CAUCHY DATA FOR 1D SINGULAR SCHRÖDINGER OPERATORS

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Abstract. We study semiclassical 1-D Schrödinger operators of the form $Pu = -h^2u'' + x^\gamma W(x)u$ on a finite interval $[0, b]$ for $0 < \gamma \in \mathbb{R} \setminus \mathbb{Q}$. We show that that the WKB expansions of solution can be extended on $[h^{1-\varepsilon}, b]$, for any $\varepsilon > 0$. Using a different approximation near 0 and a matching procedure, we obtain the Cauchy Data at 0 of such WKB solutions. This allows us to derive singular Bohr-Sommerfeld rules. We also pay special attention to uniformity in W for our expansions.

1. Introduction

We consider a self-adjoint realization of the one dimensional semiclassical Schrödinger operator

$$P_h u = -h^2u'' + V(x)u$$

that is defined on a interval $I = [0, b]$ with some boundary condition at 0, b and the potential V is defined by $x \mapsto x_+^\gamma W(x)$ for some $0 < \gamma \in \mathbb{R} \setminus \mathbb{Q}$ and smooth W. The eigenvalue equation

$$P_h u_h = E_h u_h, \tag{1}$$

can be studied by asking that the Cauchy data at 0 and b of a solution u_h (i.e. $(u_h(0), hu_h'(0))$ and $(u_h(b), hu_h'(b))$) satisfy the boundary conditions. This approach requires to relate the two Cauchy data at both ends of the interval. In smooth settings, WKB expansions can be used to make this relation explicit leading to the so-called Bohr-Sommerfeld rules, see [OB78, CdV05] e.g.. In settings for which 0 exhibits some kind of singularity, it is sometimes useful to split the interval $[0, b]$ and to use expansions in $[0, b_h]$ and $[a_h, b]$ respectively. Typically, the methods used to obtain the expansions on $[0, b_h]$ and $[a_h, b]$ will be different and valid only in some regime (i.e. for some choice of a_h, b_h). In order to obtain a full answer, it is then crucial that the latter intervals overlap for some choice of a_h and b_h. When the singularity is at 0, the interval $[0, b_h]$ is usually called *the interior region* or *the boundary layer*, the interval $[a_h, b]$ is *the exterior region* and the interval $[a_h, b_h]$ is *the matching region*.

Obtaining the matching region usually requires to go beyond standard analysis. More precisely, in our case, we will use WKB expansions in $[a_h, b]$. For the latter, the regime for which a_h of order h^0 is standard. The main task is thus to push the method further so as to obtain expansions valid on $[h^\alpha, b]$ for some positive

α. On $[0, b_h]$ we will use the classical variation of constants method. In both cases, we will obtain joint asymptotic expansions for $(u(x_h), hu(x_h))$ when h and x_h go to zero.

Our main result is as follows. Let ϕ_h be a solution to our Schrödinger equation. Since the space of solutions is of dimension 2, there is a linear relation between the Cauchy datum at 0 and the Cauchy datum at b. The idea of matching is to use an intermediate interval $[h^{1-\varepsilon_0}, h^{1-\varepsilon_1}]$ where $0 < \varepsilon_1 < \varepsilon_0 < 1$. For x in this interval, we use the interior solutions that we will construct to relate the Cauchy data at 0 and x_h and WKB solutions to relate the Cauchy data at x_h and b. Basic linear algebra and careful asymptotic analysis will then yield the following theorem.

Theorem 1. *Take $0 < \gamma \notin \mathbb{Q}$. There exist matrices $\mathbb{A}_h^{\pm}(E)$ that admit an asymptotic expansion with exponent set $\{m\gamma + n,\ m \geqslant 0, n \geqslant 0\} \setminus \{0\}$ such that for any solution ϕ to the Schrödinger equation* (1)*, the following relation holds:*

$$\begin{pmatrix} E^{\frac14}\phi(0) \\ E^{-\frac14} h\phi'(0) \end{pmatrix} = \Big(D_h + \cos(\frac{\sigma_E}{h}) \cdot \mathbb{A}_h^{+}(E) + \sin(\frac{\sigma_E}{h}) \cdot \mathbb{A}_h^{-}(E)\Big) \begin{pmatrix} (E-V(b))^{\frac14}\phi(b) \\ (E-V(b))^{-\frac14} h\phi'(b) \end{pmatrix}$$

where

$$D_h = \begin{pmatrix} \cos\frac{\sigma_E}{h} & -\sin\frac{\sigma_E}{h} \\ \sin\frac{\sigma_E}{h} & \cos\frac{\sigma_E}{h} \end{pmatrix} \quad \textit{and} \quad \sigma_E = \int_0^b \sqrt{E - V(y)}\, dy.$$

Remark 1.1. The case $\gamma \notin \mathbb{Q}$ is a technical requirement to keep the different exponent sets that appear from including the value -1. If $\gamma \in \mathbb{N}$, the result follows from standard WKB analysis. If $\gamma \in \mathbb{Q} \setminus \mathbb{N}$, similar results should hold including $x^{m\gamma+n}\log x$ terms in the representative expansion.

The study of Schrödinger operators is a standard, very classical problem and many properties of their eigenvalues and eigenfunctions can be found in the literature on Sturm-Liouville problems and semiclassical analysis (see Titchmarsch [Tit46], Olver [Olv74], Hörmander [Hör03, Hör05, Hör07, Hör09], Maslov [MA72], Helffer-Robert [HR83], Dimassi-Sjöstrand [DS99], Zworski [Zwo12]).

Related rules for smooth potentials ($\gamma \geqslant 2$) in the semiclassical literature for a sequence of eigenvalues $(E_h)_{h>0}$ that converges to a non-critical energy $E_0 > 0$ with a connected energy surface can be found in Section 10.5 in [OB78] or [HMR87, CdV05, Yaf11].

This is a follow-up result and related to the authors' previous results on eigenvalue spacings for Schrödinger operators with rough potentials [HM23]. There we considered $b = +\infty$ and analyzed the eigenvalue spacings resulting from boundary conditions at $x = 0$ using very different techniques, such as the construction of semiclassical defect measures. Notably, the spacings found in [HM23] depend upon the singular potential parameter γ in a natural way, related to how the exponent set determines the behavior of the matrix $\mathbb{A}_h^{\pm}$. The Eigenvalue spacings for different boundary conditions can also be inferred from the Bohr-Sommerfeld rules through the matrix equation we establish in Theorem 1.

