ON THE SPECTRUM OF SCHRÖDINGER OPERATORS INTERACTING AT TWO DISTINCT SCALES

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Abstract. Schrödinger operators of the form $\Delta - W$ on $L^2_{\text{rad}}(\mathbb{R}^3)$, the space of radially symmetric square integrable functions are relevant in a variety of physical contexts. The potential W is taken to be radially symmetric (i.e. $W(x) = W(|x|)$) and to decompose into two components with distinct spatial scales: $W = W_\varepsilon = V_0 + V_{1,\varepsilon}$. The second component $V_{1,\varepsilon}(|x|) = \varepsilon^2 V_1(\varepsilon|x|)$ represents a scaled potential that becomes increasingly delocalized as $\varepsilon \to 0$. We will assume that both potentials $V_0(r), V_1(r)$ exhibit certain decay properties as $r \to \infty$. We show how the eigenvalue count on the positive real axis is built out of the spectra associated with the two reduced eigenvalue problems on their separate scales. The result is that the total number of eigenvalues of $\Delta - W$ is the sum of the number of positive eigenvalues of $\Delta - V_0$ and $\Delta - V_1$. Our analysis combines dynamical systems techniques with a separation of scales argument, providing a novel framework for studying spectral properties of differential operators where multiple spatial scales interact.

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1. Introduction

1.1. **Mathematical framework and setup.** The Schrödinger operator $\Delta - W$ on $L^2(\mathbb{R}^3)$ with certain conditions on the potential function W arises in the classical study of quantum mechanics and in a variety of other physical contexts. In particular, the behavior of a quantum particle confined in a well described by the function W is determined by the spectrum of the Schrödinger operator $\Delta - W$, denoted $\sigma(-\Delta + W)$, where the proper notions of discrete (or point) spectrum and essential spectrum are as described for instance in Reed-Simon [RS78b], Chapter XIII and Kato [Kat95]. As the problem is already sufficiently interesting, we work here in the setting of radially symmetric potentials. I.e., we will restrict ourselves to the weighted space $L^2_{\text{rad}}(\mathbb{R}^3)$ consisting of radial functions $f(x) = f(|x|)$ with the norm

$$\|f\|^2_{L^2_{\text{rad}}} = 4\pi \int_0^\infty |f(r)|^2 r^2 dr. \tag{1.1}$$

Following Simon [Sim82] and Agmon [Agm82], we know the operator $\Delta - W$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$, with its closure defining a self-adjoint operator with domain

$$\mathcal{D} = \{f \in L^2_{\text{rad}}(\mathbb{R}^3) : \Delta f - Wf \in L^2_{\text{rad}}(\mathbb{R}^3)\}, \tag{1.2}$$

where Δf is understood in the distributional sense. Given the orientation of our Schrödinger operator $\Delta - W$, we see that the essential spectrum, denoted

$\sigma_{\text{ess}}(\Delta - W)$, will consist of the negative real numbers, while the discrete spectrum, denoted $\sigma_p(\Delta - W)$, will consist of positive, isolated eigenvalues with associated eigenfunctions in $L^2_{\text{rad}}(\mathbb{R}^3)$. Our focus here will be on quantifying the number of positive, eigenvalues for Schrödinger operators of a particular form.

We specifically consider the interesting (and challenging) case $\Delta - W$ on $L^2_{\text{rad}}(\mathbb{R}^d)$, the space of radial functions, where $W = W_\varepsilon = V_0(|x|) + V_{1,\varepsilon}(|x|)$ is a sum of two radially symmetric potentials, sufficiently smooth and decaying which interact on different scales. In particular, we take

$$V_{1,\varepsilon}(|x|) = \varepsilon^2 V_1(\varepsilon|x|),$$

to be a potential acting in the far field (adiabatically in space) with respect the potential V_0. For the radial problem, the operator takes the explicit form:

$$\Delta - W_\varepsilon = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - V_0 - V_{1,\varepsilon} \tag{1.3}$$

with eigenvalue equation

$$u_{rr} + \frac{2}{r}u_r - (V_0 + V_{1,\varepsilon})\, u = \lambda u. \tag{1.4}$$

We will assume the following decay condition on the potentials.

(A1) *Both V_0 and V_1 are C^∞ on $r \geq 0$ and satisfy the decay condition that there exists a $C_1 > 0$ such that*

$$|\frac{d}{dr}V_i(r)| \leq \left(\frac{C_1}{r^{3+\gamma}}\right) \quad \textit{for some } \gamma > 0, i = 0, 1$$

for all $r \geq 1$.

To facilitate our error estimates being sufficiently strong in connecting our two scales, we will further assume more rapid decay than guaranteed by (A1) on the near field potential V_0.

(A2) *There exists a $C_0 > 0$ such that*

$$|V_0(r)| \leq \left(\frac{C_0}{r^{4+\gamma}}\right) \quad \textit{for some } \gamma > 0$$

for all $r \geq 1$.

The assumption on the potentials (A1) will be used to guarantee the validity (well-posedness) of the representation of the eigenvalue problem (1.4) as a dynamical system on a compact phase space. Although, for this representation, decay like $1/r^{2+\gamma}$ would suffice, we also perform a blow-up procedure inside the compactified phase space and this requires an extra degree of decay. The second assumption (A2) ensures that the interaction between the two potentials is not too strong as $\varepsilon \to 0$.

We consider each of the operators $\Delta - V_0$, $\Delta - V_1$, and $\Delta - W_\varepsilon$ on the space:

$$\mathcal{D} = \{u \in H^2(\mathbb{R}^3) \cap L^2_{\text{rad}}(\mathbb{R}^3) : \Delta u \in L^2(\mathbb{R}^3)\}. \tag{1.5}$$

The Lieb-Thirring inequalities [LT01], and especially the Cwikel-Lieb-Rozenbljum (CLR) bounds, see for instance [Cwi77, Lie76, Roz76], guarantee finiteness of the number of positive eigenvalues. For a potential $V = V(r)$, let us define

$$m(V) := \#\text{ of positive eigenvalues of the Schrödinger operator } \Delta - V. \tag{1.6}$$

The Cwikel-Lieb-Rozenbljum bound gives that there exists a constant C_3 independent of V such that

$$m(V) \leq C_3 \int_{\mathbb{R}^3} V_-^{\frac{3}{2}} dx, \tag{1.7}$$

where $V = V_+ - V_-$ with $V_+ \geq 0$ and $V_- > 0$ so V_- is the component of the potential responsible for any trapping effects. For these results, in addition to $V_- \in L^{\frac{3}{2}}$, one typically assumes that $V_+ \in L^1_{\text{loc}}$ to ensure that spectrum is well-defined, see [HKRV23]. It is clear that the $\frac{3}{2}$ power on the right hand side of (1.7) then means that

$$\int_{\mathbb{R}^3} V_{1,\varepsilon,-}^{\frac{3}{2}} dx = \int_{\mathbb{R}^3} V_{1,-}^{\frac{3}{2}} dx. \tag{1.8}$$

The exact value of the constant in these bounds is in and of itself a topic of great interest, see the recent result [HKRV23].

It follows from assumption (A1) that both V_0 and V_1 are in $L^{\frac{3}{2}}$ and L^1_{loc}. We can thus conclude that both spectral problems have only a finite number of positive eigenvalues. The scale separation in W_ε introduces interesting spectral phenomena for self-adjoint operators, building on classical spectral theory for Schrödinger operators, see [RS78b, Sim82, Kat95]. For eigenfunctions corresponding to discrete (point) spectrum, the eigenfunctions are "localized", in that they have exponential decay, however the rate will depend upon the distance to the essential spectrum.

Schrödinger operators also arise in the study of linear stability theory in a variety of contexts. In particular, Schrödinger operators of this type appear in reaction-diffusion systems in $\mathbb{R}^3$ with radially symmetric steady states. Consider the equation:

$$\frac{\partial u}{\partial t} = D\Delta u + f(u), \tag{1.9}$$

where $D > 0$ is the diffusion coefficient and $f(u)$ represents reaction kinetics. When seeking steady states and linearizing around them, one obtains eigenvalue problems involving $\Delta - W$ where W emerges from the linearized reaction terms.

A particular motivation that brought us to this work is a scalar problem related to the study of nonlinear stability of solitons inside potential wells as considered in the papers of Nakanishi, [Nak17a, Nak17b]. When considering the nonlinear Schrödinger (NLS), which is the equation (1.9) with an i in front of the time derivative term on the left hand side, interacting with potentials at different scales, the operators are non-self-adjoint and consist of a more complicated matrix system of operators. Nevertheless, scalar spectral considerations turn out to be important to establish first, see for instance [KS06, NS11]. The study presented here, while of independent interest and focusing on scalar Schrödinger operators, grew out of the Ph.D. thesis [Gol21], which was directly motivated by the works of Nakanishi on NLS dynamics.

1.2. **Theorems.** It follows from the finiteness of the number of positive eigenvalues for the spectral problems (for each potential separately) that there is a spectral gap in each case of the form $[0, \hat{\mu}]$. It also follows from our assumptions that the essential spectrum is exactly $(-\infty, 0]$ for all three operators: $\Delta - V_0$, $\Delta - V_1$, and $\Delta - W_\varepsilon$ (any $\varepsilon > 0$).

We adopt the following notation:

$$\mu_i = \min_{\mu>0}\{\mu \in \sigma\left(\Delta - V_i\right)\}. \tag{1.10}$$

This is well-defined by the above considerations. The spectral gaps of interest are then $(0, \mu_0)$ and $(0, \mu_1)$ for $\Delta - V_0$ and $\Delta - V_1$ respectively.

We need one further assumption on the spectra of the two individual operators.

(A3) *The point $\mu = 0$ is not an end-point resonance of either $\Delta - V_0$ or $\Delta - V_1$*

Another way of stating (A3) would be to say that the kernels of these two operators are trivial on the space $\mathcal{D}$ given in (1.5).

Theorem 1. *Under the assumptions (A1-A3), there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the operator $\Delta - W_\varepsilon$ on the domain $\mathcal{D}$, given in* (1.5)*, has exactly $m(V_1)$ eigenvalues in the spectral gap $(0, \mu_0)$.*

Moreover, for all $\varepsilon \in (0, \varepsilon_0)$, the total number of eigenvalues in $\sigma(\Delta - W_\varepsilon)$ contained in $(0, \infty)$ is given by

$$m(W) = m(V_0) + m(V_1).$$

Theorem 1 shows that this structure, with its scale separation through $V_{1,\varepsilon}$, leads to rich spectral properties, particularly regarding eigenvalues in the gap between the essential spectrum $(-\infty, 0]$ and the discrete spectrum associated with $\Delta - V_0$. To the best of our knowledge, the precise description of the spectrum generated by two potentials separated by scales in this manner has not been described before in the literature.

Remark We actually show that the eigenvalues in the spectral gap $(0, \mu_0)$ inherited from the far field potential are all of $\mathcal{O}(\varepsilon^2)$. The eigenvalues inherited from the near field potential only perturb a small amount when ε is turned on.

Many other results exist to give integrability conditions on W (and hence V_0, V_1) such that no discrete eigenvalues could exist, see [Bar52, Ber82] and Theorem XIII.9 of [RS78b] for an overview. Interestingly, our results in Theorem 1 then prove that related conditions on V_1 make this potential still globally perturbative to V_0 in a natural way.

