EIGENVALUE SPACING FOR 1D SINGULAR SCHRÖDINGER OPERATORS

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ABSTRACT. The aim of this paper is to provide uniform estimates for the eigenvalue spacings of one-dimensional semiclassical Schrödinger operators with singular potentials on the half-line. We introduce a new development of semiclassical measures related to families of Schrödinger operators that provides a means of establishing uniform non-concentration estimates within that class of operators. This dramatically simplifies analysis that would typically require detailed WKB analysis near the turning point and several gluing type results to connect various regions in the domain.

1. INTRODUCTION

We consider a (self-adjoint) one dimensional semiclassical Schrödinger operator

$$P_h u = -h^2 u'' + V(x) u$$

that is defined on the half-line $I = [0, +\infty)$. The potential V is defined by $x \mapsto x_+^\gamma W(x)$ for some $\gamma > 0$ and a smooth, positive W. We will be interested in the eigenvalue equation

$$P_h u_h = E_h u_h, \tag{1}$$

for an energy E_h in a certain regime that is a, possibly h-dependent, compact interval $K_h \subset \mathbb{R}$ that we call the energy window. If the spectrum of P_h is discrete in K_h we define, for E in $\operatorname{spec} P_h$,

$$d_h(E) \stackrel{\text{def}}{=} \inf\{|E - \tilde{E}|, \ \tilde{E} \in \operatorname{spec} P_h, \tilde{E} \neq E\},$$

and we aim at giving lower bounds on $d_h(E)$ as uniform as possible.

Studying Schrödinger operators is a standard problem in spectral theory and many results on eigenvalues and eigenfunctions can be extracted from the literature on Sturm-Liouville problems and semiclassical analysis (Titchmarsch [Tit46], Olver [Olv74], Hörmander [Hör03, Hör05, Hör80, Hör09], Maslov [MA72], Helffer-Robert [HR83], Dimassi-Sjöstrand [DS99], Zworski [Zwo12]).

In particular, Bohr-Sommerfeld rules for smooth potentials in the semiclassical literature imply that, for a sequence of eigenvalues $(E_h)_{h>0}$ that converges to a non-critical energy E_0 with a connected energy surface, then the spacing is of order h (see Section 10.5 in [OB78] or [dV05, Yaf11] for instance). In most cases, semiclassical techniques allow one to work in any dimension but, often, only for smooth potentials.

Singular potentials have also been studied (see among others [Ber82, Chr15]). Often, the "bottom-of-the-well" regime is considered, i.e. when E_h goes to 0 at a certain rate. The latter rate can be obtained by a scaling argument by deciding for which power α the change of variables $x \leftarrow h^\alpha x$ transforms the problem into a non-semiclassical second order differential equation. It can then be proved that the k-th eigenvalue of P_h behaves like $a_k h^{\frac{2\gamma}{\gamma+2}}$ from which we infer that the spacing in this regime is also of order $h^{\frac{2\gamma}{\gamma+2}}$ (see [FS09], and also [Sim83] for a much more complete study of the bottom of the well for quadratic potentials, or [BP19] for even more degenerate situations). We also advertise the recent paper [GW18] that lays the foundations for a systematic semiclassical study of a class of singular potentials.

The intermediate regime, which is neither the non-critical energies nor the bottom of the well is known in the semiclassical literature as *semi-excited states* and has been initiated by Sjöstrand [Sjö92].

Our main result is stated as follows and can be seen as an estimate unifying all the preceding regimes.

Theorem 1. *Assume that $\gamma > 0$ and W is smooth and positive on $[0, +\infty)$. Let $V = x^\gamma W$ and P_h the Dirichlet or Neumann realization of $-h^2 u'' + V$ on $[0, +\infty)$. If $\liminf_{x \to +\infty} V(x) > 0$, there exist $M, h_0, \mathbf{c} > 0$ such that*

(1) For all $h \leqslant h_0$, $\operatorname{spec} P_h \cap [0, M]$ is purely discrete,
(2) For any $h \leqslant h_0$ and any E in $\operatorname{spec} P_h \cap [0, M]$,

$$d_h(E) \geqslant \mathbf{c} h \cdot E^{\frac{\gamma-2}{2\gamma}}.$$

Such a theorem is actually equivalent to answering the following question: consider a sequence $(E_h)_{h \geqslant 0}$ going to some limit E_0 as h goes to 0 and study the behavior of the sequence $(d_h(E_h))_{h \geqslant 0}$. When E_0 is non-critical then our result recovers the usual order h separation. This is completely standard if γ is an integer, for, in that case, the potential is smooth and the full semiclassical machinery can be used. If γ is not an integer, the energy surface is not smooth anymore and it must be proved that the singularity is not strong enough to perturb the order h spacing of eigenvalues. This can perhaps be done by rather soft techniques such as some Dirichlet-Neumann bracketing argument. We have chosen a different, also well known, technique that relies in estimating how fast the semiclassical Cauchy datum of the fundamental solution at $x = 0$ winds around the origin. We will observe that this winding is related to non-concentration at the singular point. One motivation for studying this kind of potential comes from the adiabatic ansatz in a stadium-like billiard (see [HM12]). In the latter, the potentials that come up are of the form $x \mapsto x_+^\gamma W(x)$ on the half-line $[-B, +\infty)$ for $B > 0$ and the eigenvalue problem can be restated as a gluing problem that involves the fundamental solution on the half-line that we study here. We also point out that our assumptions imply that the energy surface is connected so

that no tunneling effect has to be taken into account (see [HS84, MR88] for the more delicate case involving such tunneling effects).

Organization of the paper. In Section 2, we will treat the bottom of the well regime. All the results of this section can be found in the literature but we will outline a proof so as to make this paper self-contained.

In Section 3, we will first give a general strategy of proof to obtain the eigenvalue spacing for $1D$ Schrödinger operators. Our assumptions will imply that the vector space of L^2 solutions to $(P_h - E)u = 0$ is one dimensional so that the eigenvalue spacing will follow from the study of $G_h(\cdot\,; E)$ which is a conveniently normalized solution to this equation. We will in particular observe that the winding argument that leads to h-spacing in the non-critical case can be reduced to a concentration estimate. We will also show that, using an energy-dependent scaling, the latter estimate in the intermediate regime can be obtained from estimates in the non-critical regime that are uniform with respect to the potential.