As seen in [HM12], the potentials we consider here arise from the adiabatic ansatz in a stadium-like billiard. In addition, semiclassical Schrödinger operators of this sort appear in the study of waveguides with corners [RS95, DR12], of flat triangles [OB15, HJ11], and of diffractive trapping for conormal potentials [GW18]. Singular potentials have also been studied in for instance [LR79, Ber82, Chr15, Fil23]. See also [FS09] and [Sim83] for a much more complete study of the bottom of the well for quadratic potentials ($\gamma = 2$), or [BP19] for even more degenerate situations. The study of semi-excited states in Sjöstrand [Sjö92] is also related.

The paper will proceed as follows. In Section 2, we clearly define the problem and lay out the notation necessary to proceed. Then, in Section 3, we describe a WKB expansion that is valid up to an h dependent neighborhood of 0, i.e. the *exterior region*. Then, we control the eigenfunctions in a small h-dependent neighborhood of 0 in Section 4, i.e. the *interior region*. In Section 5, we prove Theorem 1 by gluing solutions together on interface of the exterior and interior regions. Lastly, in Section 6, we apply Theorem 1 to two key settings of computing singular Bohr-Sommerfeld rules.

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2. Setting

We consider the semiclassical Schrödinger equation

$$-h^2 u'' + V(x)u = E_h u. \tag{2}$$

on the interval $[0, b]$.

Let $(u_h)_{h\leqslant 1}$ be a family of solutions to (2), and $(x_h)_{h\leqslant 1}$ be such that x_h decreases to 0 for h going to 0. We define the semiclassical Cauchy datum at x_h by

$$C_h = \begin{pmatrix} u_h(x_h) \\ hu_h'(x_h) \end{pmatrix}.$$

For a compact subinterval $I \subset [0, b]$, we equip the space $C^\infty(I)$ with its classical Fréchet topology associated with the family of norms $(p_N)_{N\geqslant I}$ defined by

$$\forall u \in C^\infty(I), \quad p_N(u) \,=\, \max\left\{\sup\{|u^{(k)}(x)|,\ x \in I\},\ 0 \leqslant k \leqslant N\right\}.$$

Observe that the notation does not reflect the dependence on I that will be clear from the context.

Assumptions
2.1. We make the following assumptions on V:

- The potential V is smooth and increasing on $(0, b]$ and continuous on $[0, b]$.
- $V(0) = 0$ and there exist $\gamma > 0$ and $W > 0$ smooth on $[0, b]$ such that $\forall x > 0,\ V(x) = x^\gamma W(x)$.

We fix $\mathcal{K}$ a compact set in $C^\infty([0, b]; \mathbb{R})$, and denote by $\mathcal{V}$ the set of potentials V that satisfy the preceding assumptions with $W \in \mathcal{K}$. We fix K a compact interval in $(0, +\infty)$ and we assume that

$$\left\{\begin{array}{l} \exists \delta > 0, \quad \forall (V, E) \in \mathcal{V} \times K, \\ \forall x \in [0, b], \quad E - V(x) \,\geqslant \delta. \end{array}\right. \tag{3}$$

This assumption says that for any energy $E \in K$, the interval $[0, b]$ is in the classically allowed region. We can thus perform a WKB approximation for u_h. It is expected that, when the potential is not smooth at $x = 0$, the WKB method will give a good approximation of the solution u_h only for $x \geqslant a_h > 0$. In the following section, we provide the necessary estimates to give quantitative statements about a_h and the corresponding asymptotic expansions for C_h.

3. WKB Ansatz in the exterior region

The WKB method (see for instance Dyatlov-Zworski [DZ19], Zworski [Zwo12] and many others) gives asymptotic expansions for any solution to the second order ODE

$$h^2 u_h'' \,+\, q(x) u_h \,=\, 0,$$

on some interval $I_h \stackrel{\text{def}}{=} [a_h, b_h]$, where q is a smooth potential that is positive on I_h. In our setting, we have $q = E - V$.

The strategy consists in first constructing two independent ($O(h^\infty)$) quasi-modes $u_h^\pm$ and then proving that any true solution is $O(h^\infty)$ close to a linear combination of the $u_h^\pm$. It is usually performed with a smooth potential q on a fixed interval I (i.e. I_h independent on h). In that case, both $O(h^\infty)$ remainder terms can be estimated using the sup-norm over I of q^{-1} and of the derivatives of q. A rather crude estimate (or even knowing that such an estimate exists) is enough to ensure that the method works.

With our approach, it will be crucial to let a_h go to zero so that, when γ is not an integer, the sup-norm on I_h of high-order derivatives of q will blow-up. As a result, the sequence a_h cannot decrease too fast to zero but it is crucial to our method that a_h does not decrease too slow either. Indeed, the main task here is to determine the greatest α such that the WKB expansion holds on $[h^\alpha, b]$.

For the convenience of the reader, we have found it clearer to present the basics of the WKB method so as to see what estimate is needed. The WKB Ansatz consists in writing u_h under the following form :

$$u_h(x) \sim \exp(\frac{i}{h}S(x)) \sum_{k\geqslant 0} h^k A_k(x).$$

Plugging into the equation and putting together the terms with the same power of h, we obtain the following set of equations.