It is of note that the scalings we consider here that are sharp with respect to the (CLR) bound make the problem critical. In particular, for a scaling of the form $\varepsilon^r V_1(\varepsilon r)$, if $r > 2$ then the problem would be globally perturbative to V_0, while if $r < 2$ it would be globally perturbative to V_1.

1.3. **Comparative analysis of dynamics on two distinct scales - Outline of the proof.** At its heart, the proof uses a version of Sturm-Liouville Theory expressed through an angular flow. We transform the system so as to realize the eigenvalue problem as finding a heteroclinic connection in an appropriate phase space. We then analyze two distinct model dynamical systems; one related to V_0 and the other to V_1. The key steps are as follows:

(1) **Domain compactification:** To study the behavior near spatial infinity and near the origin within a unified framework, we compactify the radial domain $r \in [0, \infty)$ by introducing a new dependent variable realted to r, and then transform it in its role as the independent variable. We perform this operation in two distinct ways adapted to each scale. To focus on the near field, we use $\sigma = \frac{r}{1+r} \in [0, 1)$, while for the outer scaling regime we

work with $\rho = \varepsilon r$, and in its compactified form $\tau = \frac{\rho}{1+\rho} \in [0, 1)$. These allow us to separate and analyze the distinct dynamics induced by the leading-order potentials V_0 and V_1 on their respective scales.

(2) **Matched regime approximation:** We identify an intermediate region, mapped to a small neighborhood in the compactified variables, outside of which solutions can be compared to model problems.
   - Lemma 5 below shows that solutions closely follow the dynamics governed by V_0 up to this intermediate region, with an error of size $\mathcal{O}(\varepsilon)$.
   - Lemma 8 below shows that solutions re-enter a regime governed by V_1, with error $\mathcal{O}(\varepsilon^2)$, provided $\mu \notin \sigma_p(\Delta - V_1)$.

   These two Lemmas provide a justification for treating the domain as approximately partitioned into regions where either V_0 or V_1 dominates the dynamics.

(3) **Spectral gluing:** In Section 4, we use topological and spectral arguments to combine the eigenvalue counts obtained in the V_0 and V_1-dominated regimes. This yields the full spectral information and completes the proof of Theorem 1.

1.4. **Outline of the paper.** The paper is organized as follows. Section 2 formulates the eigenvalue count through the two key scalings. We introduce the model systems and define a threshold for the spatial variable at which we match the model problems. Section 3 develops the necessary machinery for tracking the relevant trajectories, determined by the asymptotic invariant manifolds, in the compactified phase space. Section 4 completes the proof of Theorem 1 through a gluing argument at the threshold. Section 5 provides numerical examples validating our theoretical results across different classes of potentials satisfying our hypotheses. Finally, Section 6 discusses our findings and future directions.

2. A geometric framework for the spectral problem

The eigenvalue problem on radial L^2 functions is a boundary value problem with a condition of regularity at $r = 0$ and decay as $r \to \infty$. The eigenvalue equation is thus $Lp = \lambda p$ with

$$Lp = p_{rr} + \frac{2}{r} p_r - (V_0 + V_{1,\varepsilon})\, p = \lambda p, \tag{2.1}$$

where $V_0 = V_0(r)$ and $V_{1,\varepsilon}(r) = \varepsilon^2 V_1(\varepsilon r)$ The full eigenvalue problem for λ being in the point spectrum of L thus becomes:

$$p_{rr} + \frac{2}{r} p_r - \lambda p - (V_0 + V_{1,\varepsilon})\, p = 0 \tag{2.2}$$

together with boundary conditions

$$\lim_{r\to 0} p_r(r) = 0, \quad \text{and} \quad \lim_{r\to\infty} p(r) = 0. \tag{2.3}$$

We will refer to this as the *inner system* as the potential in the far field is small, and it has a well-defined limit when $\varepsilon \to 0$ where only the inner potential V_0 remains.

An analogous system will be derived that focuses on the outer potential, called the *outer system*. This system will not have a limit as $\varepsilon \to 0$, and so we will always refer back to (2.2) and (2.3) for solving the eigenvalue problem.

To formulate the outer system, we scale the independent variable $r = \frac{\rho}{\varepsilon}$, then

$$Lp = \varepsilon^2 p_{\rho\rho} + \varepsilon^2 \frac{2}{\rho} p_\rho - \left(V_0\left(\frac{\rho}{\varepsilon}\right) + \varepsilon^2 V_1(\rho) \right) p = \lambda p. \tag{2.4}$$

Using $V_{0,\varepsilon^{-1}}(\rho) = \frac{1}{\varepsilon^2} V\left(\frac{\rho}{\varepsilon}\right)$ and $\lambda = \varepsilon^2 \mu$, we can rewrite (2.4) as

$$\tilde{L}p := p_{\rho\rho} + \frac{2}{\rho} p - \left(V_{0,\varepsilon^{-1}}(\rho) + V_1(\rho) \right) p = \mu p, \tag{2.5}$$

where we have introduced the modified Schrödinger operator $\tilde{L}$. The boundary conditions are, analogously,

$$\lim_{\rho \to 0} p_\rho(\rho) = 0 \text{ and } \lim_{\rho \to \infty} p(\rho) = 0. \tag{2.6}$$

If $\varepsilon > 0$ then(2.5)-(2.6) is equivalent to (2.2)-(2.3).

2.1. **Eigenvalue problem as a dynamical system.** Returning to the inner formulation of the eigenvalue problem (2.2), and setting $p_r = v$, we can write (2.2) as a first-order system

$$\begin{aligned} p' =& v, \\ v' =& -\frac{2}{r} v + \lambda p + (V_0 + V_{1,\varepsilon}) p, \end{aligned} \tag{2.7}$$

where $' = \frac{d}{dr}$.

The system (2.7) is a non-autonomous system on $\mathbb{R}^2$ with independent variable $r \in (0, \infty)$. A standard trick in dynamical systems is to make it autonomous by introducing r as a new dependent variable. We will perform a variant of this strategy that has the effect of also compactifying the phase space. To that end, we set $\sigma = \frac{r}{r+1}$, and note that $\sigma' = (1-\sigma)^2$. The phase space of the resulting equation is $\mathbb{R}^2 \times (0,1)$, with the independent variable still being r. The system is singular at $\sigma = 0$ since $\frac{2}{r} = 2\left(\frac{1-\sigma}{\sigma}\right)$. To desingularize the system, we introduce a new independent variable $s = r + \ln r$. In these variables, the augmented version of(2.7) becomes

$$\begin{aligned} \dot{p} =& \sigma v, \\ \dot{v} =& 2(\sigma - 1)v + \sigma\left(\lambda p + (V_0 + V_{1,\varepsilon}) p\right), \\ \dot{\sigma} =& \sigma(1-\sigma)^2, \end{aligned} \tag{2.8}$$

where $\dot{} = \frac{d}{ds}$. The eigenvalue condition now becomes one of finding a trajectory $(p(s), v(s), \sigma(s))$ of (2.8) for which:

$$\lim_{s \to -\infty} v(s) = 0 \tag{2.9}$$

and

$$\lim_{s \to +\infty} p(s) = 0, \tag{2.10}$$

where we also require that $\sigma(s) \neq 0$ or 1 for any finite s.

This last condition requires some explanation which can also serve to illuminate this dynamical systems viewpoint. The phase space for (2.8) is $\mathbb{R}^2 \times [0,1]$. With suitable extensions of the potentials to $\sigma = 1$ (see (2.13) and (2.14) below for formal definitions), the planes $\sigma = 0$ and $\sigma = 1$ are both invariant, with the former corresponding to $r = 0$ and the latter the limit $r \to \infty$. The great benefit of this approach is that the asymptotic behavior, both at 0 and ∞, is now represented

within the phase space. The trajectory we want to get that corresponds to the eigenfunction lives in between these invariant planes. The condition that $\sigma \neq 0$ guarantees that we would not get a trajectory in the limit planes.

To recover the eigenfunction, the parameterization needs to be set correctly. This is achieved by choosing $\sigma = \frac{1}{2}$ when $s = 1$. This compactification is an example of a general construction that is explored in some detail in [WXJ21]. It follows from [WXJ21] that (2.8) is a C^1 system on its phase space because of the assumed decay of the potentials. In particular, the assumptions (A1)-(A3) ensure that the vector field extends smoothly to the compactified boundaries and that the number of positive eigenvalues is finite.

The analogous compactification and desingularization for the outer system involves setting $\tau = \frac{\rho}{\rho+1}$ and $t = \rho + \ln \rho$. It reads very similarly to (2.8)

$$\begin{aligned} \dot{p} =& \tau w, \\ \dot{w} =& 2(\tau - 1)w + \tau \left(\mu p + (V_{0,\varepsilon} + V_1)\, p\right), \\ \dot{\tau} =& \tau(1-\tau)^2, \end{aligned} \tag{2.11}$$

where now $\dot{} = \frac{d}{dt}$. The boundary conditions (2.9) and (2.10) are the same with s replaced by t and v by w. The variable p is the same in (2.8) and (2.11) and $v = \varepsilon w$.

2.2. **The angular system.** Because the original eigenvalue equation is linear, system (2.7) is linear, as are the first two equations of (2.8) in (p, v). We can then express (p, v) in polar coordinates and the angular equations will decouple from the radial part.

Setting the angular variable to be $\theta = \arctan(\frac{v}{p})$, we obtain the following system

$$\begin{aligned} \dot{\theta} =& (\sigma - 1)\sin 2\theta + \sigma \left(\left(\lambda + \tilde{V}_0(\sigma) + \tilde{V}_{1,\varepsilon}(\sigma)\right)\cos^2\theta - \sin^2\theta\right), \\ \dot{\sigma} =& \sigma(1-\sigma)^2, \end{aligned} \tag{2.12}$$

where $\dot{} = \frac{d}{ds}$. The functions $\tilde{V}_0$ and $\tilde{V}_{1,\varepsilon}$ are given by

$$\tilde{V}_0 = \begin{cases} V_0\left(\frac{\sigma}{1-\sigma}\right) & \text{if} \quad \sigma \neq 1, \\ 0 & \text{if} \quad \sigma = 1, \end{cases} \tag{2.13}$$

and

$$\tilde{V}_{1,\varepsilon} = \begin{cases} V_{1,\varepsilon}\left(\frac{\varepsilon\sigma}{1-\sigma}\right) & \text{if} \quad \sigma \neq 1, \\ 0 & \text{if} \quad \sigma = 1. \end{cases} \tag{2.14}$$

We can take θ to lie in either $\mathbb{R}$ or S^1. The latter is achieved by taking $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, mod π, which corresponds to taking a wrapped branch of arctan. The fact that system (2.1) is C^1 on either phase space follows from assumption (A1).

Another way to think of the angular system is for the angle of a subspace as it evolves under the linear dynamics of the first two equations of (2.8). Since scalar multiples of solutions are also solutions, subspaces are carried to subspaces under the dynamics, which is why the angular equation decouples from the radial part.

On $\sigma = 0$ the system reduces to $\dot{\theta} = -\sin 2\theta$. On $\sigma = 1$ the equation is $\dot{\theta} = \lambda\cos^2\theta - \sin^2\theta$. The boundary conditions for the eigenvalue problem will be realized as asymptotic conditions to the relevant fixed points in these reduced equations.