This will lead us to standard problems in semiclassical analysis with the twist that the potential is not fixed but lives in some set $\mathcal{V}$ of functions. In Sections 4.1 and 4.2 we tackle the problems of exponential decay and semiclassical measures from this point of view and we prove essentially that the usual statements remain true with constants that are uniform in $\mathcal{V}$ provided the latter set exhibits some compactness. These two sections address the way the function $G_h(\cdot\,; E)$ may concentrate in the classically not allowed region and near the turning point so that the singularity at 0 actually does not play any role. It then remains to address the classically allowed region and this will be done in Section 4.3 in which we will combine WKB expansions with a Volterra type approach. We will need only the first order approximation but we will have to treat the cases $\gamma < 1$ and $\gamma \geqslant 1$ separately. In the latter case, the first order correction is of magnitude h and we obtain directly a WKB-approximation for G down to $x = 0$. when $\gamma < 1$, we will have to perform a matching at $x = h$ and the first order correction will be of magnitude h^γ.

In the final section, we will patch all the different regimes to obtain the proof of Theorem 1.

2. Bottom of the well

We recall that we consider the following Schrödinger equation

$$-h^2 u'' + V(x)u = Eu$$

on the half-line $I = [0, +\infty)$ with either Dirichlet or Neumann boundary condition at 0. Before proceeding, we outline the conditions we will place on the potential V moving forward.

Assumptions 2.1. The following properties of V hold:
- The potential V is smooth on $(0, \infty)$ and continuous on I.
- $V(0) = 0$ and there exist $\gamma > 0$ and W smooth on $[0, \infty)$ such that $\forall x > 0$, $V(x) = x^\gamma W(x)$, $W(0) > 0$.
- There exists some $d > 0$ such that

$$\forall x \geqslant d, \ V(x) \geqslant V(d),$$

$$\forall x \in (0, d], \ V'(x) > 0.$$

The latter assumption implies that for any $E < V(d)$, the energy surface

$$\left\{ (x, \xi) \in I \times \mathbb{R}, \ \xi^2 + V(x) = E \right\}$$

is compact and connected. It follows that the spectrum of P_h that lies below $V(d)$ consists of eigenvalues of finite multiplicity ([BS12], Ch. 10.6 or [RS78b], Ch. XIII). Moreover, since the potential is of limit-point type near infinity any eigenvalue in the preceding regime is necessarily simple (see Titchmarsch [Tit46] or [RS78a, GZ06, Tes09]).

Proposition 2.1. *Under Assumptions 2.1, for any M, there exists $c > 0$ and $h_0 > 0$ such that*

$$\forall h \leqslant h_0, \ \forall E \in [0, Mh^{\frac{2\gamma}{\gamma+2}}] \cap \operatorname{spec} P_h, \ d_h(E) \geqslant ch^{\frac{2\gamma}{\gamma+2}}.$$

Proof. As this result is somewhat classical, we only outline the proof and refer the reader to [FS09, Hil18] for complete details.

We use a scaling argument: set $\alpha = \frac{2}{\gamma+2}$ and define $v_h(y) = u_h(h^\alpha y)$. The function v_h is a solution to

$$-v'' + (y^\gamma W(h^\alpha y) - e_h) v = 0,$$

where we have put $e_h = h^{-\frac{2\gamma}{\gamma+2}} E_h$.

One can then argue by min-max arguments that e_h is close to an eigenvalue of the operator

$$A(v) = -v'' + W(0)x^\gamma v,$$

with the same boundary condition. In order to estimate the error term, we can introduce the point $x_h = h^{\alpha - \varepsilon}$ for some $\varepsilon > 0$, and then use the exponential decay for $x > x_h$ (see Section 4.1 below).

The eigenvalues of A are spaced at order 1 and this gives the result. □

3. GENERAL STRATEGY AND SCALING

3.1. **Energy spacing and eigenfunction concentration.**

It is well-known that the spacing between eigenvalues of a semiclassical 1-D Schrödinger operator around non-critical energies with a connected energy surface is of order h. This fact is classically derived from the Bohr-Sommerfeld quantization rules (cf section 10.5 in [OB78] or [dV05, Yaf11]). We present here a strategy that, in the end, relies on a concentration estimate for eigenfunctions. Showing this estimate uniformly with respect to the potential will be the key to the spacing in the intermediate regime.

Consider the eigenvalue equation

$$(P_h - E)u_h = 0,$$

in which the potential satifies the same assumptions as before and E is in some compact set $K \subset (0, V(d))$. Since this equation is of limit point type near ∞, we know that

$$\dim \left\{ u \in C^\infty \cap L^2(0, +\infty), \ (P_h - E)u = 0 \right\} = 1,$$

so that there is a unique solution $G(\,\cdot\,;\, E)$ that satisfies

$$(P_h - E)G_h(\,\cdot\,;\, E) = 0$$
$$\int_0^{+\infty} |G_h(x\,;\, E)|^2 \, dx = 1,$$
$$\forall x \geqslant d, \ G_h(x\,;\, E) > 0.$$

It is also standard that the mapping $E \mapsto G_h(\,\cdot\,;\, E)$ is analytic from $(0, V(d))$ into $L^2((0, +\infty))$. If we denote by $\dot{G}_h(\,\cdot\,;E)$ the derivative of G with respect to E, then, by differentiating the eigenvalue equation, we obtain

$$(P_h - E)\dot{G}(\,\cdot\,;\, E) = G(\,\cdot\,;\, E). \tag{2}$$

We define

$$Z_h(E) = G_h(0\,;\, E) + ihG_h'(0, E),$$

which we can write, in polar coordinates, as

$$Z_h(E) = |Z_h(E)|e^{i\theta_h(E)}$$

in which $E \mapsto \theta_h(E)$ is analytic.

A straightforward computation yields

$$|Z_h(E)|^2 \, \dot{\theta}_h(E) = \operatorname{Im}(\overline{Z_h(E)}\dot{Z}_h(E))$$
$$= \mathcal{W}_0 \left[G_h, \dot{G}_h \right],$$

where $\mathcal{W}$ is the (semiclassical) Wronskian that is defined by

$$\mathcal{W}_x \left[f, g \right] = hf(x)g'(x) - hf'(x)g(x).$$

The semiclassical Wronskian of G_h and $\dot{G}_h$ can also be computed by multiplying equation (2) by G, integrating, and making two integration by parts (the contribution of $+\infty$ vanishes since the equation is of limit-point type there and both functions are L^2). We obtain

$$\int_0^{+\infty} G_h^2(x\,;\,E)\,dx \;=\; h^2\dot{G}_h'(0\,;\,E)G_h(0\,;\,E) - h^2\dot{G}_h(0\,;\,E)G_h'(0\,;\,E)$$

$$= h\mathcal{W}_0\left[G_h, \dot{G}_h\right].$$

Finally, we obtain

$$|Z_h(E)|^2\,\dot{\theta}_h(E) \;=\; \frac{1}{h}\int_0^{+\infty} G_h^2(x\,;\,E)\,dx.$$

That, we rewrite as

$$dE \;=\; h|Z_h(E)|^2 d\theta, \tag{3}$$

since G is normalized.