- The eikonal equation:

$$\forall x \in I_h,\ S'(x)^2 \ =\ q(x).$$

- The homogeneous transport equation:

$$\forall x \in I_h,\ 2S'(x)A_0'(x) \ +\ S''(x)A_0(x) \ =\ 0.$$

- The inhomogeneous transport equations:

$$\forall k \geqslant 0,\ 2S'A_{k+1}' \ +\ S''A_{k+1} \ =\ -iA_k''.$$

If this system can be solved, then for any solution $(S,(A_k)_{k\geqslant 0})$ and any N, we can define $u_{h,N}^+ \ =\ \exp(\frac{i}{h}S)\sum_{k=0}^N h^k A_k$ and this function then satisfies

$$h^2 {u_{h,N}^+}'' \ +\ q\cdot u_{h,N}^+ \ =\ h^{N+2}A_N'' \exp(\frac{i}{h}S).$$

The eikonal equation can be solved because q is positive. The homogeneous and inhomogenous transport equations are linear first order ODE that thus can also be solved. We choose the following solution

$$\begin{gathered}
\forall x \in I_h,\quad S'(x) \ =\ \sqrt{q(x)},\quad S(x) \ =\ -\int_x^b S'(y)\,dy,\\
A_0(x) \ =\ [S'(x)]^{-\frac{1}{2}} \ =\ [q(x)]^{-\frac{1}{4}},\\
\forall k \geqslant 0,\ A_{k+1}(x) \ =\ \frac{i}{2}A_0(x)\cdot \int_x^b A_k''(y)(A_0(y))^{-1}\,dy. \qquad (4)
\end{gathered}$$

With this choice, we define $u_{h,N}^+$ as above and set $u_{h,N}^- \overset{\mathrm{def}}{=} \overline{u_{h,N}^+}$.

We now proceed to make estimates in the case that $q(x) = E - V(x)$ and the interval $I_h \subset [a_h, b]$. When the potential V is smooth on $[0,b]$ the functions A_k determined by the preceding formulas are also smooth on $[0,b]$. It is no longer the case when $\gamma \notin \mathbb{N}$ for which we need a convenient setting of asymptotic expansions at 0.

3.1. **Generalized Taylor expansions.** The idea behind dealing with singular potentials is to keep track separately of the singular behavior when x goes to zero. When differentiating again and again, new singular powers of x appear. In order to be able to follow each of them, it is convenient to set the following definition.

Definition 3.1. Let u be a continuous function on some interval $(0, b]$. We will say that u admits a generalized Taylor expansion at 0 if there exists a discrete set $\mathcal{A} \subset \mathbb{R}$ that is bounded from below, and a collection of complex numbers $(a_\alpha)_{\alpha\in\mathcal{A}}$ such that

$$\forall N \geqslant 0,\ \exists C_N,\ \forall x \in (0, b],\ \left| u(x) - \sum_{\alpha\in\mathcal{A},\alpha<N} a_\alpha x^\alpha \right| \leqslant C_N x^N.$$

We will say that $\mathcal{A}(u)$ (or simply $\mathcal{A}$ if there is no ambiguity) is the exponent set of u.

Remark 3.1. If we wish to deal with $\gamma \in \mathbb{Q}$, then we would need to use generalized Taylor expansion with respect to a *scale* of functions that also include the functions $x \mapsto x^\alpha \log x$.

Observe that there is a small ambiguity in the set $\mathcal{A}$. Indeed, we can artificially add exponents and say that the corresponding coefficient vanishes. However, the set of α for which $a_\alpha \neq 0$ is determined by u. Indeed, either $u = O(x^\infty)$ or we have

$$\min\{\alpha \in \mathcal{A}, a_\alpha \neq 0\} = \sup\{\alpha \in \mathbb{R},\ \lim_{x\to 0} x^{-\alpha} u(x) = 0\}.$$

Once α_0 is determined, we get $a_0 = \lim_{x\to 0} x^{-\alpha_0} u(x)$. Inductively, we then obtain a sequence α_n and the corresponding a_n. This argument shows that if we know that u has some generalized Taylor expansion then we can find some exponent set that is associated to it. Alternatively, if we know *a priori* the exponent set then, as for regular Taylor expansions, each coefficient is determined by u (including those that vanish).

We record here a few facts that generalize the corresponding statements for usual Taylor expansions. The proofs are left to the reader.

- If u and v admit generalized Taylor expansions then so does any linear combination of u and v and

$$\forall \lambda, \mu \in \mathbb{C},\ \mathcal{A}(\lambda u + \mu v) = \mathcal{A}(u) \cup \mathcal{A}(v).$$

- If u and v admit generalized Taylor expansions, then so does the product uv and

$$\mathcal{A}(uv) = \big\{\alpha + \beta,\ \alpha \in \mathcal{A}(u),\ \beta \in \mathcal{A}(v)\big\}.$$

- If u admits a generalized Taylor expansion, and $-1 \notin \mathcal{A}(u)$, then the function U defined on $(0,b]$ by $U(x) = \int_x^b u(y)\,dy$ admits a generalized Taylor expansion and
$$\mathcal{A}(U) = \big\{\alpha+1,\ \alpha \in \mathcal{A}(u)\big\} \cup \{0\}.$$
This property implies that if u and u' admit generalized Taylor expansion then
$$\mathcal{A}(u') = \big\{\alpha-1,\ \alpha \in \mathcal{A}(u)\setminus\{0\}\big\}.$$
In particular, -1 is never in $\mathcal{A}(u')$.
- If u admits a generalized Taylor expansion at 0 and $\alpha_0 = \min\big\{\alpha \in \mathcal{A}(u)\setminus\{0\}, a_\alpha \neq 0\big\}$. If $\alpha_0 \geqslant 0$, $0 \in \mathcal{A}(u)$, and $a_0 \neq 0$ then there exists a function v that is continuous on $[0,b]$ such that
$$\forall x \in (0,b],\ u(x) = (1+x^{\alpha_0})v(x).$$
In any other cases, there exists a function v that is continuous on $[0,b]$ such that
$$\forall x \in (0,b],\ u(x) = x^{\alpha_0}v(x).$$

We observe that such a definition is quite common in asymptotic analysis. It is also closely linked with asymptotic expansions in symbol classes for conormal potentials (see [GW18] Gannot-Wunsch).

EXAMPLE 3.1. For any $(W,E) \in \mathcal{K}\times K$ and any $\alpha \in \mathbb{R}$, the function q_α defined on $(0,b]$ by $q_\alpha(x) = (E - x^\gamma W(x))^\alpha$ admits a generalized asymptotic expansion with exponent set $\{m\gamma+n, m \geqslant 1, n \geqslant 0\}\cup\{0\}$. Indeed, using the fact that $(E - x^\gamma W)$ never vanishes, we can make a Taylor expansion
$$q_\alpha(x) \sim E + \sum_{m\geqslant 1} c_{\alpha,m}x^{m\gamma}(W(x))^m.$$
The claim follows by making a Taylor expansion of W^m.