Taking $\psi = \arctan(\frac{w}{p})$, there is an analogous angular system for the outer equation:

$$\begin{aligned} \dot{\psi} =& (\tau - 1)\sin 2\psi + \tau\left(\left(\mu + \tilde{V}_{0,\varepsilon^{-1}}(\tau) + \tilde{V}_1(\tau)\right)\cos^2\psi - \sin^2\psi\right), \\ \dot{\tau} =& \tau(1-\tau)^2, \end{aligned} \tag{2.15}$$

where $\dot{} = \frac{d}{dt}$ and $\tilde{V}_{0,\varepsilon^{-1}}$ and $\tilde{V}_1$ are extended in a similar fashion to (2.13) and (2.14). Note that

$$\theta = \arctan\left(\frac{v}{p}\right) = \arctan\left(\frac{\varepsilon w}{p}\right), \tag{2.16}$$

and so

$$\tan\theta = \varepsilon \tan\psi. \tag{2.17}$$

2.3. **Boundary conditions and invariant manifolds.** For (2.12) on $S^1 \times [0,1]$, there are two equilibria in each of $\sigma = 0$ and $\sigma = 1$. In $\sigma = 0$, we have

$$\hat{\theta}_{0,-} = (0,0), \quad \hat{\theta}_{0,+} = \left(0, \frac{\pi}{2}\right), \tag{2.18}$$

and in $\sigma = 1$:

$$\hat{\theta}_{1,\pm}(\lambda) = \left(1, \arctan(\mp\sqrt{\lambda})\right). \tag{2.19}$$

The regularity boundary condition at $r = 0$ will be satisfied by tending to $\hat{\theta}_{0,-}$, as that corresponds to $v = p_r = 0$. For the boundary condition as $r \to +\infty$, it helps to see that the equation governing the dynamics in $\sigma = 1$ is exactly the angular equation for the eigenvalue problem in one space dimension. The angle $\arctan\left(-\sqrt{\lambda}\right)$ is that of the subspace of decaying solutions, and the boundary condition is to find a trajectory tending to this value.

Within each of their respective invariant planes, i.e., $\sigma = 0$ or 1, the fixed points with a $+$ superscript are unstable (positive eigenvalue), while those with a $-$ superscript are stable (negative eigenvalue). Note the reversal of signs from the argument of the arctan. Also, note the irony that the angle of the space of decaying solutions is actually unstable inside $\sigma = 1$ as solutions that do not decay must grow and hence their angles are pushed away.

Since $\sigma = 0$ is repelling for (2.12) and $\sigma = 1$ is attracting, albeit not exponentially, the fixed points $\hat{\theta}_{0,-}$ and $\hat{\theta}_{1,+}$ are both saddle-like, each with one attracting direction and one unstable direction. Although for $\hat{\theta}_{1,+}$ the attracting direction is a center direction (zero eigenvalue).

We will, on occasion, include or suppress the dependence of θ on one or both of λ and ε. From these considerations, the following proposition easily follows.

Proposition 2. *A value $\lambda \in \mathbb{R}$ is an eigenvalue of L for the space of radial L^2 functions iff* (2.12) *has a heteroclinic orbit $(\theta(\lambda, s), \sigma(s))$ connecting from $\hat{\theta}_{0,-}$ to $\hat{\theta}_{1,+}$, i.e.,*

$$(\theta(\lambda,s),\sigma(s)) \to \begin{cases} \hat{\theta}_{0,-} & \text{as} \quad s \to -\infty, \\ \hat{\theta}_{1,+}(\lambda) & \text{as} \quad s \to +\infty. \end{cases}$$

This eigenvalue condition can now be expressed in terms of invariant manifolds associated with these fixed points.

The boundary conditions (2.9) and (2.10) are thus satisfied if and only if

$$(\theta(\lambda, s), \sigma(s)) \in W^u\left(\hat{\theta}_{0,-}\right), \tag{2.20}$$

and

$$(\theta(\lambda, s), \sigma(s)) \in W^c\left(\hat{\theta}_{1,+}(\lambda)\right), \tag{2.21}$$

respectively. Note that both $W^u\left(\hat{\theta}_{0,-}\right)$ and $W^c\left(\hat{\theta}_{1,+}\right)$ are $1-$dimensional and hence each consists of exactly one trajectory.

While, in general, center manifolds are not uniquely defined, $W^c\left(\hat{\theta}_{1,+}\right)$ is well-defined and unique as the complementary direction is (exponentially) unstable.

We will denote the trajectory in $W^u\left(\hat{\theta}_{0,-}\right)$ by

$$\hat{\theta}_-(\lambda, s) = (\theta_-(\lambda, s), \sigma(s)), \tag{2.22}$$

and the one in $W^c\left(\hat{\theta}_{1,+}\right)$ by

$$\hat{\theta}_+(\lambda, s) = (\theta_+(\lambda, s), \sigma(s)). \tag{2.23}$$

In both cases, we parameterize $\sigma(s)$ so that $\sigma(1) = \frac{1}{2}$, consistently with $\sigma = \frac{r}{r+1}$.

Both $\hat{\theta}_-(\lambda, s)$ and $\hat{\theta}_+(\lambda, s)$ are (at least) C^1 in s and λ from the unstable and center manifold theorems.

2.4. **Matching condition and proof strategy.** The proof proceeds by finding λ so that

$$\hat{\theta}_-(\lambda, s) = \hat{\theta}_+(\lambda, s), \tag{2.24}$$

for some $s \in \mathbb{R}$. Note that if (2.24) holds for some s, it does so for all s as both $\hat{\theta}_-$ and $\hat{\theta}_+$ are solutions of the same equation.

A key point will be to choose a value of s where the underlying trajectory hits a σ-section that lies between the inner and outer regions. Since the potentials are originally functions of r, it is most natural to set an r value first. To that end, we set $r_\varepsilon = \varepsilon^\alpha$ and so $\rho_\varepsilon = \varepsilon^{\alpha+1}$ where $\alpha \in (-1, 0)$. We observe later in Remark 16 that we require $\alpha > -\frac{1}{2}$ to complete the proof.[1] The corresponding s and t will be $s_\varepsilon = \varepsilon^\alpha + \alpha \ln \varepsilon$ and $t_\varepsilon = \varepsilon^{\alpha-1} + (\alpha - 1) \ln \varepsilon$

The σ-section will then be

$$\sigma_\varepsilon = \frac{\varepsilon^\alpha}{1 + \varepsilon^\alpha}, \tag{2.25}$$

and, in terms of τ

$$\tau_\varepsilon = \frac{\varepsilon^{\alpha-1}}{1 + \varepsilon^{\alpha-1}}. \tag{2.26}$$

An eigenvalue λ corresponds to a zero of the quantity

$$\Sigma(\lambda, \varepsilon) = \theta_-(\lambda, s_\varepsilon, \varepsilon) - \theta_+(\lambda, s_\varepsilon, \varepsilon), \tag{2.27}$$

which is the difference between the two quantities in (2.24) where the trajectories both hit the σ_ε-section. From the Implicit Function Theorem and the second

[1] Later, we will need to take $\alpha > -\frac{1}{2}$ sufficiently close to $-\frac{1}{2}$ such that $(4 + \gamma)(-\alpha) > 2 + \frac{\gamma}{4}$ with γ as in assumption (A2).

equation in (2.12) evaluated at σ_ε, it follows that $\Sigma(\lambda, \varepsilon)$ is, at least, C^1 in its arguments.

3. Estimates for eigenvalue parameters close to zero

The hardest part is detecting eigenvalues close to zero, and we develop the needed estimates for this case first. Set $\lambda = \varepsilon^2 \mu$ with μ order one. We need to estimate $\theta_-(\lambda, s)$ from $s \to -\infty$ up to the threshold $s = s_\varepsilon$. Similarly, we will obtain an estimate of $\theta_+(\lambda, s)$ backwards from $s \to +\infty$ to the threshold $s = s_\varepsilon$.

Each will be estimated by the solution of a model problem. For the inner problem, the outer potential $V_{1,\varepsilon}$ is omitted. Similarly, for the outer problem, the inner potential $V_{0,\varepsilon}$ is omitted.

As will be seen in the proofs below, there is a region around the threshold where the estimates are harder to establish and we call this the *plateau*.

3.0.1. *Inner problem.* The inner model problem we use here is obtained by setting $\lambda = 0$ in the angular equation, and keeping only the inner potential. In its compactified and desingularized form, it is

$$\begin{aligned} \dot{\theta} =& (\sigma - 1) \sin 2\theta + \sigma \left(\tilde{V}_0(\sigma) \cos^2 \theta - \sin^2 \theta \right), \\ \dot{\sigma} =& \sigma (1 - \sigma)^2. \end{aligned} \tag{3.1}$$

The regularity boundary condition at $r = 0$ leads us to consider

$$\hat{\theta}_0(s) = (\theta_0(s), \sigma(s)),$$

where $\theta_0(s) \to \theta_{0,-}$ as $s \to -\infty$. In other words, $\hat{\theta}_0(s)$ is the trajectory in $W^u(\hat{\theta}_{0,-})$ for the system (3.1) with the usual parameterization.

The goal is to estimate

$$|\theta_-(\lambda, s) - \theta_0(s)|. \tag{3.2}$$

The quantity in (3.2) tends to 0 as $s \to -\infty$ and we want to show that it remains small, in terms of ε, up to $s = s_\varepsilon$. We will divide the interval $(-\infty, s_\varepsilon]$ into two subintervals. Fix an $s_0 \in (0, 1)$ independently of ε. We shall think of s_0 as close to 1, but it is key that it will stay fixed as $\varepsilon \to 0$. The first estimate is on $(-\infty, s_0]$.

Lemma 3. *There is a constant C_1 and an $\varepsilon_0 > 0$ so that*

$$\left|\theta_-(\varepsilon^2 \mu, s) - \theta_0(s)\right| \leq C_1 \varepsilon^2$$

when $\varepsilon \leq \varepsilon_0$ and $s \in (-\infty, s_0]$

Proof. If we append the equation $\dot{\varepsilon} = 0$ to both (2.12) and (3.1), then the union of $W^u(0, 0, \varepsilon)$ over ε small can be viewed as a center-unstable manifold of $(0, 0, 0)$ in $(\theta, \sigma, \varepsilon)$ space. Since such a manifold is C^1 and the perturbation to the vector field is $O(\varepsilon^2)$, the estimate holds for s near $-\infty$. The standard result on continuity in parameters then implies the estimate holds up to $s = s_0$. □

Remark The technique in this proof cannot be used to push the estimate to s_ε as $s_\varepsilon \to \infty$ as $\varepsilon \to 0$. The strategy will be to estimate $\eta_0(\varepsilon, , s) = \theta_-(\varepsilon^2 \mu, s) - \theta_0(s)$ up to the threshold $s = s_\varepsilon$ through an understanding of the dynamics of the solution $\theta_0(s)$ of the model inner problem.

There is only one fixed point of (3.1) in $\sigma = 1$, namely $\theta_{1,\pm} = 0$. It is degenerate with two zero eigenvalues, and to analyze the dynamics in a full neighborhood of the fixed point will require a blow-up procedure. We shall prove the following lemma.