Since being an eigenvalue is equivalent to asking that $G_h(\,\cdot\,;\,E)$ satisfies Dirichlet or Neumann boundary condition at 0, it follows that, between two consecutive eigenvalues $\int d\theta \;=\; \pi$. We will thus get the spacing of order h provided that there exists some positive constant c such that

$$\forall E \in K,\; |Z_h(E)|^2 \;\geqslant\; c.$$

One way to obtain this inequality is by using WKB expansions and semiclassical measures. Indeed, the WKB expansion near 0 will yield that, for some small a

$$|Z_h(E)|^2 \;\asymp\; \int_0^a G_h^2(x\,;\,E)\,dx$$

and a semiclassical measure argument will yield that

$$\int_0^a G_h^2(x\,;\,E)\,dx \;\asymp\; \int_0^{+\infty} G_h^2(x\,;\,E).$$

Both these arguments are standard for a smooth potential for non-critical energies. In the next section we show that an energy-dependent scaling allows to get the estimate for the intermediate regime by following the same method of proof but for families of potentials. Showing that the estimates are uniform with respect to both the potential and the energy will finally yield Theorem 1.

3.2. **Energy scaling for the intermediate region.**

Choose a sequence $(E_h, u_h)_{h \geqslant 0}$ that is a solution to (1) under the standing assumptions on V. Recall that E_h is in the intermediate regime if neither E_h is non-critical, nor E_h is in the bottom of the well regime. Equivalently, this reads as

$$E_h \xrightarrow[h \to 0]{} 0, \text{ and } h^{-\frac{2\gamma}{\gamma+2}} E_h \xrightarrow[h \to 0]{} +\infty.$$

We perform a E-dependent scaling on the equation by setting $\tilde{v}_h(z) = \tilde{u}_h(E_h^{\frac{1}{\gamma}} z)$. We obtain

$$-h^2 \tilde{u}_h'' + (x^\gamma W(x) - \tilde{E})\tilde{u}_h = 0 \iff$$

$$-h^2 E_h^{-1-\frac{2}{\gamma}} \tilde{v}_h'' + (z^\gamma W(E_h^{\frac{1}{\gamma}} z) - \frac{\tilde{E}_h}{E_h})\tilde{v}_h = 0.$$

Since E_h is in the intermediate regime :

- $W(E_h^{\frac{1}{\gamma}} \cdot)$ converges to the constant function $W(0)$ (uniformly on every compact set),
- $\bar{h} \overset{\text{def}}{=} h E_h^{-\frac{2+\gamma}{2\gamma}}$ tends to 0.

We may thus take $\bar{h}$ as a new genuine semiclassical parameter. By construction, we are now working near the energy 1 which is non-critical. Assuming we have a spacing of order $\bar{h}$ uniformly for the sequence of potentials $z \mapsto z^\gamma W(E_h^{\frac{1}{\gamma}} z)$, we obtain that any eigenvalue $\tilde{E}_h \neq E_h$ must satisfy

$$|\frac{\tilde{E}_h}{E_h} - 1| \geqslant c\bar{h}.$$

Thus, we obtain the bound

$$|\tilde{E}_h - E_h| \geqslant ch \cdot E^{\frac{\gamma-2}{2\gamma}}.$$

Consequently, we see that Theorem 1 will follow from the usual semiclassical estimates at a non-critical energy provided the latter are proven to hold for singular potential and uniformly. This approach is interesting in its own and we will develop it after having made the setting precise.

3.3. **Global Assumptions.** We fix $\gamma > 0$, $0 < b < c < d$, $\mathcal{K}$ a compact set in $C^\infty([0,d]; \mathbb{R})$ equipped with its Fréchet topology and K a compact set in $(0, +\infty)$. We denote by $\mathcal{V}$ the set of potentials such that the following assumptions hold.

Assumptions 3.1.

- The conditions on V from Assumptions 2.1 hold.
- The restriction of W to $[0,d]$ belongs to $\mathcal{K}$.
- The following estimates hold

$$\forall (V, E) \in \mathcal{V} \times K, \ \forall x \in [0, b], \ E - V(x) > 0,$$
$$\forall (V, E) \in \mathcal{V} \times K, \ \forall x \in [c, d], \ V(x) - E > 0.$$

Let us observe that these assumptions imply that

$$\forall E \in K, \ \forall x \geqslant d, \ V(x) - E \geqslant V(d) - E > 0,$$

so that the operator $P_h - E$ is of limit-point type near ∞ which allows us to define $G_h(\cdot\,;\,E)$ for any $E \in K$ and $V \in \mathcal{V}$. Observe that the notation does not reflect the fact that the function G also depends on V.

We want to prove the following theorem.

Theorem 2. *Under the preceding assumptions, there exists* $\mathbf{c}$ *and* h_0 *such that for any* $h \leqslant h_0$*, for any* $V \in \mathcal{V}$ *and any* E_h *eigenvalue of* P_h:

$$E_h \in K \implies d_h(E_h) \geqslant \mathbf{c}h.$$

The results in Theorem 2 will follow from the following proposition.

Proposition 3.1. *There exist* $\mathbf{c}, h_0 > 0$ *such that*

$$\forall\,(V, E) \in \mathcal{V} \times K,\ \forall h \leqslant h_0,\ |Z_h| \geqslant \mathbf{c}.$$

The proof of this proposition is somewhat technical and is the main result of this section. Hence we postpone it until we have discussed how the proof of Theorem 2 follows.

Remark 3.1. We have renamed h the semiclassical parameter, although, in the scaling argument, we use this bound for the rescaled semiclassical parameter $\bar{h}$.

Proof of Theorem 2. The same computation as that yielding (3) gives us

$$h|Z_h|^2\dot{\theta}_h = 1.$$

We recall that, for any (V, E) in $\mathcal{V} \times K$,

$$Z_h \overset{\text{def}}{=} G_h(0\,;\,E) + ihG_h'(0\,;\,E) = |Z_h|e^{i\theta_h}.$$

The claim thus follows from Proposition 3.1.

$\square$

The proof of Proposition 3.1 will proceed by estimating $G_h(\cdot\,;\,E)$ in different regions of the half-line, uniformly with respect to the potential. To this end, we will need several uniform quantities that we now define.

Remark 3.2. Observe that the order h spacing at non-critical energies follow from Thm 2 by considering $\mathcal{V} = \{V\}$.