The rest of this section is devoted to prove the following proposition.

Proposition 3.2. *In the setting described above, let $(A_k)_{k\geqslant 0}$ be the sequence of functions defined on $(0,b]$ by (4).*

(1) For any $k,\ell \geqslant 0$, $A_k^{(\ell)}$ admits a generalized Taylor expansion at 0. Defining $\mathcal{A}_{k,\ell}$ the exponent set of $A_k^{(\ell)}$, we have
$$\mathcal{A}_{k,0} \overset{\text{def}}{=} \{m\gamma+n-k,\ m\geqslant 1,\ n \geqslant 0\}\cup\{0\},$$
$$\forall \ell \geqslant 1,\ \mathcal{A}_{k,\ell} \overset{\text{def}}{=} \{m\gamma+n-k-\ell,\ m\geqslant 1,\ n\geqslant 0\}.$$

(2) For any $k \geqslant 0$, there exists $C_k \in \mathbb{R}$, N_k, $\sigma_k \in \mathbb{N}$ such that, for all $W \in \mathcal{K}$ and $E \in K$,
$$\forall x \in (0,b],\ |A_k(x)| \leqslant C_k(1+x^{\gamma-k})(1+p_{N_k}(W))^{\sigma_k}.$$

(3) For any $k \geqslant 0$, for any $\ell \geqslant 1$, there exists $C_{k,\ell} \in \mathbb{R}$, $N_{k,\ell}$, $\sigma_{k,\ell} \in \mathbb{N}$ such that, for all $W \in \mathcal{K}$ and $E \in K$,

$$\forall x \in (0,b],\ |A_k^{(\ell)}(x)| \leqslant C_{k,\ell} \cdot x^{\gamma-k-\ell} \cdot (1+p_{N_{k,\ell}}(W))^{\sigma_{k,\ell}}.$$

The proof of this proposition will be by induction and we begin by studying the case $k=0$.

3.2. **A preliminary estimate.** The following lemma will allow us to control the derivatives of A_0.

Lemma 3.3. *Let W_0 be a smooth function defined in a neighborhood of 0 and $\gamma \in \mathbb{R} \setminus \mathbb{Q}$. Let E and b be such that the function $x \mapsto E - x_+^\gamma W_0(x)$ is positive on $[0,b]$. For $\alpha \in \mathbb{R}$, define q_α on $[0,b]$ by $q_\alpha(x) = [E - x_+^\gamma W_0(x)]^\alpha$. Then there exist functions $W_{\alpha,\ell,j,m,n}$ that are smooth on $[0,b]$ such that, for all $\ell \geqslant 1$ and all $x \in (0,b]$ we can write*

$$q_\alpha^{(\ell)}(x) = \sum_{j=1}^{\ell} [E - x_+^\gamma W_0(x)]^{\alpha-j} \sum_{m=1}^{\ell} \sum_{n=0}^{\ell} x^{m\gamma-n} W_{\alpha,\ell,j,m,n}(x).$$

If we define

$$p_N(W_{\alpha,\ell,\bullet}) = \max\{p_N(W_{\alpha,\ell,j,m,n}),\ 1 \leqslant j,m \leqslant \ell,\ 0 \leqslant n \leqslant \ell\},$$

then for any ℓ and any N there exists a constant $C \stackrel{\text{def}}{=} C(\ell,N)$ such that

$$p_N(W_{\alpha,\ell,\bullet}) \leqslant C(1+p_{N+1}(W_0))(1+p_{N+2}(W_0)) \cdots (1+p_{N+\ell}(W_0)) p_{N+\ell}(W_0).$$

Moreover, if γ is an integer, then $W_{\alpha,\ell,j,m,n}$ vanishes as soon as $m\gamma - n < 0$.

In the sequel, we will need this lemma only for $\alpha = \pm\frac{1}{4}$.

Proof. In order to make the notations a bit lighter, we omit the dependence with respect to α below. Thus we set $W_{\ell,j,m,n} = W_{\alpha,\ell,j,m,n}$.

The proof is by induction on ℓ. The fact that $q_\alpha^{(\ell)}$ has the given expression is obtained by a straightforward derivation. Indeed, we find that

$$\begin{aligned} W_{\ell+1,j,m,n} &= W'_{\ell,j,m,n} + (m\gamma - n + 1) W_{\ell,j,m,n-1} \\ &\quad + (\alpha - j + 1)\left(W_0' \cdot W_{\ell,j-1,m-1,n} + \gamma \cdot W_0 \cdot W_{\ell,j-1,m-1,n-1}\right) \end{aligned}$$

with the convention that if j,m,n is not in the range given for the sum defining $q_\alpha^{(\ell)}$ then the corresponding $W_{\ell,j,m,n} = 0$. Using the Leibniz derivation rule, the preceding expression also gives some C (that depends only on ℓ and N) such that

$$\begin{aligned} p_N(W_{\ell+1,\bullet}) &\leqslant C\big(p_{N+1}(W_{\ell,\bullet}) + p_{N+1}(W_0) p_N(W_{\ell,\bullet})\big) \\ &\leqslant C\big(1 + p_{N+1}(W_0)\big) \cdot p_{N+1}(W_{\ell,\bullet}). \end{aligned}$$

The estimate follows by induction. □

3.3. **Proof of Proposition 3.2.**

3.3.1. *The case $k=0$.* Using the notations of Lemma 3.3, we have

$$A_0 = q_{-\frac{1}{4}}.$$

The fact that A_0 admits a generalized Taylor expansion follows from Example 3.1 and the bound follows from the fact that

$$\forall x \in [0,b], \quad |A_0(x)| \leqslant \delta^{-\frac{1}{4}},$$

where δ has been defined in (3).