Lemma 4. *If* $0 \notin \sigma_p(\Delta - V_0)$, *then*

$$\theta_0(r) = O\left(\frac{1}{r^2}\right)$$

as $r \to +\infty$

Proof. Set $\xi = \sigma - 1$ in (3.1), which (3.1) then becomes

$$\begin{aligned} \dot{\theta} =& \xi \sin 2\theta + (\xi+1)\left(\left(\tilde{V}_0(\xi+1)\right)\cos^2\theta - \sin^2\theta\right), \\ \dot{\xi} =& \xi^2(\xi+1). \end{aligned} \tag{3.3}$$

System (3.3) has a fixed point at $(0,0)$ which is easily seen to be doubly degenerate (double zero eigenvalue). To blow this point up, we set $\theta = z\bar{\theta}$ and $\xi = z\bar{\xi}$, where $z \geq 0$, and $\bar{\theta}^2 + \bar{\xi}^2 = 1$. Condition (A1) guarantees that this can be done with a resulting C^1 vector field on the blown-up space.

Rather than writing out the equations for $\bar{\theta}$ and $\bar{\xi}$, which are unwieldy, it is common to calculate the equations directly in a local chart (see [Dum93, DR96, GS09b, GS09a]). The relevant chart is $\bar{\xi} = 1$ and the equations have the form:

$$\begin{aligned} \bar{\theta}' =& -\bar{\theta} - \bar{\theta}^2 + O(z^2), \\ z' =& z^2(1-z), \end{aligned} \tag{3.4}$$

where we have changed independent variable so that $\dot{\{\}} = z\{\}'$. The set $z = 0$ is the blown-up fixed point and carries the flow of

$$\bar{\theta}' = -\bar{\theta} - \bar{\theta}^2, \tag{3.5}$$

which has fixed points at $\bar{\theta} = 0$ and $\bar{\theta} = -1$. Each corresponds to a different type of decay of the original p. The former with decay to a constant and, the latter, decay like $1/r$. In the former case, $0 \notin \sigma_p(\Delta - V_0)$ whereas $0 \in \sigma_p(\Delta - V_0)$ in the latter case. Since we are assuming in (A3) that $0 \notin \sigma_p(\Delta - V_0)$, the lemma follows. □

This phenomenon was first discovered by Sandstede and Scheel [SS04], where it is discussed in the context of contact defects and extensions of the Evans Function. Note there is an irony here, as when considered in terms of the angle θ, the more rapid decay corresponds to not being in the spectrum!

The asymptotics established in these two lemmas will allow us to show that, up to the threshold σ_ε, $\theta_-(\varepsilon^2\mu, s)$ is well approximated by $\theta_0(s)$. Setting $\eta_0(\varepsilon, s) = \theta_-(\varepsilon^2\mu, s) - \theta_0(0, s)$, using various trigonometric identities, we can calculate

$$\dot{\eta}_0 = \left[(\sigma-1)\cos 2\theta_0 - \sigma(\tilde{V}_0(\sigma)+1)\sin 2\theta_0)\right]\eta_0 + h(\eta_0, \varepsilon), \tag{3.6}$$

where $\dot{\eta}_0 = \frac{d\eta_0}{ds}$, as usual, and $h(\eta_0, \varepsilon) = O(\eta_0^2) + O(\varepsilon^2)$.

Set $\tilde{\eta}_0(r) = \eta_0(\mu, r + \ln r)$, and, with an abuse of notation, we drop the $\tilde{}$ and recast (3.6) with r as the independent variable

$$\eta_0' = -\left[\frac{2}{r}\cos 2\theta_0 + (V_0(r)+1)\sin 2\theta_0)\right]\eta_0 + h(\eta_0, \varepsilon), \tag{3.7}$$

where $\eta_0' = \frac{d\eta_0}{dr}$ and $h(\eta_0, \varepsilon) = O(\eta_0^2) + O(\varepsilon^2)$.

We recall that the threshold, in terms of r is at $r_\varepsilon = \varepsilon^\alpha$. the following lemma holds under the standing assumption that $0 \notin \sigma_p(\Delta - V_0)$.

Lemma 5. *There is a constant $C > 0$, independent of ε, so that*

$$|\eta_0(r_\varepsilon)| \leq C\varepsilon. \tag{3.8}$$

Proof. Set r_0 so that $s_0 = r_0 + \ln r_0$, then we can invoke Lemma 3, to obtain an even better estimate at r_0. The region $[r_0, r_\varepsilon]$ is called the inner plateau and we need to push the estimate across the plateau from r_0 to r_ε. This is achieved using a bootstrapping argument. The bootstrap assumption will be that $|\eta_0(r)| \leq C\varepsilon$ for all $r \in [r_0, r_\varepsilon]$.

Set $g(r) = \frac{2}{r}\cos 2\theta_0 + (V_0(r) + 1)\sin 2\theta_0$, which is the coefficient of η_0 in (3.7). Using an integrating factor

$$\Gamma(r) = e^{\int_{r_0}^r g(s)ds}, \tag{3.9}$$

we can write

$$\eta_0(r) = \Gamma^{-1}(r)\Gamma(r_0)\eta_0(r_0) + \Gamma^{-1}(r)\int_{r_0}^r \Gamma(z)h(\eta_0(z), \varepsilon)dz. \tag{3.10}$$

The integrating factor $\Gamma(r)$ can be estimated as follows. Adjusting our choice of r_0 sufficiently large if necessary, but still independent of ε, we can find a small constant $0 < \delta < \frac{1}{4}$ depending only on r_0 and V_0 such that

$$1 - \frac{\delta}{4} < \cos 2\theta_0(r) < 1 + \frac{\delta}{4},$$

and

$$\left|\frac{r}{2}(V_0(r) + 1)\sin 2\theta_0(r))\right| < \frac{\delta}{4}$$

hold for all $r \geq r_0$ from Lemma 4. It follows that

$$\frac{2-\delta}{r} < g(r) < \frac{2+\delta}{r} \tag{3.11}$$

for all $r \geq r_0$, and so

$$\left(\frac{r}{r_0}\right)^{2-\delta} < \Gamma(r) < \left(\frac{r}{r_0}\right)^{2+\delta}. \tag{3.12}$$

Plugging into (3.10), we estimate

$$|\eta_0(r)| \leq |\eta_0(r_0)| + c_1\left(\frac{r}{r_0}\right)^{\delta-2}\int_{r_0}^r \Gamma(z)dz\left(\eta^2 + \varepsilon^2\right), \tag{3.13}$$

for some constant c_1. Using (3.12) again and integrating, there is a constant $c_2 = c_2(r_0, \delta)$ so that

$$|\eta_0(r)| \leq |\eta_0(r_0)| + c_2 r^{1+2\delta}\left(\eta^2 + \varepsilon^2\right). \tag{3.14}$$

The second term on the right hand side will be largest on the interval $[r_0, r_\varepsilon]$ at $r = r_\varepsilon = \varepsilon^\alpha$, and so

$$|\eta_0(r)| \leq |\eta_0(r_0)| + c_2\varepsilon^{\alpha(1+2\delta)}\left(\eta^2 + \varepsilon^2\right). \tag{3.15}$$

By the bootstrapping hypothesis, there is a constant c_3 and

$$|\eta_0(r)| \leq |\eta_0(r_0)| + c_3\varepsilon^{\alpha(1+2\delta)+2}. \tag{3.16}$$

We choose $-1 < \alpha < 0$ so that $\alpha(1+2\delta) > -1$, i.e

$$2 + \alpha(1+2\delta) = 1 + \nu, \tag{3.17}$$

for some sufficiently small $\nu > 0$ so that $-1/(1+2\delta) < \alpha < 0$,

$$|\eta_0(r)| \leq |\eta_0(r_0)| + c_3\varepsilon^{1+\nu}. \tag{3.18}$$

Recalling the bootstrapping assumption that $|\eta_0(r)| \leq C\varepsilon$ for all $r \in [r_0, r_\varepsilon]$, we could estimate the right-hand side by a constant times ε. But that is not good enough, and we need to have a stronger estimate on $|\eta_0(r)|$. If we set ε_0 small enough that $c_3\varepsilon^{2\nu} < \frac{C}{2}$, then from (3.18) and choosing ε_0 so that $|\eta_0(r_0)| \leq \frac{C}{2}\varepsilon$ for a constant C, by Lemma 3.2,

$$|\eta_0(r)| \leq \frac{C}{2}\varepsilon, \tag{3.19}$$

which closes the bootstrapping argument. □

3.0.2. *Outer problem.* The angular equation for the outer model problem that is analogous to (3.1) is.

$$\begin{aligned} \dot{\psi} =& (\tau - 1)\sin 2\psi + \tau\left(\left(\mu + \tilde{V}_1(\tau)\right)\cos^2\psi - \sin^2\psi\right), \\ \dot{\tau} =& \tau(1-\tau)^2, \end{aligned} \tag{3.20}$$

where $\dot{} = \frac{d}{d\rho}$.

For the outer problem, we consider solutions that satisfy the boundary condition at $\tau = 1$. To that end, set $\hat{\psi}_1(\mu, t) = \psi_1(\mu, t), \tau(t))$ to be a solution of (3.20) for which $\psi_1(\mu, t) \to \psi_{1,+}(\mu)$ as $t \to +\infty$. We will compare ψ_1 to the solution of the full (outer) problem (2.15) that satisfies the same boundary condition at $\tau = 1$. We denote this by $\hat{\psi}_-(\mu, t) = (\psi_-(\mu, t), \tau(t))$. Set

$$\eta_1(\mu, \varepsilon, t) = \psi_1(\mu, t) - \psi_-(\mu, t, \varepsilon).$$

The goal of this section is to obtain an estimate on η_1 at $t = t_\varepsilon$ where $t_\varepsilon = \varepsilon^\kappa + \kappa \ln \varepsilon$. We will actually need an estimate on the analogous quantity where ψ is replaced by θ. The relationship between the two is given by (2.17).

The equation satisfied by η_1 is

$$\dot{\eta}_1 = \left[(\tau - 1)\cos 2\psi_1 - \tau(\tilde{V}_1(\tau) + \mu + 1)\sin 2\psi_1\right]\eta_1 + \frac{\tau}{\varepsilon^2}\tilde{V}_0(\tau) + O(\eta_1^2). \tag{3.21}$$

As in the previous section, we will switch to expressing η_1 in terms of the independent variable, which here will be ρ. Recall that $\rho = \varepsilon r$, and setting $\kappa = \alpha + 1$ makes this match with the thresholds as given in (2.25) and (2.26). We write $\hat{\eta}_1(\mu, \varepsilon, \rho) = \eta_1(\mu, \varepsilon, \rho + \ln \rho)$. Dropping the $\hat{}$ again and the explicit dependence on μ and ε, (3.21) becomes

$$\eta_1' = -\left[\frac{2}{\rho}\cos 2\psi_1 + (\tilde{V}_1(\tau) + \mu + 1)\sin 2\psi_1\right]\eta_1 + \frac{1}{\varepsilon^2}V_0(\frac{\rho}{\varepsilon}) + O(\eta_1^2), \tag{3.22}$$

where $\eta_1' = \frac{d\eta_1}{d\rho}$.

We will need information on the asymptotics of the solution of the model outer problem (3.20) as $\rho \to 0$, or, equivalently, as $t \to -\infty$.

Lemma 6. *If $\mu \notin \sigma_p(\Delta - V_1)$ then*

$$\psi_1(\mu, t) \to (2k+1)\frac{\pi}{2}$$

for some integer k as $t \to -\infty$.

The proof of the lemma follows from considering the dynamics of (3.20) near $\tau = 0$ and the fact that tending to $(2k+1)\frac{\pi}{2}$ corresponds to the boundary condition at $\rho = 0$ not being satisfied, as it should not be when μ is not an eigenvalue.