3.4. **Uniform Bounds.** For any $(V, E) \in \mathcal{V} \times K$, the assumptions imply:
- There is a unique solution x_E to the equation $V(x_E) = E$ (the turning point).
- $[0, b]$ is in the classically allowed region and $(E - V)$ is uniformly bounded below on it.

$$\exists \kappa_o > 0,\ \forall\,(V, E) \in \mathcal{V} \times K,\ \forall h \leqslant h_0,$$
$$\forall x \in [0, b],\ E - V(x) \geqslant \kappa_o. \tag{4}$$

(The o stands for oscillating since, in the classically allowed region, G_h exhibits highly oscillating behaviour).

- $[c, +\infty)$ is in the classically not allowed region, and $(V - E)$ is uniformly bounded below on it.

$$\exists \kappa_e > 0, \ \forall \ (V, E) \in \mathcal{V} \times K, \ \forall h \leqslant h_0,$$
$$\forall x \in [c, +\infty), \ V(x) - E \geqslant \kappa_e. \tag{5}$$

  (The e stands for exponential).
- The turning point x_E always belong to $[b, c]$. Since, on $[b, c]$, V' is uniformly bounded below, the turning point is non-degenerate. We also have the following estimate from below:

$$\forall a \leqslant b, \ \exists \delta_a > 0, \ \forall (V, E) \in \mathcal{V} \times K, \ \forall h \leqslant h_0,$$
$$\forall x \in [a, c], \ V'(x) \geqslant \delta_a.$$

  We will also use the shortcut $\delta \overset{\text{def}}{=} \delta_b$.
- Finally, for any ℓ, $W^{(\ell)}$ is, uniformly on $[0, d]$, bounded above by some C_ℓ.
- If γ is an integer, $W^{(\ell)}$ can be replaced by $V^{(\ell)}$ in the latter statement.

Remark 3.3. The point c should not be confused with the (different) constant **c** that appears in the estimates.

4. Uniform concentration estimates

In this section we aim at showing that the mass of $G_h(\cdot \,; E)$ in the classically allowed region is bounded below uniformly for $(V, E) \in \mathcal{V} \times K$.

4.1. **In the classically not-allowed region.** In this section, we prove that the function G is exponentially small in the region $x \geqslant c$ with constants that are uniform with respect to $V \in \mathcal{V}$ and $E \in K$. Such exponential estimates are well-known for a fixed pair (V, E). We present here a (classical) rudimentary proof that has the advantages of assuming very little on the potential and of making it very easy to track the constants.

Proposition 4.1. *Under the assumptions 3.1 and using the preceding notations, for any $(V, E) \in \mathcal{V} \times K$ and for any $\eta > 0$, we have*

$$\forall x, z > 0, \ z \geqslant x \geqslant x_E + \frac{\eta}{2}, \ G(z) \leqslant e^{-\frac{\sqrt{\delta \eta}}{2h}(z-x)} G(x), \tag{6}$$

in which we recall that

$$\delta \overset{\text{def}}{=} \inf\{V'(x), \ V \in \mathcal{V}, \ x \in [b, c]\} > 0.$$

Proof. First we observe that, for any $x \in [x_E + \frac{\eta}{2}, c]$, we have, uniformly for $(V, E) \in \mathcal{V} \times K$,

$$V(x) - E = V(x) - V(x_E) \geqslant \delta \cdot \frac{\eta}{2}.$$

Since V is increasing on $[c, d]$, the same estimate is true on $[c, d]$ and then on $[d, +\infty)$ since $V(x) \geqslant V(d)$ on this interval. Finally, we obtain:

$$\forall (V, E) \in \mathcal{V} \times K, \ \forall x \geqslant x_E + \frac{\eta}{2}, \ V(x) - E \geqslant \delta \cdot \frac{\eta}{2}.$$

From the equation

$$-h^2 G'' + (V - E)G = 0,$$

we thus infer

$$\forall x \geqslant x_E + \frac{\eta}{2}, \ (G^2)''(x) \geqslant 2G(x)G''(x) \geqslant \frac{\delta \eta}{h^2} G^2(x).$$

We set $\omega = \frac{\sqrt{\delta \eta}}{h}$ and, for any $x_E + \frac{\eta}{2} \leqslant x < y$, we denote by ϕ the solution to $\phi'' = \omega^2 \phi$ that takes the same values as G^2 at x and y. Since $G^2 - \phi$ vanishes at x and y and satisfies $(G^2 - \phi)'' \geqslant \omega^2 (G - \phi)$, a maximum principle argument shows that

$$\forall z \in [x, y], \ G^2(z) \leqslant \phi(z).$$

By making ϕ explicit, we find

$$\forall\, x, y, z, \ x_E + \frac{\eta}{2} \leqslant x < z < y,$$

$$G^2(z) \leqslant G^2(x) \frac{\sinh(\omega(y - z))}{\sinh(\omega(y - x))} + G^2(y) \frac{\sinh(\omega(z - x))}{\sinh(\omega(y - x))}. \tag{7}$$

In this inequality, we fix x and z and integrate with respect to y in $[z + 1, z + 2]$, we find

$$\forall\, x, z \ x_E + \frac{\eta}{2} \leqslant x < z,$$

$$G^2(z) \leqslant G^2(x) \int_{z+1}^{z+2} \frac{\sinh(\omega(y - z))}{\sinh(\omega(y - x))} \, dy + \int_{z+1}^{z+2} G^2(y) \, dy.$$

It follows that $G^2(z)$ goes to zero when z goes to ∞. So we may let y go to $+\infty$ in the estimate (7) and obtain

$$\forall\, x, z, \ x_E + \frac{\eta}{2} \leqslant x < z, \ G^2(z) \leqslant G^2(x) e^{\omega(x - z)}.$$

The claim follows by taking the square root, since, by choice, G is positive in the classically not-allowed region. $\qquad\square$

We use this proposition to prove uniform exponential estimates for the mass of G and for the semiclassical Cauchy data in the classically not-allowed region.