We now estimate the derivatives $A_0^{(\ell)}$, starting from the expression in Lemma 3.3. Since, for any j, $z \mapsto (E-z)^{\alpha-j}$ has a power series expansion in a neighborhood of 0, we can expand $(E-x^\gamma W(x))^{\alpha-j} = \sum_{m\geqslant 0} a_{j,n} x^{m\gamma}(W(x))^m$. Expanding $x \mapsto W(x)^m$ in Taylor series gives a generalized Taylor expansion for $x \mapsto (E-x^\gamma W(x))^{\alpha-j}$ whose exponent set is $\{m\gamma+n,\ m \geqslant 1, n \geqslant 0\} \cup \{0\}$. The claim then follows using the properties of functions with generalized Taylor expansions.

We now prove the estimate for $A_0^{(\ell)}, \ell \geqslant 1$. Using properties of generalized Taylor expansions, there exists a function $B_{0,\ell}$ that is continuous on $[0,b]$ and such that

$$A_0^{(\ell)}(x) \,=\, x^{\gamma-\ell} B_{0,\ell}(x).$$

We set $b_{0,\ell} \,=\, \|B_{0,\ell}\|_\infty$.

The expression in Lemma 3.3 implies that

$$\forall x \in (0,b],\ |A_0^{(\ell)}(x)| \leqslant p_0(W_{\ell,\bullet}) \cdot \sum_{j=1}^{\ell} \delta^{-\frac{1}{4}-j} \sum_{m=1}^{\ell}\sum_{n=0}^{\ell} x^{m\gamma-n}.$$

We observe that the smallest power that appears is $\gamma-\ell$ so that we can factorize it. The remaining sum is then bounded by $Cp_0(W_{\ell,\bullet})$. This gives the result we want for $b_{0,\ell}$, given the estimate on $p_0(W_{\ell,\bullet})$ provided by Lemma 3.3.

3.3.2. *The induction step.* Let us now prove that if for any ℓ, $A_k^{(\ell)}$ has a generalized Taylor expansion, then so does $A_{k+1}^{(\ell')}$, for any ℓ'.

Using the notations of Lemma 3.3, we have $A_0^{-1} = q_{\frac{1}{4}}$ and the latter admits a generalized Taylor expansion (see Example 3.1). So, using the induction hypothesis, $y \mapsto A_k''(y)A_0^{-1}(y)$ has a generalized Taylor expansion. Since $\gamma \notin \mathbb{Q}$, -1 is not in the exponent set of the latter function, it follows that $x \mapsto \int_x^b A_k''(y)A_0^{-1}(y)\,dy$ also has a generalized Taylor expansion Moreover, the exponent set is seen to be

$$\{m\gamma+n-k-1,\ m \geqslant 1, n \geqslant 0\} \cup \{0\}.$$

We now use Leibniz derivation rule and observe that $A_{k+1}^{(\ell')}(x)$ can be written as a linear combination of terms of the following form :

$$A_0^{(\ell')}(x) \cdot \int_x^b A_k''(y) A_0^{-1}(y)\,dy,$$
$$A_0^{(\ell_1)}(x) A_k^{(2+\ell_2)}(x) (A_0^{-1})^{(\ell_3)}(x), \quad \ell_1+\ell_2+\ell_3+1 = \ell'.$$

All these terms have a generalized Taylor expansion using the induction hypothesis, Lemma 3.3 the statement for $A_0^{(\ell)}$ and $(A_0^{-1})^{(\ell)}$ and the preceding argument for the integral. The exponent set is easily derived. We thus obtain the first claim of the proposition.

We now move to prove the remaining estimate. Using the properties of functions with generalized Taylor expansion, we define continuous functions $(B_{k,\ell})_{k,\ell \geqslant 0}$ such that

$$\forall k \geqslant 0,\ \forall x \in (0,b], A_k(x) \;=\; (1+x^{\gamma-k})B_{k,0}(x),$$
$$\forall k \geqslant 0, \forall \ell \geqslant 1,\ \forall x \in (0,b],\ A_k^{(\ell)}(x) \;=\; x^{\gamma-k-\ell} B_{k,l}(x).$$

We will also denote by $b_{k,\ell} \stackrel{\text{def}}{=} \|B_{k,\ell}\|_\infty$. Observe that the statement of the proposition is equivalent to proving that there exist constant C, N, σ, independent of $W \in \mathcal{K}$ and $E \in K$ such that

$$b_{k,\ell} \;\leqslant\; C(1+p_N(W))^\sigma.$$

This is again proved by induction on k.

From the definition of A_{k+1}, we derive

$$A_{k+1}(x) \leqslant \delta^{-\frac{1}{4}} (\sup K + b^\gamma \sup_{W \in \mathcal{K}} \|W\|_\infty)^{\frac{1}{4}} \int_x^b y^{\gamma-k-2} b_{k,2}\,dy.$$

Observe that since $\gamma - k$ is never 0, for any k, $x \mapsto (1+x^{\gamma-k-1})^{-1} \int_x^b y^{\gamma-k-2}$ is bounded on $[0,b]$. This gives the relation

$$b_{k+1,0} \;\leqslant\; C \cdot b_{k,2} p_0(W)$$

with a constant C that depends only on δ, and $\gamma - k$. We now assume $\ell' \geqslant 1$ and address $b_{k+1,\ell'}$. We address all the terms that appear in the formula for $A_{k+1}^{(\ell)'}$, namely

$$A_0^{(\ell')}(x) \cdot \int_x^b A_k''(y) A_0^{-1}(y)\,dy \;\leqslant\; C \cdot x^{\gamma-\ell'}(1+x^{\gamma-k-1}) b_{0,\ell'} b_{k,2} p_0(W). \tag{5}$$

For the terms of the form $A_0^{(\ell_1)}(x) A_k^{(2+\ell_2)}(x) A_0^{(\ell_3)}(x)$, $\quad \ell_1+\ell_2+\ell_3+1 = \ell'$, we study four cases, depending on whether ℓ_1, ℓ_3 vanish or not. We obtain the following bounds (up to a uniform multiplicative constant)

$$\begin{cases} x^{\gamma-k-\ell'-1}b_{k,\ell'+2}, & \ell_1=0,\ \ell_3=0,\\ x^{2\gamma-k-\ell'-1}b_{k,\ell_2+2}b_{0,\ell_3}, & \ell_1=0,\ \ell_3\neq 0,\\ x^{2\gamma-k-\ell'-1}b_{0,\ell_1}b_{k,\ell_2+2}, & \ell_1\neq 0,\ \ell_3=0,\\ x^{3\gamma-k-\ell'-1}b_{0,\ell_1}b_{k,\ell_2+2}b_{0,\ell_3}, & \ell_1\neq 0, \ell_3\neq 0.\end{cases}$$

Comparing all the terms, we see that, if we factorize $x^{\gamma-\ell'-k-1}$, the remaining powers of x will be non-negative. Finally, we obtain the crude estimate

$$b_{k+1,\ell'} \leqslant C\big(\max\{b_{0,\ell_1},\ \ell_1\leqslant \ell'\}\big)^2 \max\{b_{k,\ell_2},\ \ell_2\leqslant \ell'+1\}.$$

This estimate is good enough to obtain the claimed result by induction on k.