From assumption (A2) on the decay of $V_0(r)$, we can set a fixed ρ_1 so that

$$\left|V_0\left(\frac{\rho_1}{\varepsilon}\right)\right| \leq C\varepsilon^{4+\gamma}$$

for chosen $\gamma > 0$ and some $C > 0$, and we then have an analog of Lemma 3.

Lemma 7. *There is a constant $C_1 > 0$ and an ε_0 so that*

$$|\eta_1(\rho_1)| \leq C_1\varepsilon^{2+\gamma}$$

when $\varepsilon \leq \varepsilon_0$, and γ is as determined above.

The proof is similar to that of Lemma 3 with the stable manifold at $\sigma = 0$ replaced by the center manifold at $\tau = 1$. Because we are working with ψ and need to convert back to θ for the main proof, we need a stronger estimate for the outer system.

Lemma 8. *Assume $\mu \notin \sigma_p(\Delta - V_1)$, then there is a constant C, independent of ε, so that*

$$|\eta_1(\rho_\varepsilon)| \leq C\varepsilon^2, \tag{3.23}$$

where $\varepsilon \leq \varepsilon_0$ for prescribed ε_0 depending μ.

Proof. We use the same methodology as for the inner system by expressing η_1 as a solution of an integral equation. The first step is then to obtain estimates on the integrating factor.

Proceeding as before, we will need estimates on

$$g(\rho) = \frac{2}{\rho}\cos 2\psi_1 + (\tilde{V}_1(\tau) + \mu + 1)\sin 2\psi_1,$$

which is the parenthetical quantity in (3.22). With $\mu \notin \sigma_p(\Delta - V_1)$, the base solution ψ_1 tends to $\frac{\pi}{2} + k\pi$, as $\rho \to 0$ for some integer k. It follows that $\cos 2\psi_1 \to -1$ and $\sin 2\psi_1 \to 0$ as $\rho \to 0$. Since this is independent of ε, for fixed $0 < \delta < \frac{1}{4}$, let us take $\rho_1 = \rho_1(V_1) > 0$ small enough so that

$$\frac{-2-\delta}{\rho} < g(\rho) < \frac{-2+\delta}{\rho}, \tag{3.24}$$

holds for $\rho \leq \rho_1$. By integrating, we obtain

$$\ln\left(\frac{\rho_1}{\rho}\right)^{2+\delta} > \int_{\rho_1}^{\rho} g(z)dz > \ln\left(\frac{\rho_1}{\rho}\right)^{2-\delta},$$

and exponentiating, with $\Gamma(\rho) = \exp\{\int_{\rho_1}^{\rho} g(z)dz\}$ being the integrating factor we want,

$$\left(\frac{\rho_1}{\rho}\right)^{2-\delta} < \Gamma(\rho) < \left(\frac{\rho_1}{\rho}\right)^{2+\delta}.$$

Using the integrating factor, we obtain an expression for η

$$\eta_1(\rho) = \Gamma^{-1}(\rho)\Gamma(\rho_1)\eta_1(\rho_1) + \Gamma^{-1}(\rho)\int_{\rho_1}^{\rho}\Gamma(z)h(z,\eta_1,\varepsilon)dz, \tag{3.25}$$

where $h(z,\eta_1,\varepsilon) = \frac{1}{\varepsilon^2}V_0\left(\frac{\rho}{\varepsilon}\right) + O(\eta_1^2)$. From the assumption on the decay of V_0, and recalling that $\rho_\varepsilon = \varepsilon^\kappa$, we can estimate

$$\left|\frac{1}{\varepsilon^2}V_0\left(\frac{\rho}{\varepsilon}\right)\right| \leq C\varepsilon^{(1-\kappa)(4+\gamma)-2}, \tag{3.26}$$

provided $\rho \geq \rho_\varepsilon$.

We also have

$$\Gamma^{-1}(\rho)\int_{\rho}^{\rho_1}\Gamma(z)dz \leq \frac{\rho_1}{1+\delta}, \tag{3.27}$$

for any $\rho \leq \rho_1$. Plugging (3.26) and (3.27) into (3.25) and estimating, we obtain

$$|\eta_1(\rho)| \leq |\eta_1(\rho_1)| + C\frac{\rho_1}{1+\delta}\left[\varepsilon^{(1-\kappa)(4+\gamma)-2} + (\eta_1(\rho))^2\right] \tag{3.28}$$

for some (new) constant $C > 0$.

Again, we make a bootstrap argument and assume that $|\eta_1(\rho)| \leq C_1\varepsilon^2$, for all $\rho \in [\rho_\varepsilon, \rho_1]$. From Lemma 7, the fact that $(4+\gamma) > 4$, and by choosing κ sufficiently close to $\frac{1}{2}$ (and hence α sufficiently close to $-\frac{1}{2}$) such that $(4+\gamma)(1-\kappa) > 2 + \frac{\gamma}{4}$, we can set the parameters so that the right-hand side of (3.28) is dominated by a constant times $\varepsilon^{2+\frac{\gamma}{4}}$, thus allowing closure of the bootstrap. □

For the proof of the main result, we will need an estimate in terms of θ.

Corollary 9. *Assume $\mu \notin \sigma_p(\Delta - V_1)$, then there is a constant C, independent of ε, so that*

$$|\theta_+(\rho_\varepsilon) - \theta_1(\rho_\varepsilon)| \leq C\varepsilon, \tag{3.29}$$

where $\varepsilon \leq \varepsilon_0$ for prescribed ε_0 depending μ.

Proof. Noting that ψ_- lies in a neighborhood of $\frac{\pi}{2}$ in the outer plateau $[\rho_\varepsilon, \rho_1]$, and that we can write $\cot\psi = \varepsilon\cot\theta$, we can find a constant $C > 0$ so that

$$|\theta_+ - \theta_1| \leq \frac{C}{\varepsilon}|\psi_+ - \psi_1|.$$

Combining this estimate with Lemma 8 yields the result. □

4. Proof for small eigenvalue parameters

Throughout this section, we will work with angles θ in all of $\mathbb{R}$, rather than S^1, and use the notation $\tilde{\theta}$ to distinguish it. If the covering map is denoted by $\Pi : \mathbb{R} \to S^1$ then, $\Pi(\tilde{\theta}) = \theta$.

We introduce new notation for the trajectories of interest in the covering space $\mathbb{R}$. The lift of $\theta_0(s)$ will be denoted $\tilde{\theta}_0(s)$ if it satisfies $\lim_{s\to-\infty}\tilde{\theta}_0(s) = 0$, called the primary lift. Other lifts will be denoted $\tilde{\theta}_0^k(s) = \tilde{\theta}_0(s) + k\pi$. The (primary) lift of $\theta_-(\lambda, s)$, given in (2.22), will be denoted $\tilde{\theta}_-(\mu, s)$, where $\lambda = \varepsilon^2\mu$. The other lifts will be denoted $\tilde{\theta}_-^k(\mu, s) = \tilde{\theta}_-(\mu, s) + k\pi$. Analogously, we use the notation $\tilde{\psi}_1(\mu, t)$ and $\tilde{\psi}_+(\mu, t)$ for the primary lifts of the model and full outer problems respectively; note we are suppressing explicit mention of the dependence of ψ_- on ε.

We set $\tilde{\mu}$ so that if $\mu \in \sigma(\Delta - V_1)$ and $\mu \geq 0$ then $\mu \in (0, \tilde{\mu})$. Recall that we evaluate the angles of the two candidate trajectories at $\sigma = \sigma_\varepsilon$. In terms of the independent variables, the candidate trajectories intersect the section when $s = s_\varepsilon$ or $t = t_\varepsilon$. It then follows from Lemma 6 that there are k_0 and k_1 so that

$$\begin{aligned} \tilde{\psi}(\tilde{\mu}, t) &\to (2k_1 + 1)\frac{\pi}{2}, \text{ and} \\ \tilde{\psi}(0, t) &\to (2k_0 + 1)\frac{\pi}{2}, \end{aligned} \tag{4.1}$$

as $t \to -\infty$. From Sturm-Liouville theory, we know that $k_0 \geq k_1$, and we have the following lemma, recalling that $m(V_1)$ is the number of positive eigenvalues of $\Delta - V_1$ on L_2 radial functions in $\mathbb{R}^3$.

Lemma 10. $m(V_1) = k_0 - k_1$.

The proof is a shooting argument: there must be $k_0 - k_1$ values of μ for which

$$\tilde{\psi}_1(\mu, t) \to k\pi$$

for some integer k. Since the eigenvalues are simple, the count is exact.

The remainder of the argument is to show that $k_0 - k_1$ also counts the eigenvalues of the full problem. First, we need estimates on the evaluation of the solutions of the two model problems at the threshold.

Lemma 11. *Assuming $0 \notin \sigma(\Delta - V_0)$, there is an $\varepsilon_0 > 0$ and $C > 0$ so that for $\varepsilon \leq \varepsilon_0$,*

$$\left|\tilde{\theta}_0(s_\varepsilon) - m\pi\right| \leq C\varepsilon^{-2\alpha}, \tag{4.2}$$

for some integer m, where $\alpha \in (-\frac{1}{2}, 0)$ is as chosen consistently with prior conditions.

Proof. From Lemma 4, we know that any lift $\tilde{\theta}_0(s)$ must tend to an integer multiple of π as $s \to +\infty$. Recalling that $r_\varepsilon = \varepsilon^\alpha$, and that $s_\varepsilon = r_\varepsilon + \ln r_\varepsilon$ we can conclude that $s_\varepsilon \to \infty$ as $\varepsilon \to 0$. The lemma then follows from the asymptotic relation in Lemma 4. □

For the outer problem, we have:

Lemma 12. *If $\mu \notin \sigma(\Delta - V_1)$ then there is and $\varepsilon_0 > 0$ and $C > 0$ so that*

$$\left|\tilde{\theta}_1(\mu, t_\varepsilon) - (2k + 1)\frac{\pi}{2}\right| \leq C\varepsilon^{2\kappa - 1}, \tag{4.3}$$

where $\kappa = \alpha + 1$ is chosen appropriately, and so $\kappa > \frac{1}{2}$.

Proof. From Lemma 6, $\tilde{\psi}_1(\mu, t)$ tends to an odd multiple of $\frac{\pi}{2}$. By linearizing (3.20), reset in the covering space, at the fixed point $\left((2k + 1)\frac{\pi}{2}, 0\right)$, and calculating the unstable eigenvalue we can conclude there is a $C > 0$ for which

$$\left|\tilde{\psi}_1(\mu, t_\varepsilon) - (2k + 1)\frac{\pi}{2}\right| \leq C\varepsilon^{2\kappa},$$

and the statement in the lemma follows by switching to θ.

□

The estimates in Lemmas 11 and 12 pertain to the respective model problems. We need similar estimates for the two full problems.

Lemma 13. *There is an $\varepsilon_0 > 0$ and $C > 0$ so that for $\varepsilon \le \varepsilon_0$,*

$$\left|\tilde{\theta}_-(s_\varepsilon) - m\pi\right| \le C\left(\varepsilon + \varepsilon^{-2\alpha}\right), \tag{4.4}$$

for some integer m.

Proof. This follows easily from Lemma 11 put together with Lemma 5, using the triangle inequality. □

The lemma addresses the fate of the primary lift, a straightforward corollary gives the related estimate for other lifts.