Lemma 4.2.

$$\forall\, \eta > 0, \ \exists\, \kappa, h_0 > 0, \ \forall\, (V, E) \in \mathcal{V} \times K, \ \forall\, h \leqslant h_0,$$

$$\int_{x_E + \eta}^{+\infty} |G_h(x\,;\, E)|^2 \, dx \leqslant e^{-\kappa/h}.$$

Proof. We start from the estimate

$$\forall\, x, z,\ x_E + \frac{\eta}{2} \leqslant x < z,\ G^2(z) \leqslant G^2(x)e^{\omega(x-z)},$$

in which we recall that $\omega = \frac{\sqrt{\delta\eta}}{h}$. For any $x \in [x_E + \frac{\eta}{2}, x_E + \eta]$ we integrate this equality over $z \in [x_E + \eta, +\infty)$, we find

$$\forall\, x \in [x_E + \frac{\eta}{2}, x_E + \eta],\ \int_{x_E+\eta}^{+\infty} G^2(z)\,dz \leqslant \frac{1}{\omega}e^{-\omega(x_E+\eta-x)}G^2(x).$$

We may now integrate this inequality over $x \in [x_E + \frac{\eta}{2}, x_E + \frac{3\eta}{4}]$. Using that G is L^2 normalized, we obtain

$$\frac{\eta}{4}\int_{x_E+\eta}^{+\infty} G^2(z)\,dz \leqslant \frac{1}{\omega}e^{-\frac{\omega\eta}{4}}.$$

We obtain finally

$$\int_{x_E+\eta}^{+\infty} G^2(z)\,dz \leqslant \frac{4}{\eta\omega}e^{-\frac{\omega\eta}{4}}.$$

The claim follows if we set $\kappa = \frac{\delta^{\frac{1}{2}}\cdot\eta^{\frac{3}{2}}}{4}$ and choose h_0 small enough so that the prefactor is bounded by 1. $\qquad\square$

We now proceed to give an estimate for the semiclassical Cauchy datum in the classically not-allowed region, using the proposition and the eigenvalue equation for G.

Proposition 4.3. *There exist $h_1 > 0$ and a constant κ_1 such that, for any $(V, E) \in \mathcal{V} \times K$ and any $h \leqslant h_1$, we have*

$$G(c) \leqslant e^{-\frac{\kappa_1}{h}},$$
$$|hG'(c)| \leqslant e^{-\frac{\kappa_1}{h}}.$$

Proof. First we observe that, due to compactness, there exists $\bar{\eta} > 0$ such that

$$\forall\, (V, E) \in \mathcal{V} \times K,\ x_E + \bar{\eta} < c.$$

Choosing $\eta = \frac{\bar{\eta}}{2}$ and h small enough, we may thus make sure that

$$\forall\, (V, E) \in \mathcal{V} \times K,\ \forall\, h \leqslant h_1,\ [c - h, c + h] \subset [c - \frac{\bar{\eta}}{2}, d] \subset [x_E + \eta, d].$$

Using Proposition 4.1, we thus obtain that

$$\forall\, x \in [x_E + \frac{\eta}{2}, x_E + \frac{3\eta}{4}],\ \forall\, z \in [c - h, c + h],\ G^2(z) \leqslant e^{-\delta^{\frac{1}{2}}\eta^{\frac{3}{2}}/4h}G^2(x),$$

since $(z - x) \geqslant \frac{\eta}{4}$ for this range of values of x and z. Integrating with respect to x and taking the square-root we find:

$$\forall\, z \in [c - h, c + h],\ |G(z)| \leqslant \sqrt{\frac{4}{\eta}}e^{-\delta^{\frac{1}{2}}\eta^{\frac{3}{2}}/8h}.$$

This gives the result if we take $\kappa < \delta^{\frac{1}{2}} \eta^{\frac{3}{2}}/8$ and h small enough so that

$$\sqrt{\frac{4}{\eta}} e^{-(\delta^{\frac{1}{2}} \eta^{\frac{3}{2}} - \kappa)/8h} \leqslant 1.$$

Setting

$$M = \sup\{V(x) - E, \ (V, E) \in \mathcal{V} \times K, \ x \in [b, d]\},$$

which is finite by compactness, and using the eigenvalue equation, we also have

$$\forall z \in [c - h, c + h], \ |h^2 G''(z)| \leqslant M \cdot \sqrt{\frac{4}{\eta}} e^{-\delta^{\frac{1}{2}} \eta^{\frac{3}{2}}/8h}.$$

Using Taylor-Lagrange expansions, there exist $\theta_- \in [c - h, c]$ and $\theta_+ \in [c, c+h]$ such that

$$G(c - h) = G(c) - h \cdot G'(c) + \frac{h^2}{2} G''(\theta_-),$$

$$G(c + h) = G(c) + h \cdot G'(c) + \frac{h^2}{2} G''(\theta_+).$$

By combining these two equations, we obtain

$$|hG'(c)| \leqslant \frac{|G(c - h)| + G(c + h)|}{2} + \frac{1}{4}(h^2|G''(\theta_+)| + h^2|G''(\theta_-)|).$$

It then follows from the preceding estimate that there exist some constant C such that

$$|hG'(C)| \leqslant C e^{-\delta^{\frac{1}{2}} \eta^{\frac{3}{2}}/8h}.$$

The claim follows by taking the same κ as above and a smaller h_1 if needed. $\qquad\square$

Remark 4.1. Arguing similarly, we could get an estimate replacing c by any $x \in [c, d]$.

A consequence of this estimate is that the Cauchy data of G at c is exponentially small uniformly for $(V, E) \in \mathcal{V} \times K$. More precisely, setting $Z_h(\cdot\,;\, E) = G_h(\cdot\,;\, E) + ihG'(\cdot\,;\, E)$, we have

$$\exists\, h_0, \ \kappa > 0, \ \forall\, h \leqslant h_0, \ \forall\, (V, E) \in \mathcal{V} \times K,$$

$$|Z_h(c\,;\, E)| \leqslant e^{-\frac{\kappa}{h}}. \tag{8}$$

The latter estimates allow control of G in the classically forbidden region. We will see that, in the classically allowed region, WKB expansions will also provide us with enough control. It thus remains to address the turning point. There are several ways to do so (using a Maslov or a Airy Ansatz for instance [MA72, Yaf11]). We have chosen a semiclassical measure approach since we think it is a nice generalization of the usual theory.

4.2. **Semiclassical measures for families of potential.** Let $(V_h, E_h)_{h \geqslant 0}$ be a family in $\mathcal{V} \times K$. For each smooth observable a that is compactly supported in $(0, d) \times \mathbb{R}$, we define

$$\mu_h(a) \overset{\text{def}}{=} \langle \mathrm{Op}_h(a)G_h, G_h \rangle,$$

where Op_h is some semiclassical quantization procedure (see [Zwo12] for instance). A standard argument shows that, up to extracting a subsequence, there exists a limiting measure μ_0. Using compactness, we may extract again and assume that V_h converges to V_0 and E_h converges to E_0.

We then have the following proposition that generalizes the known results when the potential is fixed.

Proposition 4.4. *Under the preceding assumptions, the support of the semiclassical measure μ_0 is a subset of the energy surface*

$$\left\{ \xi^2 + V_0(x) = E_0, (x, \xi) \in (0, d) \times \mathbb{R} \right\}.$$

The measure μ_0 is invariant by the hamiltonian flow of $p_0(x, \xi) \overset{\text{def}}{=} \xi^2 + V_0(x)$.

