha \psi_\alpha\|^2 + \|\operatorname{Im} P_\alpha \psi_\alpha\|^2 + \langle i[\operatorname{Re} P_\alpha, \operatorname{Im} P_\alpha] \psi_\alpha, \psi_\alpha \rangle.$$

Hence, one uses that $i[\operatorname{Re} P_\alpha, \operatorname{Im} P_\alpha]$ is a positive commutator term. We require that

$$[\Delta, 2\partial_r + (\partial_r \log A)] \geq c \frac{1}{r^2} \Delta_\theta + R$$

for R in the calculus of first order conic vector fields.

B.3. The Boundary Pairing Formula. Following a suggestion of Dean Baskin, we can interpret embedded resonances for more general conic Schrödinger operators through the boundary pairing formula of Melrose.

B.3.1. Existence of the Boundary Pairing Formula on Cones. We outline the necessary generalizations to the presentation of the Boundary Pairing formula from the book of Melrose [54]. First, we must consider the radial compactification of a cone to a compactified manifold $\hat{C} = X \times [0, 1]$ for X the link of the cone, which is similar to that in [2]. We again consider solutions of the equation

$$(-\Delta_{\mathbb{R}^n} + V - \lambda^2)u = f$$

for $f \in \mathcal{S}$ for instance say a Schwartz class function.

Let

$$u = u_+ + u_-,$$

namely a sum of the outgoing and incoming solutions, with

$$u_\pm = e^{\pm i\lambda/x} x^{\frac{n-1}{2}} w_\pm \tag{B.1}$$

for $w_\pm \in C^\infty(\hat{C})$. This formula appears in a variety of settings in the literature, starting with the foundational work of Melrose [53] on asymptotically Euclidean manifolds, then Melrose-Zworski [52] on general scattering manifolds with smooth boundary, Hassell-Vasy [35] on scattering manifolds with conic points, and the corresponding discussion of

Guillarmou-Hassell-Sikora in [32], Section 5. See also [3, 67] for a recent discussion on product cones that contains formulae from which such a decomposition can be obtained.

Then, we claim that

$$2i\lambda \int_X [v_+^{(1)}\overline{v_+^{(2)}} - v_-^{(1)}\overline{v_-^{(2)}}]dv_h = \int_{C(X)} [f^{(1)}\overline{u^{(2)}} - u^{(1)}\overline{f^{(2)}}]drdv_h \quad (\text{B.2})$$

with $v_{\pm}^{(i)} = w_{\pm}^{(i)}|_X$. We will need to consider behaviors both at $x = 0$ and 1 whereas on Euclidean space the compactification really only sees ∞ . See Ch. 2.3 and Ch. 6 of Melrose [54]. Compactify via the stereographic projection to the quarter circle

$$S_{+,+}^1 = \{(z_1, z_2) \in S^1 \subset \mathbb{R}^2 | z_1 \geq 0, z_2 \geq 0\}$$

with $r \rightarrow (r, 1)/\sqrt{1+r^2}$. This is a manifold with boundaries of the form

$$z_1 = r/\sqrt{1+r^2}, \quad z_2 = 1/\sqrt{1+r^2}.$$

Define $M = S_{+,+}^1 \times X$ and we get the compactified structure.

The formula (B.2) then follows from integration by parts on the expression

$$\int_{C(X)} [f^{(1)}\overline{u^{(2)}} - u^{(1)}\overline{f^{(2)}}]\chi(\varepsilon r)drdv_h$$

for χ a smooth cut-off function localized near 0. Integrating by parts in r , applying formula (B.1), and taking the limit as $\varepsilon \rightarrow 0$ the formula follows after an application of the Riemann-Lebesgue lemma.

Note, the boundary pairing formula is strongly related to structure of the Jost solutions through the existence of polyhomogeneous expansions (power series solutions near the boundary).

B.3.2. Outline of the remaining arguments. To prove the absence of embedded resonances, we use the boundary pairing formula with $f = 0$ to prove that any outgoing solution vanishes to leading order on the $x_1 = 0$ boundary. This may be iterated to show that in fact the power series vanishes to arbitrary order and hence the solution is indeed Schwarz on the cone. To see this, differentiate the equation with respect to x_1 and look at the resulting inhomogeneous equation in the boundary pairing. Otherwise, the power series solution depends uniquely on the first terms in the expansion.

In most circumstances, eliminating embedded eigenvalues requires unique continuation. In the radial problem, we may apply the Theorem XIII.56 of [59] built around Jost solutions as mentioned above. For conic metrics, we must follow the procedure of Froese-Herbst [21] built around separable metric, which has been extended and formalized by Vasy [64, 65].

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