Corollary 14. *There is an $\varepsilon_0 > 0$ and $C > 0$ so that, if $\varepsilon \le \varepsilon_0$, then*

$$\left|\tilde{\theta}^k_-(s_\varepsilon) - (k+m)\pi\right| \le C\left(\varepsilon + \varepsilon^{-2\alpha}\right) \tag{4.5}$$

For the outer problem, there are two estimates, one at each end of the interval $[0, \tilde{\mu}]$ which encloses the (positive) point spectrum of $\Delta - V_1$.

Lemma 15. *There is an $\varepsilon_0 > 0$ and $C > 0$ so that, if $\varepsilon \le \varepsilon_0$, then*

$$\begin{aligned} \left|\tilde{\theta}_+(\tilde{\mu}, t_\varepsilon) - (2k_1+1)\frac{\pi}{2}\right| &\le C\left(\varepsilon + \varepsilon^{2\kappa-1}\right), \\ \left|\tilde{\theta}_+(0, t_\varepsilon) - (2k_0+1)\frac{\pi}{2}\right| &\le C\left(\varepsilon + \varepsilon^{2\kappa-1}\right), \end{aligned} \tag{4.6}$$

where k_0 and k_1 are given by (4.1).

Proof. The proof follows similarly to that of Lemma 15 and Corollary 14 but applying Lemma 8 and Lemma 12. □

Remark 16. *Note, we need here that $2\kappa - 1 > 0$, which since $\kappa = 1 + \alpha$ is the condition that $\alpha > -\frac{1}{2}$.*

We can now complete the proof of the main result for eigenvalue parameter values of size $O(\varepsilon^2)$. Set

$$\Sigma^k(\mu, \varepsilon) = \tilde{\theta}^k_-(\mu, s_\varepsilon) - \tilde{\theta}_+(\mu, t_\varepsilon). \tag{4.7}$$

If $\Sigma^k(\mu, \varepsilon) = 0$ for any integer k, then $\lambda = \varepsilon^2\mu$ is an eigenvalue of $\Delta - (V_0 + V_{1,\varepsilon})$ for that value of ε.

The following lemma summarizes the first part of this result.

Lemma 17. *If ε is sufficiently small, then there are at least $m(V_1)$ values of $\mu > 0$ for which there is an integer k so that*

$$\Sigma^k(\mu, \varepsilon) = 0.$$

Proof. The proof proceeds in a similar fashion to that of Lemma 10. From Lemma 15, there will be $k_0 - k_1$ values of $\mu \in [0, \tilde{\mu}]$ for which $\tilde{\theta}_+(\mu, t_\varepsilon) = k\pi$. Using Corollary 14, it follows easily that there will be $k_0 - k_1$ values of μ for which $\Sigma^k(\mu, \varepsilon) = 0$ for some k, again with ε sufficiently small, The number $k_0 - k_1$ can be replaced by $m(V_1)$ from Lemma 10, thus proving the lemma. □

It remains to prove that there are at most $m(V_1)$ such small eigenvalues, i.e., that the eigenvalue count is exact.

Lemma 18. *If ε is sufficiently small and $\mu = \hat{\mu}$ is a zero of Σ^k, then*

$$\left.\frac{\partial}{\partial\mu}\right|_{\mu=\hat{\mu}} \Sigma^k(\mu, \varepsilon) > 0.$$

Proof. We first show that $\frac{\partial}{\partial\mu}\tilde{\theta}_+(\mu,t) < 0$ for any μ and t. It is equivalent to show that $\frac{\partial}{\partial\mu}\tilde{\psi}_+(\mu,t) < 0$ for any μ and t. As $t \to +\infty$, $\tilde{\psi}_+(\mu,t) \to \arctan(-\sqrt{\mu})$. Since $\frac{d}{d\mu}\arctan(-\sqrt{\mu}) < 0$, $\frac{\partial}{\partial\mu}\tilde{\psi}_+(\mu,t) < 0$ for t large. To show that this quantity stays negative, we calculate from (2.15):

$$\frac{d}{dt}\left(\frac{\partial\psi}{\partial\mu}\right) = R(\tau,\psi,\mu)\frac{\partial\psi}{\partial\mu} + \tau\cos^2\psi, \tag{4.8}$$

where $R = 2(\tau-1)\cos 2\psi - \tau\left((\mu + \tilde{V}_{0,\varepsilon}(\tau) + \tilde{V}_1(\tau) + 1)\sin 2\psi\right)$. Since $\tau\cos^2\psi \geq 0$, solving (4.8) using an integrating factor, we can conclude that $\frac{\partial}{\partial\mu}\tilde{\psi}_+(\mu,t) < 0$ for any μ and t.

Similarly, we show that $\frac{\partial}{\partial\mu}\tilde{\theta}_-(\mu,t) < 0$ for any μ and t by first showing that $\frac{\partial}{\partial\mu}\tilde{\theta}_-(\mu,t) > 0$ for large negative t. Using that the unstable manifold of $(0,0)$ in (2.12) is tangent to the unstable eigenvector, we can expand

$$\theta = \left(\mu + V_0(0) + \varepsilon^2 V_1(0) - 1\right)\sigma + O(\sigma^2),$$

from which we can conclude that $\frac{\partial}{\partial\mu}\tilde{\theta}_-(\mu,t) > 0$ if $t \ll 0$. A similar argument using a version of (4.8) in terms of θ then shows that $\frac{\partial}{\partial\mu}\tilde{\theta}_-(\mu,t) < 0$ for any μ and t. □

It follows that there is at most one μ for each integer k for which $\Sigma^k(\mu,\varepsilon) = 0$. The number of eigenvalues of the full problem is therefore exactly $k_0 - k_1$ and hence $m(V_1)$. This ostensibly still allows for these eigenvalues to have multiplicity greater than 1. However, from classical Sturm-Liouville theory, see for instance [RS78a], Theorem XI.53, these eigenvalues must be simple in the class of radial functions.

4.1. **The case of $O(1)$ eigenvalues.** The eigenvalues that do not tend to 0 with ε come from the inner potential V_0 and it is considerably easier to capture them as the outer potential manifests as just a small perturbation. We work with the solution $\theta_-(\lambda,s,\varepsilon)$ of (2.12) on the unstable manifold of $(0,0)$ and compare it with the analogous solution of with a model problem based on just V_0. Whereas the model problem in the case of $O(\varepsilon^2)$ had the eigenvalue parameter evaluated at 0, we need to include it here for all positive values:

$$\begin{aligned}\dot{\theta} =& (\sigma-1)\sin 2\theta + \sigma\left(\lambda + \tilde{V}_0(\sigma)\cos^2\theta - \sin^2\theta\right),\\ \dot{\sigma} =& \sigma(1-\sigma)^2.\end{aligned} \tag{4.9}$$

Let $\theta_0(\lambda,s)$ denote the solution of (4.9) on $W^u(0,0)$, suitably parameterized. Its asymptotics as $s \to +\infty$ are given by the following lemma.

Lemma 19. *If $\lambda \notin \sigma(\Delta - V_0)$ and $\lambda > 0$, then*

$$\theta_0(\lambda,s) \to \theta_{1,-} = \arctan\left(\sqrt{\lambda}\right). \tag{4.10}$$

The purely perturbative nature of $V_{1,\varepsilon}$ then allows us to prove the lemma comparing the model and full problems for $\lambda = O(1)$.

Lemma 20. *If $\lambda \notin \sigma(\Delta - V_0)$ and $\lambda > 0$, then there are ε_0 and $C > 0$ so that*

$$|\theta_-(\lambda,s,\varepsilon) - \theta_0(\lambda,s)| \leq C\varepsilon \tag{4.11}$$

for all $s \in \mathbb{R}$.

The new piece here is that the estimate holds as $s \to +\infty$. This can be seen by checking the bootstrapping estimates or looking at the dynamics of (2.12) in a neighborhood of $\sigma = 1$ for small ε.

It can now be seen that the number of positive eigenvalues of L on L_2 radial functions that are bounded away from zero as $\varepsilon \to 0$ is exactly $m(V_0)$ by repeating a similar argument to that used in the previous section for Lemma 10. The exactness of the count also follows from a similar argument using $\frac{\partial \theta}{\partial \mu}$.

Combining these results, we complete the proof of Theorem 1, demonstrating that the total number of eigenvalues in $\sigma_p(\Delta - W_\varepsilon)$ contained in $(0, \infty)$ is precisely $m(V_0) + m(V_1)$.

5. Numerical Examples

We present three numerical scenarios that illustrate Theorem 1, demonstrating how different potential combinations affect the behavior of systems (2.12) and (2.15). In all cases, we fix $\mu \in (0, 1)$, $\varepsilon = 0.1$, and use thresholds (2.25) and (2.26) for σ and τ respectively. All numerical simulations were executed on a MacBook Pro equipped with an Apple M1 chip (8-core CPU at 3.2 GHz) and 8GB of unified memory. The computations were completed with minimal processing time.[2]

Each scenario satisfies assumptions (A1-A2). In particular:

- All potentials are $C^\infty(\mathbb{R}^+)$ and radially symmetric;
- Each $V_{1,\varepsilon}$ satisfies the scaling structure described in assumption (A1);
- Each V_0 satisfies the integrability and decay assumptions required to ensure that $\Delta - V_0$ has essential spectrum $(-\infty, 0]$ and a finite number of eigenvalues in $(0, \infty)$.

In all Figures in scenarios 1,2, and 3, left plots are solutions of (2.12), while right plots are solutions of (2.15). We are plotting the unstable manifolds of 3 different equilibrium solutions for the σ system for scenarios 1 and 3, in particular $(0, 0)$, $(0, \pi)$ and $(0, 2\pi)$ while varying $\mu \in (0, 1)$. For scenario 2, in addition to the first 3 mentioned, we are also plotting the unstable manifold of $(0, 3\pi)$. For the τ system, we are plotting the center manifold of $(1, \arctan\sqrt{\gamma} - 2\pi)$ while also varying μ from $(0, 1)$. To compute the eigenvalue counts above a given μ value, we consider $\lfloor \frac{\theta(0)-\theta(1)}{\pi} \rfloor$ in the (σ, θ) system and $\lfloor \frac{\psi(0)-\psi(1)}{\pi} \rfloor$ in the (τ, ψ) system for the given μ.