Proof. We follow the standard proofs. For the support property, we need to show that if a vanishes on a neighbourhood of the energy surface, then

$$\mu_h(a) \underset{h \to 0}{\longrightarrow} 0.$$

We denote by P_h^0 the operator

$$P_h^0 u = -h^2 u'' + V_0(x)u.$$

We write

$$\begin{aligned}
\mu_h(a) &= \langle \mathrm{Op}_h(a)G_h, G_h \rangle \\
&= \langle \mathrm{Op}_h(\frac{a}{p_0 - E_0})(P_h^0 - E_0)G_h, G_h \rangle + o(1) \\
&= \langle \mathrm{Op}_h(\frac{a}{p_0 - E_0})(V_0 - V_h + E_0 - E_h)G_h, G_h \rangle + o(1) \\
&\underset{h \to 0}{\longrightarrow} 0,
\end{aligned}$$

where we have used that a vanishes on the energy surface so that $\frac{a}{p_0 - E_0}$ is smooth with compact support, and in the latter stage the fact that $(V_0 - V_h + E_0 - E_h)$ converges to 0 on $[0, d]$ and G_h is exponentially small on $[d, +\infty)$.

For the invariance property, we write

$$\begin{aligned}
\frac{h}{i}\left[(\langle \mathrm{Op}_h(\{p_0, a\})G_h, G_h \rangle + o(1)\right] &= \langle [P_h^0, \mathrm{Op}_h(a)]G_h, G_h \rangle \\
&= \langle [P_h, \mathrm{Op}_h(a)]G_h, G_h \rangle \\
&\quad + \langle [V_h - V_0, \mathrm{Op}_h(a)]G_h, G_h \rangle \\
&= \frac{h}{i}\left[(\langle \mathrm{Op}_h(\{V_h - V_0, a\})G_h, G_h \rangle + o(1)\right]).
\end{aligned}$$

We now use the fact that the norm of a pseudodifferential operator on L^2 depends on the uniform norm of a finite number of derivatives of the symbol and that $\{V_h - V_0, a\}$ and all its derivatives converge uniformly to 0 on the support of a. $\qquad\square$

The semiclassical measure can be extended to symbols that are not compactly supported in ξ, in particular to symbols that only depend on x.

In dimension 1, μ_0 is thus determined up to a factor. More precisely, according to the assumptions, there exists ϕ_0 defined by

$$\xi^2 + V(\phi_0(\xi)) = E_0$$

and there exists c such that $\mu_0 = c\nu$ where ν is defined by

$$\nu(a) = \int a(\phi_0(\xi), \xi) \frac{d\xi}{V_0'(\phi_0(\xi))}.$$

For a smooth function χ whose support is a subset of $(0, x_{E_0})$, we have the alternative expression:

$$\nu(\chi) = \int \chi(x) \frac{dx}{\sqrt{E - V(x)}}.$$

Using the semiclassical measure, we obtain that the mass of G_h is uniformly bounded below in the classically allowed region $V(x) \leqslant E$.

Proposition 4.5. *There exists positive constants* $\mathbf{c}$ *and* h_0 *such that*

$$\forall\, h \leqslant h_0,\ \forall\, (V, E) \in \mathcal{V} \times K,\ \int_0^b |G_h(x\,;\,E)|^2\, dx\ \geqslant\ \mathbf{c}. \tag{9}$$

Proof. The proof is a typical application of using semiclassical measures to prove (non-)concentration estimates. By contradiction, we assume that the estimate (9) does not hold. We can thus find a sequence (V_h, E_h) with h going to 0 such that

$$\int_0^b |G_h(x\,;\,E)|^2\, dx \underset{h \to 0}{\longrightarrow} 0. \tag{10}$$

Using compactness, we may first extract subsequences and also assume that (V_h, E_h) tends to a limiting (V_0, E_0). We then extract a subsequence again to obtain a semiclassical measure μ_0. The preceding argument implies that there exists $\lambda \geqslant 0$ such that $\mu_0 = \lambda\nu$. Next, we observe that the assumption (10) implies that $\lambda = 0$. Indeed, for any non-negative function χ that has compact support in $(0, b)$ and that is bounded above by 1 we have

$$\langle \mathrm{Op}_h(\chi) G_h, G_h \rangle = \int_0^b \chi(x)\, |G_h(x\,;\,E)|^2\, dx \leqslant \int_0^b |G_h(x\,;\,E)|^2\, dx.$$

It follows that

$$\lambda \int_0^b \chi(x) \frac{dx}{\sqrt{E - V(x)}} = 0,$$

and hence $\lambda = 0$.

By choosing an appropriate symbol, this implies that for any closed interval $[x_0, x_1] \subset (0, +\infty)$, we have

$$\int_{x_0}^{x_1} |G_h(x\,;\,E)|^2 \, dx \xrightarrow[h \to 0]{} 0.$$

Setting $x_0 = b$, summing and using (10), we obtain that, for any $x_1 > b$

$$\int_0^{x_1} |G_h(x\,;\,E)|^2 \, dx \xrightarrow[h \to 0]{} 0.$$

Since G_h is normalized, this implies that the mass of G_h escapes to $+\infty$ but this is in contradiction with the estimates in the classically not allowed region. $\qquad\square$

4.3. **In the classically allowed region.** We now work on $[0, b]$. In this interval, we know that $E - V$ is uniformly bounded from below so that we can perform WKB approximation of solutions. For the estimate we are looking for only a first order WKB approximation is needed, but the lack of smoothness at $x = 0$ creates small additional complications. In particular, we will first make the assumption that $\gamma \geqslant 1$ and then explain how to modify the proof for $\gamma \in (0, 1)$.

Remark 4.2. We actually conjecture that the following full asymptotic expansion for Z_h holds:

$$Z_h = \sum_{\substack{m,n \geqslant 0, \\ m+n \geqslant 1}} a_{m,n} h^{m\gamma + n}.$$

The leading term in that expansion is thus h^γ if $\gamma \in (0, 1)$ and h if $\gamma \geqslant 0$. This also explains the two cases. Proving such a uniform expansion will be a topic of future work and is not required to the proof of the results contained here.

Let (V, E) be in $\mathcal{V} \times K$ and $G_h(\cdot\,;\,E)$ be defined as before. We define the functions S, a, $\phi_\pm$ on $[0, b]$ by

$$S(x) = \int_0^x \sqrt{E - V(y)} \, dy,$$

$$a(x) = (E - V(x))^{-\frac{1}{4}},$$

$$\phi_\pm(x) = a(x) e^{\pm \frac{i}{h} S(x)}.$$

A straightforward computation yields

$$-h^2 \phi_\pm'' + (V - E)\phi_\pm = h^2 \cdot r \phi_\pm,$$

where we have set $r \stackrel{\text{def}}{=} -\frac{a''}{a}$.