3.4. **Solutions in the exterior region.** In this section, we show that any true solution can be approximated by WKB constructions on intervals $[a_h,b]$ for good choices of a_h. Let ψ be an exact solution to

$$h^2\psi'' + (E-x^\gamma W(x))\,\psi = 0.$$

Elaborating on the variation of constants methods we look for functions $A^\pm$ such that

$$\begin{cases} \psi &= B_+u^+_{h,N} + B_-u^-_{h,N},\\ \psi' &= B_+(u^+_{h,N})' + B_-(u^-_{h,N})', \end{cases} \tag{6}$$

where $u^\pm_{h,N}$ are given by the WKB construction. More precisely, we define

$$u^+_{h,N}(x) = \exp\left(\frac{iS}{h}\right)\left(A_0(x) + \sum_{k=1}^N h^kA_k(x)\right)$$

and $u^-_{h,N}=\overline{u^+_{h,N}}$.

With this choice, we have

$$(u^\pm_{h,N})'' + (E-x^\gamma W(x))u^\pm_{h,n} = r^\pm_{h,N}$$

and

$$|r^\pm_{N,h}| = h^{N+2}|A''_N|.$$

We also define the Wronskian-like function:

$$W_{h,N} = (u^+_{h,N})'u^-_{h,N} - u^+_{h,N}(u^-_{h,N})'.$$

The following lemma records the needed estimates.

Lemma 3.4. *Fix $\varepsilon\in(0,1)$, and set $a_h=h^{1-\varepsilon}$. For any N, there exists constants C such that*

$$\|u^\pm_{h,N} - A_0\exp\left(\frac{\pm iS}{h}\right)\|_{C^0([a_h,b])} \leqslant Ch^\varepsilon,$$

$$\|h(u^\pm_{h,N})' - iA_0^{-1}\exp\left(\frac{\pm iS}{h}\right)\|_{C^0([a_h,b])} \leqslant Ch^\varepsilon.$$

In addition, we have

$$\|r^{\pm}_{h,N}\|_{C^0([a_h,b])} \leqslant Ch^{\varepsilon N+2}, \;\; \|h^2 W'_{h,N}\|_{C^0([a_h,b])} \leqslant Ch^{\varepsilon(N+2)},$$
$$\|h^2 W_{h,N} - 2ih\|_{C^0([a_h,b])} \leqslant Ch^{\varepsilon(N+2)}.$$

Proof. Using the estimates of the preceding section, we have

$$\left| u^+_{h,N}(x) - A_0(x)\exp\left(\frac{iS(x)}{h}\right)\right| \leqslant C\sum_{k=1}^{N} h^k(1+x^{\gamma-k}).$$

The first estimate follows since

$$\forall x\in[a_h,b], \quad \left|\frac{h}{x}\right| \leqslant \frac{h}{a_h} \leqslant h^{\varepsilon},$$

and $h^{\varepsilon} \gg h$. The following three estimates are obtained in a similar way, starting from Proposition 3.2. The last one follows from integrating the third estimate on $[a_h,b]$ and observing that $h^2W_{h,N}(b) = 2ih$. □

We use these estimates to prove the following, in which it is convenient to introduce the semiclassical C^1 norm defined by

$$\|v\|_{C^1(I)} = \max\{|v(x)|, |hv'(x)|, \quad x\in I\}.$$

Theorem 2. *For any $\varepsilon\in(0,1)$ and any D, there exists N, C and two independent solutions $\psi_h^{\pm,\mathrm{ext}}$ such that,*

$$\|\psi_h^{\pm,\mathrm{ext}} - u^{\pm}_{h,N}\|_{C^1([h^{1-\varepsilon},b])} \leqslant C\, h^{D}.$$

Proof. Starting from (6), we have that (B_+,B_-) satisfies

$$\left\{\begin{array}{r} h^2B'_+(u^+_{h,N})' + h^2B'_-(u^-_{h,N})' = B_+r^+_{h,N} + B_-r^-_{h,N}, \\ B'_+u^+_{h,N} + B'_-u^-_{h,N} = 0. \end{array}\right. \tag{7}$$

For $\triangleleft=\pm$ and $\triangleright=\pm$, we define the integral operator $H^{\triangleleft,\triangleright}$ on $C^0([a_h,b])$ by

$$H^{\triangleleft,\triangleright}[B](x) = \int_x^b \frac{r^{\triangleleft}_{h,N}(\xi)u^{\triangleright}_{h,N}(\xi)}{h^2W_{h,N}(\xi)}B(\xi)\,d\xi.$$

We also define the matrix operator $\mathbb{H}$ acting on $\left(C^0([a_h,b])\right)^2$ by

$$\mathbb{H} = \begin{pmatrix} H^{+,+} & H^{+,-} \\ H^{-,+} & H^{-,-}\end{pmatrix}.$$

If the system (7) can be inverted, we can express $B'_{\pm}$ depending on $B_{\pm}$. By integration, we obtain that there exist two constants β_+ and β_- such that

$$\begin{pmatrix} B_+ \\ B_- \end{pmatrix} = \begin{pmatrix} \beta_+\mathbb{1} \\ \beta_-\mathbb{1}\end{pmatrix} + \mathbb{H}\begin{pmatrix} B_+ \\ B_-\end{pmatrix}.$$