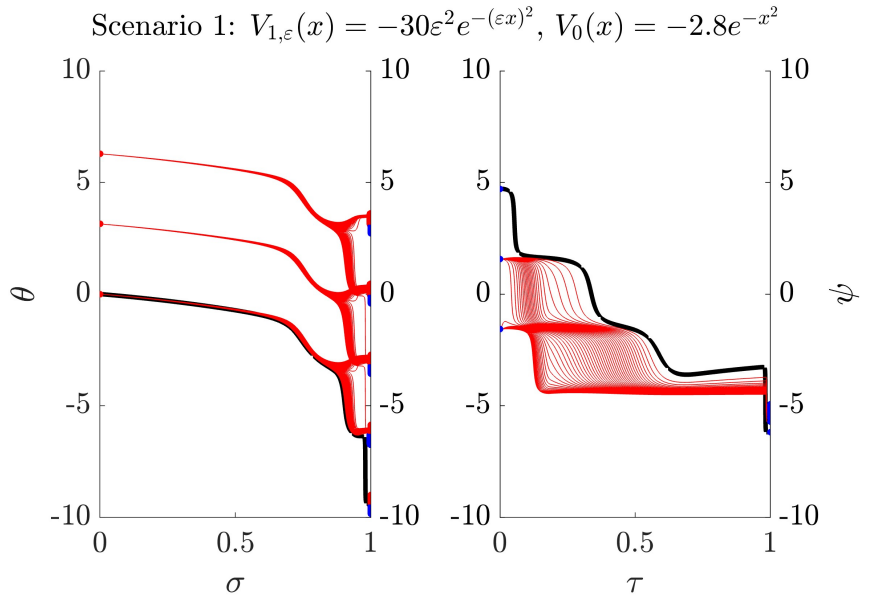
5.1. **Scenario 1: Exponential and Gaussian potentials.**

Consider the potentials

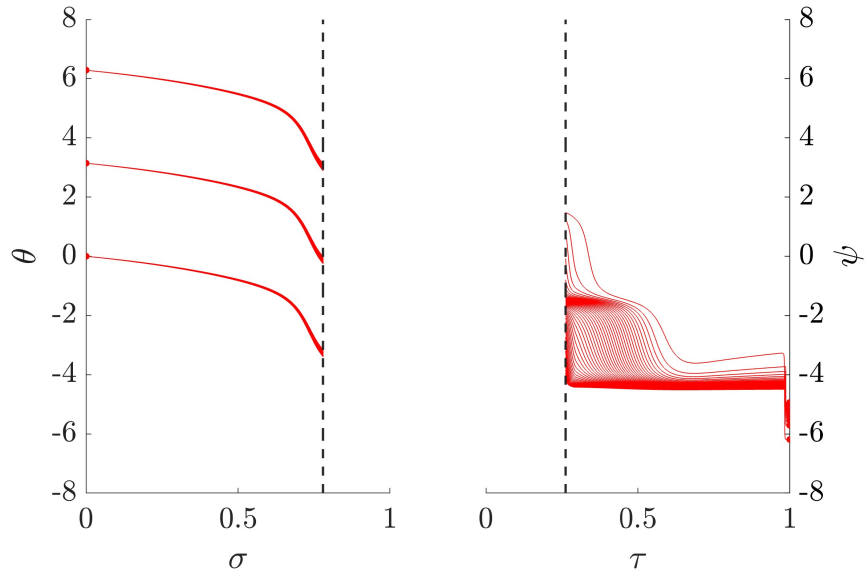
$$V_0(x) = -2.8e^{-x^2}, \quad V_{1,\varepsilon}(x) = -30\varepsilon^2 e^{-(\varepsilon x)^2}.$$

The Gaussian profile ensures that $V_{1,\varepsilon}$ is rapidly decaying and scales appropriately under $\varepsilon \to 0$. Meanwhile, $V_0(x)$ is smooth, integrable on $[0, \infty)$, and satisfies $|V_0(x)| \leq C(1 + x)^{-a}$ for any $a > 0$, thus fulfilling the assumptions on the background potential V_0. Figure 5.1 shows the corresponding phase portraits.

[2]The complete MATLAB source code for all three cases is available at `https://github.com/efleurantin2103/AlgoSpace/tree/master/Scalar_Paper_3cases`. The computational codes are subject to usage restrictions described in the repository documentation.



(a)



(b)

FIGURE 5.1. Scenario 1: Unstable manifold $W^u(\hat{\theta}_{0,-})$ (left) and center manifold $W^c(\hat{\psi}_{1,-})$ (right) from (2.12) and (2.15). Red/blue points indicate saddle/unstable equilibria at $\sigma = 0$, while blue points at $\sigma = 1$ indicate center-stable equilibria. Black curves show maximum π-jumps as $\varepsilon \to 0$ for their respective systems.

5.2. **Scenario 2: Rational and hyperbolic secant potentials.** Consider the potentials

$$V_0(x) = -\frac{2.6}{1+2x^4}, \quad V_{1,\varepsilon}(x) = -20\varepsilon^2 \operatorname{sech}(\varepsilon x).$$

Again, $V_{1,\varepsilon}$ has the correct scaling and decay properties. The rational function $V_0(x)$ is C^∞, and satisfies $|V_0(x)| \leq C(1+x)^{-4}$, which ensures square-integrability and decay sufficient to define a compact perturbation of the Laplacian. Figure 5.2 shows the results.

5.3. **Scenario 3: Hyperbolic cosine and rational potentials.** Consider the potentials

$$V_0(x) = -\frac{3}{\cosh^2(1.2x)}, \quad V_{1,\varepsilon}(x) = -\frac{30\varepsilon^2}{(1+(\varepsilon x)^2)^2}.$$

The potential $V_{1,\varepsilon}$ is smooth and scales as required, with algebraic decay. The background potential $V_0(x)$ is bounded, smooth, integrable, and satisfies $|V_0(x)| \leq 12e^{-2.4x}$, which again meets the necessary integrability and decay assumptions. Figure 5.3 displays the associated system behavior.

5.4. **Verification of spectral assumptions.** In all three scenarios, the operator $\Delta - V_0 - V_{1,\varepsilon}$ satisfies the spectral assumptions required in Theorem 1. Specifically:

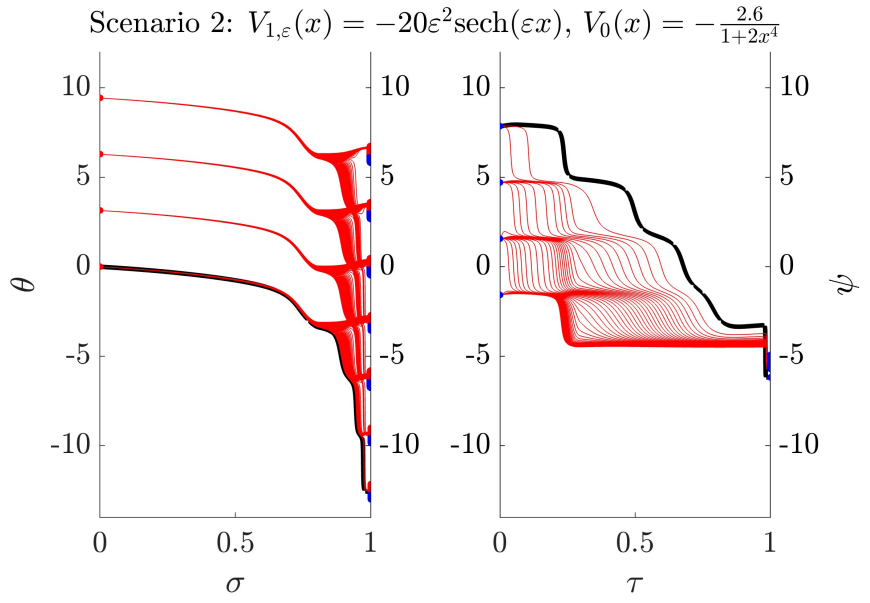
- The essential spectrum of $\Delta - V_0$ remains $(-\infty, 0]$ since V_0 is compactly supported or decaying and integrable.
- Each $V_{1,\varepsilon}$ has compact support at scale $\mathcal{O}(1/\varepsilon)$ and contributes a finite-rank perturbation, preserving the essential spectrum and modifying the point spectrum.
- The eigenvalue count $m(V_1)$ is well-defined and finite for each case.

Numerical results show at most three jumps in the angle variable ψ from (2.15) in scenarios 1 and 3, and at most four jumps in scenario 2, corresponding to topological phase shifts in θ from (2.12). The eigenvalue analysis in Figure 5.4 demonstrates that $m(V_0) = 1$ consistently across all scenarios. The eigenvalue count for the V_1 system for each scenario is illustrated in Figure 5.5. We note that for scenario 2, $m(V_1) = 3$, while in scenarios 1 and 3, $m(V_1) = 2$. Incorporating this information into the topological relationship $m(W) = m(V_0) + m(V_1)$, we obtain $m(W) = 3$ for scenarios 1 and 3, and $m(W) = 4$ for scenario 2. This observation is consistent for both the σ and τ systems.

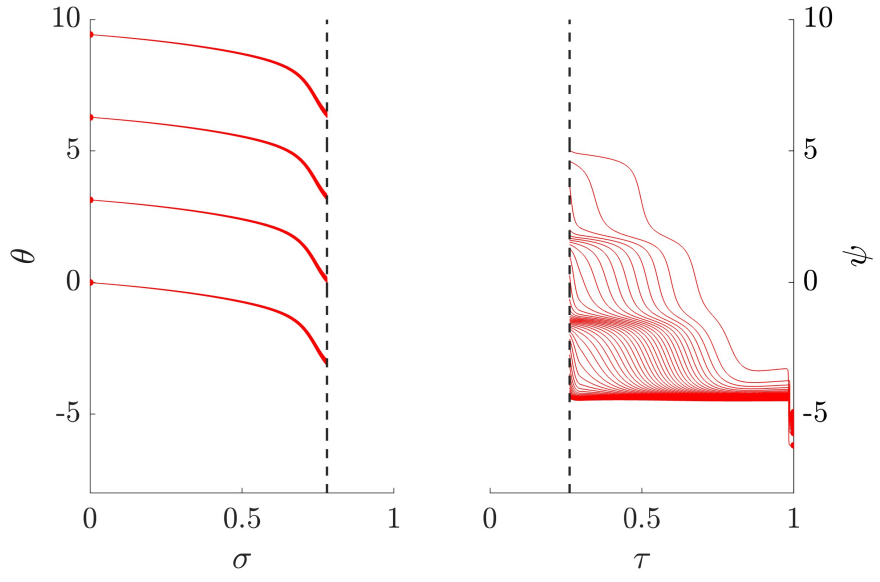
Remark 21. *The τ system trajectories in the right panels of Figures 5.1, 5.2 and 5.3 exhibit a characteristic initial transient that requires careful interpretation. Crucially, these transients contribute to the topological count given by $m(W)$, which essentially measures the net phase winding of the solution. The rapid initial transition represents genuine phase accumulation that must be included in this count.*

6. Conclusion and discussion

In this paper, we have presented a rigorous analysis of the operator $\Delta - W_\varepsilon$ with scale-separated potential $W_\varepsilon = V_0 + V_{1,\varepsilon}$, where both components are smooth, radially symmetric, and satisfy assumptions (A1-A3). Our main focus has been on understanding the spectral properties, particularly the existence and counting of eigenvalues in the spectral gap. More specifically, we have established that for

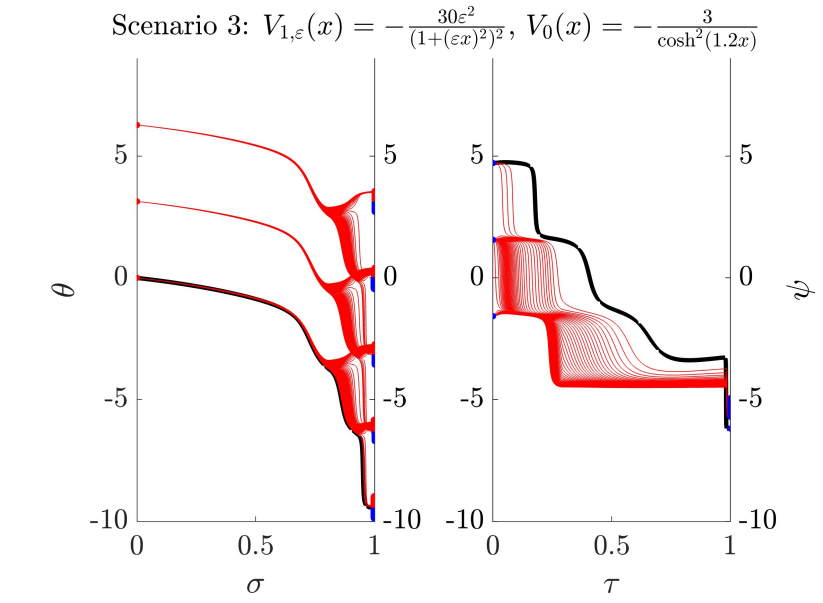


(a)