This computation implies that, on $[0, b]$, $\phi_\pm$ is a basis of solutions to the equation

$$-h^2 y'' + (V - E - h^2 r) y = 0.$$

Let u be a solution to

$$-h^2 u'' + (V - E) u = 0.$$

The classical method consists in saying that u is a solution to the former equation with an inhomogeneous term that reads $-h^2 r u$ and then in applying the variation of constants method. We find that there exists constants $\alpha_\pm$ such that, for all $x \in (0, b]$, we have

$$u(x) = \alpha_+ \phi_+(x) + \alpha_- \phi_-(x) - \frac{h}{2i} \int_x^b r(y) u(y) \left[\phi_-(y) \phi_+(x) - \phi_+(y) \phi_-(x) \right] \, dy.$$

We define the operator L_h by

$$L_h[u](x) = \frac{h}{2i} \int_x^b r(y) u(y) \left[\phi_-(y) \phi_+(x) - \phi_+(y) \phi_-(x) \right] \, dy.$$

so that the preceding equation rewrites

$$(\mathrm{id} + L_h)[u] = \alpha_+ \phi_+ + \alpha_- \phi_-.$$

The operator L_h is easily seen to be linear from $C^0([0, b]; \mathbb{C})$ into itself.

Using the compactness of $\mathcal{K}$ and K, there exist C_1 and C_2 such that, for all $(V, E) \in \mathcal{V} \times K$ and all $y \in [O, b]$:

$$|r(y)| \leqslant C_1 y^\rho,$$
$$|a(y)| \leqslant C_2,$$

where $\rho = \gamma - 2$ if $\gamma \in (0, 2) \setminus \{1\}$ and $\rho = 0$ if $\gamma = 1$ or $\gamma \geqslant 2$.

Remark 4.3. In the sequel we will denote by C a generic constant that is uniform for (V, E) in $\mathcal{V} \times K$. Observe that this constant may change from one line to the other.

We obtain that, for all $(V, E) \in \mathcal{V} \times K$,

$$\forall\, x \in (0, b), \ |L_h[u](x)| \leqslant C \cdot h \cdot \int_x^b y^\rho \, dy \cdot \|u\|_{C^0([0,b])}.$$

If $\gamma \geqslant 1$ then the integral on the right is convergent and we obtain that the operator norm of L_h is (uniformly w.r.t. (V, E)) bounded by $C \cdot h$.

Proposition 4.6. *Let $\gamma \geqslant 1$ then there exists a constant C that is uniform with respect to $(V, E) \in \mathcal{V} \times K$ and h_0 such that, for any $h \leqslant h_0$ there exists $\alpha_\pm(h)$ such that*

$$\|G_h - \alpha_+ \phi_+ - \alpha_- \phi_-\|_{C^0([0,b])} \leqslant Ch,$$
$$\|h G_h' - \alpha_+ h \phi_+' - \alpha_- h \phi_-'\|_{C^0([0,b])} \leqslant Ch.$$

Proof. According to the previous computation, there exists α_+ and α_- so that

$$(\mathrm{id} + L_h)\,[G] = \alpha_+\phi_+ + \alpha_-\phi_-,$$

and a uniform C such that

$$\|L_h\|_{\mathcal{L}(C^0([0,b]))} \leqslant C \cdot h.$$

We choose h_0 so that $C \cdot h_0 < 1$. It follows that $\mathrm{id} + L_h$ is invertible and

$$\left\|(\mathrm{id} + L_h)^{-1} - \mathrm{id}\right\|_{\mathcal{L}(C^0([0,b]))} \leqslant C \cdot h.$$

The first estimate on G_h follows. For the second one, we first observe that

$$G'(x) = \alpha_+\phi'_+(x) + \alpha_-\phi'_-(x) - \frac{h}{2i}\int_x^b r(y)G(y)\left[\phi_-(y)\phi'_+(x) - \phi_+(y)\phi'_-(x)\right]\,dy.$$

The integral is then uniformly bounded since G and $h\phi'_\pm$ are bounded in C^0 (recall that $\gamma \geqslant 1$) and r is integrable. $\qquad\square$

Corollary 4.7. *There exist uniform constant m_1, M_1, m_2, M_2 so that*

$$m_1(|\alpha_+|^2 + |\alpha_-|^2) \leqslant \left|G(0\,;\,E) + ihG'(0\,;\,E)\right|^2 \leqslant M_1^2(|\alpha_+|^2 + |\alpha_-|^2),$$

$$m_2(|\alpha_+|^2 + |\alpha_-|^2) \leqslant \int_0^b |G(x\,;\,E)|^2\,dx \leqslant M_2^2(|\alpha_+|^2 + |\alpha_-|^2).$$

Proof. We denote by $\alpha = {}^t(\alpha_+, \alpha_-)$ the (column)-vector in $\mathbb{C}^2$ and by $|\alpha|_{\mathbb{C}^2} \overset{\text{def}}{=} (|\alpha_+|^2 + |\alpha_-|^2)^{\frac{1}{2}}$ its norm. Starting from the expressions in Proposition 4.6, we first observe that

$$\|\alpha_+\phi_+ + \alpha_-\phi_-\|_{L^2([0,b])} = |\alpha|_{\mathbb{C}^2}\left(\int_0^b \frac{dx}{\sqrt{E - V(x)}} + O(h)\right)^{\frac{1}{2}},$$

where the O is uniform in $\mathcal{V} \times K$. Indeed, using an integration by parts, the fact that $\gamma \geqslant 1$ and compactness to obtain uniform estimate, we see that the cross-terms give a $O(h)$ contribution.

Using the triangle inequality then yields

$$\left(\int_0^b |G(x\,;\,E)|^2\,dx\right)^{\frac{1}{2}} = |\alpha|_{\mathbb{C}^2}\left(\int_0^b \frac{dx}{\sqrt{E - V(x)}} + O(h)\right)^{\frac{1}{2}} + O(h)$$

in which both O are uniform with respect to $(V, E) \in \mathcal{V} \times K$. Since

$$\int_0^b |G(x\,;\,E)|^2\,dx \geqslant c > 0,$$

and $\int_0^b \frac{dx}{\sqrt{E-V(x)}} \geqslant c' > 0$, both O term can be absorbed and we obtain the second line. The first line follows using the approximation on G and hG' and the fact that a uniform $O(h)$ term can be absorbed by $|\alpha_+|^2 + |\alpha_-|^2$. □

Combining the two estimates, and the fact that $\int_0^b |G(x\,;\,E)|^2\,dx$ is uniformly bounded away from 0, we obtain the proof of Proposition 3.1. It remains to address the case $\gamma \in (0,1)$.