Using Lemma 3.4, there exists a constant such that

$$\forall \triangleleft,\triangleright=\pm, \quad \|H^{\triangleleft,\triangleright}\|_{\mathcal{L}(C^0([a_h,b]))} \leqslant Ch^{\varepsilon(N+2)-1}.$$

It follows that $\mathrm{id} - \mathbb{H}$ is invertible and that, for any choice of (β_+, β_-)

$$\|\psi - \beta_+ u^+_{h,N}, -\beta_- u^-_{h,N}\|_{C^1([a_h,b])} \leqslant Ch^{\varepsilon(N+2)-1}(|\beta_+| + \beta_-|).$$

The claim follows by choosing first N so that $\varepsilon(N+2) - 1 > D$ and then $(\beta_-, \beta_+) = (1, 0)$ and $(0, 1)$. □

4. The interior region

The WKB expansion gives us a good approximation for the true solutions on intervals of the form $[h^{1-\varepsilon}, b]$ for any ε. We now show that there exists $\varepsilon_0 > 0$ such that on the interval $[0, h^{1-\varepsilon_0}]$ we get a good approximation by comparing the solution to trigonometric solutions. Since we can choose ε arbitrarily small, matching will be possible on the interval $[h^{1-\varepsilon}, h^{1-\varepsilon_0}]$.

It is convenient to make the change of independent variables by setting $z = \frac{\sqrt{E}}{h}x$. So that we look for solutions to the following equation:

$$\ddot{v} + v = h^\gamma z^\gamma \tilde{W}(hz)v(z), \tag{8}$$

where $\tilde{W}(\cdot) = E^{-(1+\gamma/2)}W(\frac{\cdot}{\sqrt{E}})$. We study this equation on $[0, b_h]$ where b_h will eventually be $h^{-\delta}$. We assume that $\delta < 1$ so that $b_h = O(h^{-1})$. This ensures that for any k, $z \mapsto \tilde{W}^{(k)}(hz)$ is uniformly bounded on $[0, b_h]$.

Remark 4.1. Despite the rescaling, we still denote by $[0, b_h]$ the interval we are working on. Observe that $b_h = h^{-\delta}$ corresponds to an interval of order $h^{1-\delta}$ in the original setting.

We define the following integral operators on $C^0([0, b_h])$:

$$K[v](z) = \int_0^z \sin(z - \zeta)h^\gamma \zeta^\gamma \tilde{W}(h\zeta)v(\zeta)\, d\zeta,$$

$$K'[v](z) = \int_0^z \cos(z - \zeta)h^\gamma \zeta^\gamma \tilde{W}(h\zeta)v(\zeta)\, d\zeta.$$

By differentiation, we have $(K[v])' = K'[v]$. A straightforward computation shows that any solution v to (8) can be written

$$v = a_+e_+ + a_-e_- + K[v],$$

where we have defined $e_\pm(z) = e^{\pm iz}$.

When $(I - K)$ is invertible, we obtain a basis of solutions to (8) by computing

$$(I - K)^{-1}e_\pm.$$

Lemma 4.1. *For any $\delta \in (0, \gamma)$, set $b_h = h^{-\frac{\gamma-\delta}{\gamma+1}}$. There exists C such that*

$$\|K\|_{\mathcal{L}(C^0([0,b_h]))} \leqslant Ch^\delta.$$

the constant C is uniform for (W,E) in $\mathcal{K}\times K$. For h small enough, the operator $(I-K)$ is an invertible endomorphism of $C^0([0,b_h])$.

Proof. Fix $\delta\in(0,\gamma)$ and set b_h as in the lemma. We observe that $b_h = O(h^{-1})$ and that $h^\gamma b_h^{\gamma+1} = O(h^\delta)$. Since $\tilde{W}$ is continuous, we have

$$\forall v\in C^0([0,b_h])\ \ \|K[v]\|_{C^0([0,b_h])} \leqslant C(\tilde{W})h^\gamma b_h^{\gamma+1}\|v\|_{C^0([0,b_h])}$$

with $C(\tilde{W}) = \sup\{|\tilde{W}(z)|,\ z\in[0,b_h]\}$. The claim follows. □

For the rest of this section, we fix some $\delta\in(0,\gamma)$ and set $\varepsilon_0 = \frac{\gamma-\delta}{\gamma+1}$. Using a Taylor expansion for W near 0, we can write, for any N

$$\tilde{W}(hz) = \sum_{j=0}^{N-1} h^j w_j z^j + h^N z^N r_N(hz),$$

where the w_j are smooth functions of E on K.

We thus define the operators L_j on $C^0([0,b_h])$ by

$$L_j[v](z) = w_j\int_0^z \sin(z-\zeta)h^{\gamma+j}\zeta^{\gamma+j}v(\zeta)\,d\zeta.$$

We will also need the operators L_j' so that $L_j'[v] = (L_j[v])'$. The kernel of L_j' is obtained by changing the sine function in the kernel of L_j to the cosine function.

A straightforward computation shows that

$$\forall N,\quad K = \sum_{j=0}^{N-1} L_j + R_N,$$

with

$$R_N[v](z) = \int_0^z \sin(z-\zeta)h^{\gamma+N}\zeta^{\gamma+N}r_N(h\zeta)v(\zeta)\,d\zeta.$$

By the same procedure as above, we estimate

$$\begin{aligned}\|R_N\|_{\mathcal{L}(C^0[0,b_h])} &\leqslant C_N h^{\gamma+N}b_h^{\gamma+N+1}\\ &\leqslant C_N h^{\delta_N},\end{aligned}$$

with $\delta_N = \gamma+N+\frac{\delta-\gamma}{\gamma+1}(\gamma+N+1)$ and a constant C_N that is uniform for $(W,E)\in\mathcal{K}\times K$. We see that

$$\delta_N = N(1+\frac{\delta-\gamma}{\gamma+1})+\delta \geqslant N\frac{1+\delta}{1+\gamma}.$$

In the same way, we also get that

$$\forall j,\quad \|L_j\|_{\mathcal{L}(C^0[0,b_h])} \leqslant C_j h^{\delta_j},\quad \text{with}\ \ \delta_j = \delta + j\frac{1+\delta}{1+\gamma}.$$

All these estimates have been obtained by crudely bounding $|\sin(z-\zeta)|$ by 1. It follows that the same estimates apply to K', L' and R'_N (with the now standard definition for the latter).