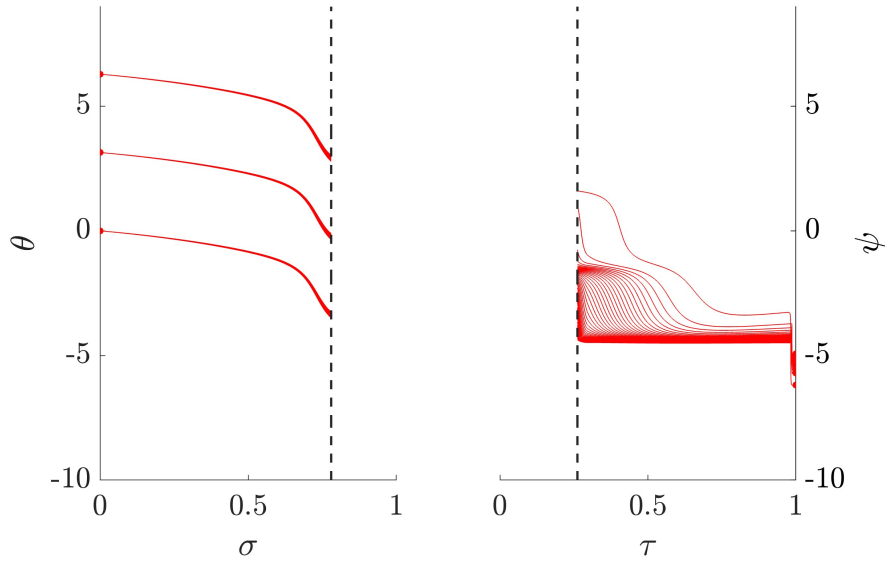


(b)

FIGURE 5.2. Scenario 2: System behavior under hyperbolic secant and rational potentials. Panel arrangement, curve colors, and equilibria properties follow the same convention as Figure 5.1.

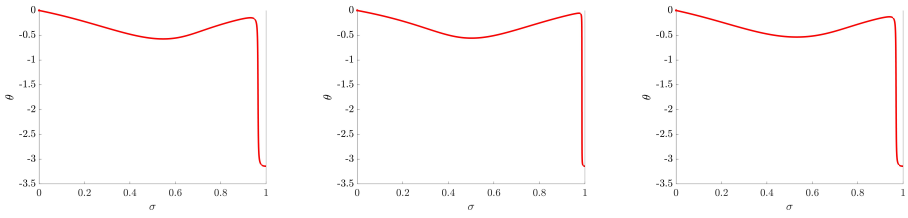


(a)



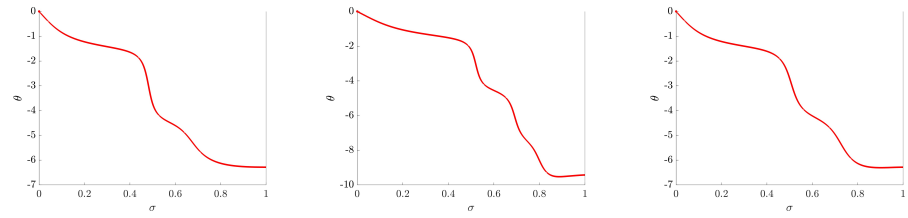
(b)

FIGURE 5.3. Scenario 3: System behavior under rational and hyperbolic potentials. Panel arrangement and equilibria properties follow the same convention as Figure 5.1.



(a) (b) (c)

FIGURE 5.4. Each panel shows the eigenvalue distribution computed using (4.9) under different scenarios: (a) Case 1, (b) Case 2, and (c) Case 3. All configurations converge to the same eigenvalue multiplicity, in that $m(V_0) = 1$.



(a) (b) (c)

FIGURE 5.5. Each panel shows the eigenvalue distribution computed using the V_1 model problem under different scenarios: (a) Case 1: $V_1 = -30e^{-x^2}$, (b) Case 2: $V_1 = -20\text{sech}(x)$, and (c) Case 3: $V_1 = \frac{-30}{(1+x^2)^2}$. For scenario 2, $m(V_1) = 3$. For scenarios 1 and 3, $m(V_1) = 2$.

sufficiently small ε, the number of eigenvalues of $\Delta - W_\varepsilon$ in the gap $(0, \mu_0)$ is exactly equal to $m(V_1)$, where $m(V_1)$ counts the eigenvalues of $\Delta - V_{1,\varepsilon}$. This result provides deep insight into how scale separation affects the spectral properties of such operators and demonstrates a principle of spectral additivity for scale-separated potentials.

Our analysis employed a careful decomposition of the problem into two distinct spatial scale regimes, allowing us to track eigenvalues through a combination of Sturm-Liouville theory and perturbation methods. This approach demonstrates broad applicability to spectral problems involving multiple characteristic length scales.

In addition, we provided a complete characterization of the discrete spectrum by showing that for sufficiently small ε, the total number of eigenvalues of $\sigma(\Delta - W_\varepsilon)$ in the interval $(0, \infty)$ is precisely the sum of the contributions from each potential component: $m(V_1)$ from the scaled potential $V_{1,\varepsilon}$ and $m(V_0)$ from the fixed potential V_0.

6.1. **Future work.** Several promising avenues present themselves:
Regularity Requirements. The smoothness assumptions on V_0 and V_1 could certainly be weakened. Recent work by Frank-Simon [FS15] suggests that finite regularity might suffice. For potentials in L^p spaces, the results of Frank-Laptev-Safronov [FLS16] provide guidance on possible extensions.

Non-radial Potentials. Perhaps the most challenging extension would be to non-radially symmetric potentials. While some results exist for such potentials in the single-scale case, as shown by Agmon [Agm82], the interaction of different scales without radial symmetry introduces substantial new difficulties. Recent techniques developed by Denisov [Den04] for handling non-radial perturbations might prove useful in this context.

The techniques developed in this paper, particularly our scale separation analysis, should provide valuable insights for these extensions. Each relaxation of assumptions will require new mathematical tools, likely combining methods from spectral theory, harmonic analysis, and dynamical systems.

A key challenge will be understanding how the eigenvalue count in the spectral gap persists under these various generalizations. The work of Simon [Sim82] on eigenfunction decay and the general framework of Reed-Simon [RS78b], Chapter XIII for essential spectra will likely play crucial roles in these investigations.

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References

[Agm82] S. Agmon. *Lectures on Exponential Decay of Solutions of Second-Order Elliptic Equations*. Princeton University Press, 1982.

[Bar52] Valentine Bargmann. On the number of bound states in a central field of force. *Proceedings of the National Academy of Sciences*, 38(11):961–966, 1952.

[Ber82] Anne Berthier. On the point spectrum of Schrödinger operators. In *Annales scientifiques de l'École Normale Supérieure*, volume 15, pages 1–15, 1982.

[Cwi77] Michael Cwikel. Weak type estimates for singular values and the number of bound states of Schrödinger operators. *The Annals of Mathematics*, 106(1):93–100, 1977.

[Den04] S. A. Denisov. On the absolutely continuous spectrum of Dirac operator. *Comm. Partial Differential Equations*, 29:1403–1428, 2004.

[DR96] Freddy Dumortier and Robert H Roussarie. *Canard cycles and center manifolds*, volume 577. American Mathematical Soc., 1996.

[Dum93] Freddy Dumortier. Techniques in the theory of local bifurcations: Blow-up, normal forms, nilpotent bifurcations, singular perturbations. *Bifurcations and periodic orbits of vector fields*, pages 19–73, 1993.

[FLS16] R. L. Frank, A. Laptev, and O. Safronov. On the number of eigenvalues of Schrödinger operators with complex potentials. *J. Math. Phys.*, 57:122101, 2016.

[FS15] R. L. Frank and B. Simon. Eigenvalue bounds for Schrödinger operators with complex potentials. II. *J. Spectr. Theory*, 2015.

[Gol21] Dmitro Golovanich. *Potential Perturbations of the 3D Nonlinear Schrödinger Equation*. PhD thesis, The University of North Carolina at Chapel Hill, 2021.

[GS09a] Ilona Gucwa and Peter Szmolyan. Geometric singular perturbation analysis of an autocatalator model. *Discrete and Continuous Dynamical Systems¿ Series S*, 2(4):783, 2009.

[GS09b] Ilona Gucwa and Peter Szmolyan. *Scaling in Singular Perturbation Problems. Blowing-up a Relaxation Oscillator*. Citeseer, 2009.

[HKRV23] Dirk Hundertmark, Peer Kunstmann, Tobias Ried, and Semjon Vugalter. Cwikel's bound reloaded. *Inventiones mathematicae*, 231(1):111–167, 2023.

[Kat95] T. Kato. *Perturbation Theory for Linear Operators*. Springer-Verlag, 1995.

[KS06] Joachim Krieger and Wilhelm Schlag. Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension. *Journal of the American Mathematical Society*, 19(4):815–920, 2006.

[Lie76] Elliott Lieb. Bounds on the eigenvalues of the Laplace and Schrödinger operators. *Bulletin of the American Mathematical Society*, 82(5):751–754, 1976.

[LT01] Elliott H Lieb and Walter E Thirring. Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities. *The Stability of Matter: From Atoms to Stars: Selecta of Elliott H. Lieb*, pages 205–239, 2001.

[Nak17a] Kenji Nakanishi. Global dynamics above the first excited energy for the nonlinear Schrödinger equation with a potential. *Communications in Mathematical Physics*, 354(1):161–212, 2017.

[Nak17b] Kenji Nakanishi. Global dynamics below excited solitons for the nonlinear Schrödinger equation with a potential. *Journal of the Mathematical Society of Japan*, 69(4):1353–1401, 2017.

[NS11] Kenji Nakanishi and Wilhelm Schlag. *Invariant manifolds and dispersive Hamiltonian evolution equations*. European Mathematical Society, 2011.

[Roz76] Grigorii Vladimirovich Rozenbljum. Distribution of the discrete spectrum of singular differential operators. *Izvestiya Vysshikh Uchebnykh Zavedenii Matematika*, 202(1):75–86, 1976.

[RS78a] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics III: Scattering Theory*, volume 3. Elsevier, 1978.

[RS78b] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics IV: Analysis of Operators*, volume 4. Elsevier, 1978.

[Sim82] B. Simon. Schrödinger semigroups. *Bull. Amer. Math. Soc.*, 7:447–526, 1982.

[SS04] Björn Sandstede and Arnd Scheel. Evans function and blow-up methods in critical eienvalue problems. *Discrete and Continuous Dynamical Systems*, 10(2004), 2004.

[WXJ21] Sebastian Wieczorek, Chun Xie, and Chris KRT Jones. Compactification for asymptotically autonomous dynamical systems: theory, applications and invariant manifolds. *Nonlinearity*, 34(5):2970, 2021.

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