4.4. **When** $\gamma \in (0,1)$. The problem when $\gamma \in (0,1)$ is that $y \mapsto y^{\gamma-2}$ is no longer integrable near 0, so we cannot work directly on $[0,b]$. It is standard in matching problems that we need to introduce an intermediate point x_h and use different approximations on $[0,x_h]$ and on $[x_h,b]$. It turns out that we can choose $x_h = h$.

We define the operator L_h as before. Its operator norm in $\mathcal{L}(C^0([h,b]))$ is bounded above (uniformly) by

$$C \cdot h \cdot \int_h^b y^{\gamma-2}\,dy,$$

so that there exists a uniform C such that

$$\|L_h\| \leqslant C \cdot h^\gamma.$$

The same proof as above yields the following proposition.

Proposition 4.8. *Let* $\gamma \in (0,1)$ *then there exists a constant* C *that is uniform with respect to* $(V,E) \in \mathcal{V} \times K$ *and* h_0 *such that, for any* $h \leqslant h_0$ *there exists* $\alpha_\pm$ *(that depend on* h*) such that*

$$\|G_h - \alpha_+\phi_+ - \alpha_-\phi_-\|_{C^0([h,b])} \leqslant Ch^\gamma,$$
$$\|hG'_h - \alpha_+h\phi'_+ - \alpha_-h\phi'_-\|_{C^0([h,b])} \leqslant Ch^\gamma.$$

On $[0,h]$, we follow the same strategy but we take as a basis of pseudosolutions the functions $\psi_\pm$ defined by

$$\psi_\pm(x) = E^{-\frac{1}{4}}e^{\pm\frac{i\sqrt{E}}{h}x}.$$

This is equivalent to treating the term Vu in the equation as some inhomogeneous term.

By following the same method, we obtain the proposition.

Proposition 4.9. *Let* $\gamma \in (0,1)$ *then there exists a constant* C *that is uniform with respect to* $(V,E) \in \mathcal{V} \times K$ *and* h_0 *such that, for any* $h \leqslant h_0$ *there exists* $\beta_\pm$ *(that depend on* h*) such that*

$$\|G_h - \beta_+\psi_+ - \beta_-\psi_-\|_{C^0([0,h])} \leqslant Ch^\gamma,$$
$$\|hG'_h - \beta_+\psi'_+ - \beta_-h\psi'_-\|_{C^0([h,b])} \leqslant Ch^\gamma.$$

Using the former proposition we obtain

$$\begin{cases} G(h) &=& \alpha_+(\phi_+(h) + O(h^\gamma)) + \alpha_-(\phi_-(h) + O(h^\gamma)) + O(h^\gamma), \\ hG'(h) &=& \alpha_+(h\phi'_+(h) + O(h^\gamma)) + \alpha_-(h\phi'_-(h) + O(h^\gamma)) + O(h^\gamma) \end{cases}$$

and using the latter proposition, we obtain

$$\begin{cases} G(h) &=& \alpha_+\psi_+(h) + \alpha_-\psi_-(h) + O(h^\gamma), \\ hG'(h) &=& \alpha_+h\psi'_+(h) + \alpha_-h\psi'_-(h) + O(h^\gamma). \end{cases}$$

We now observe that

$$S(h) = h(\sqrt{E} + O(h^\gamma)), \; a(h) = E^{-\frac{1}{4}} + O(h^\gamma), \; ha'(h) = O(h^\gamma)$$

so that $\phi_\pm(h) = \psi_\pm(h) + O(h^\gamma)$ and $h\phi'_\pm(h) = h\psi'_\pm(h) + O(h^\gamma)$. We compute

$$\begin{vmatrix} \psi_+(h) & \psi_-(h) \\ h\psi'_+(h) & h\psi_-(h) \end{vmatrix} = -2i.$$

Since this determinant is uniformly bounded away from 0 and the coefficients of the corresponding matrix are uniformly bounded above, we deduce that

$$\alpha_\pm = \beta_\pm + O(h^\gamma).$$

We now estimate the norms over $[0, h]$ and $[h, b]$:

$$\|G(x \,;\, E)\|_{L^2([0,h])} = |\beta|_{\mathbb{C}^2} \left[\frac{h}{\sqrt{E}}(1 + O(h^\gamma)) \right]^{\frac{1}{2}} + O(h^{\gamma+\frac{1}{2}}),$$

$$\|G_h(x \,;\, E)\|_{L^2([h,b])} = |\alpha|_{\mathbb{C}^2} \left[\int_h^b \frac{dx}{\sqrt{E - V(x)}} + O(h) \right]^{\frac{1}{2}} + O(h^\gamma).$$

Adding these two equalities, and using the fact that $\alpha_\pm = \beta_\pm + O(h^\gamma)$ and that $\int_0^b |G_h(x \,;\, E)|^2 dx$ is uniformly bounded away from 0, we obtain that

$$\int_0^b |G_h(x \,;\, E)|^2 dx = \left(|\alpha_+|^2 + |\alpha_-|^2\right) \int_0^b \frac{dx}{\sqrt{E - V(x)}} + O(h^\gamma)$$

$$= \left(|\beta_+|^2 + |\beta_-|^2\right) \int_0^b \frac{dx}{\sqrt{E - V(x)}} + O(h^\gamma).$$

Remarking that

$$|Z_h|^2 \asymp \left(|\beta_+|^2 + |\beta_-|^2\right)$$

completes the proof of Proposition 3.1.

5. Proof of Theorem 1

To prove part (1), let $M < \liminf_{x \to +\infty} V(x)$, then the part of the spectrum of P_h below M is discrete as follows from the fact that the set

$$\left\{ u \in H^1,\ \|u\|_{L^2} \leqslant 1,\ h^2 \int_0^{+\infty} |u'(x)|^2\, dx + \int_0^{+\infty} V(x)|u(x)|^2\, dx \leqslant M\|u\|_{L^2}^2 \right\}$$

is relatively compact in L^2.

In order to prove part (2), we argue by contradiction. If the estimate is not true then we can find two distincts eigenvalues E_h and $\tilde{E}_h$ such that

$$d_h(E) = o(h \cdot E^{\frac{\gamma-2}{2\gamma}}).$$

We may suppose that E_h has a limit E_0 and we have three cases to study.

- $E_0 = 0$ and there exists M such that $E_h \leqslant Mh^{\frac{2\gamma}{\gamma+2}}$. We obtain a contradiction using the estimate in the bottom of the well regime.
- $E_0 = 0$ and $h^{-\frac{2\gamma}{\gamma+2}}E_h \to +\infty$. We make the energy-dependent scaling and obtain a contradiction with the estimate in the intermediate regime.
- $E_0 > 0$. We obtain a contradiction with the non-critical energy regime.

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