Finally, setting $d = \frac{1+\delta}{1+\gamma}$, we have the following norm estimates, for all j, N and small enough h ,

$$\begin{aligned} \max\left(\|K\|_{\mathcal{L}(C^0[0,b_h])}, \|K'\|_{\mathcal{L}(C^0[0,b_h])}\right) &\leqslant Ch^\delta, \\ \max\left(\|L_j\|_{\mathcal{L}(C^0[0,b_h])}, \|L'_j\|_{\mathcal{L}(C^0[0,b_h])}\right) &\leqslant C_j h^{jd}, \\ \max\left(\|R_N\|_{\mathcal{L}(C^0[0,b_h])}, \|R'_N\|_{\mathcal{L}(C^0[0,b_h])}\right) &\leqslant \hat{C}_N h^{Nd}, \end{aligned} \tag{9}$$

uniformly for $(W,E) \in \mathcal{K} \times K$.

Recall that, for two operators A and B acting on a Banach space X, we have

$$\|A\|_{\mathcal{L}(X)},\ \|B\|_{\mathcal{L}(X)} \leqslant \frac{1}{2} \implies \|(\mathrm{id}-A)^{-1} - (\mathrm{id}-B)^{-1}\|_{\mathcal{L}(X)} \leqslant 4\|A-B\|_{\mathcal{L}(X)}.$$

It follows that

$$\forall D,\ \exists N,\quad \|(\mathrm{id}-K)^{-1} - (\mathrm{id} - \sum_{j=0}^{N-1} L_j)^{-1}\|_{\mathcal{L}(C^0([0,b_h])} \leqslant 4C_N h^{Nd} \leqslant h^D.$$

Using a Neumann series expansion, we have

$$(\mathrm{id} - \sum_{j=0}^{N-1} L_j)^{-1} = \sum_{n=0}^{\infty} \sum_{(j_0,j_1,\cdots j_n)\in\{0,\dots,N-1\}^n} L_{j_n}\cdots L_{j_0}.$$

It is convenient to write $\vec{j} = (j_0, \cdots, j_n)$ and

$$L_{\vec{j}} = L_{j_n}\cdots L_{j_0},$$

so that the infinite sum can be seen as a sum over all tuples of arbitrary length.

We also define $L'_{\vec{j}}$ so that $L'_{\vec{j}}[v] = (L_{\vec{j}}[v])'$. By definition, we see that $L'_{\vec{j}}$ is obtained by replacing the final L_{j_n} by L'_{j_n}. Using the preceding estimates (9), we compute

$$\max\left(\|L_{\vec{j}}\|_{\mathcal{L}(C^0([0,b_h])}, \|L'_{\vec{j}}\|_{\mathcal{L}(C^0([0,b_h])}\right) \leqslant C\, h^{(n+1)\delta + (\sum_0^n j_i)d}.$$

For any D and any j, we define n_j to the greatest integer n such that

$$(n+1)(\delta + jd) < D,$$

and we define $\mathcal{J}(D)$ to be the set of tuples such that, for any j, the index j occurs at most n_j times in $\vec{j}$ and we set

$$T_D = \sum_{\vec{j}\in\mathcal{J}_D} L_{\vec{j}}.$$

We claim that

$$\|(\mathrm{id}-\sum_{j=0}^{N-1}L_j)^{-1}-T_D\|_{\mathcal{L}(C^0([0,b_h]))}\leqslant Ch^D.$$

Indeed, we have

$$\|(\mathrm{id}-\sum_{j=0}^{N-1}L_j)^{-1}-T_D\|_{\mathcal{L}(C^0([0,a_h]))}\;\leqslant\;\sum_{\vec{j}\notin\mathcal{J}_D}\|L_{j_n}\|\cdots\|L_{j_0}\|.$$

In the product, at least one index j occurs more than n_j times. Fixing an index j_0, the sum of all terms in which there are more that n_{j_0} factors $\|L_{j_0}\|$ is bounded above by

$$\left(\sum_{n=0}^{\infty}\|L_0\|^n\right)\left(\sum_{n=0}^{\infty}\|L_1\|^n\right)\cdots\left(\sum_{n=n_{j_0}+1}^{\infty}\|L_{j_0}\|^n\right)\cdots\left(\sum_{n=0}^{\infty}\|L_{N-1}\|^n\right)\leqslant Ch^D$$

by definition of n_{j_0}. The claim follows by considering all possible j_0.

Finally, we obtain

Proposition 4.2. *For any D, there exists two independent solutions $\psi^\pm$ such that, for h small enough*

$$\|\psi^\pm-T_De^\pm\|_{C^1([0,b_h])}\;\leqslant\;Ch^D,$$

and the constant C is uniform for $(W,E)\in\mathcal{K}\times K$.

Proof. Setting $\psi^\pm=(I-K)^{-1}e^\pm$ yields two solutions. The preceding estimates give that

$$\|\psi^\pm-T_De^\pm\|_{C^0([0,b_h])}\;\leqslant Ch^D.$$

But the same proofs also yield that

$$\|(\psi^\pm)'-(T_De^\pm)'\|_{C^0([0,b_h])}\;\leqslant Ch^D$$

so that the claim follows. The fact that the two solutions are independent is obtained by computing the Wronskian at 0. The latter does not vanish since the Wronskian of $e^\pm$ does not vanish and the error made is of order h^D. □

4.1. **Asymptotic behavior of $\psi^\pm$.** As in the case for the solutions in the exterior region, we first introduce the convenient setting for our asymptotic expansions.

For $\alpha\in\mathbb{R}\setminus\mathbb{Q}$, we define the set $\mathcal{F}_\alpha$ of smooth functions on $[0,+\infty[$ that admit the following asymptotic expansion near ∞ :

$$f(z)\sim c_\alpha\;+\;\sum_{\ell\geqslant 0}a_\ell z^{\alpha-\ell}.$$

For f in $\mathcal{F}_\alpha$, there exists C such that

$$\forall z\in[0,+\infty[,\quad |f(z)|\;\leqslant\;C(1+z^\alpha).$$

Remark 4.2. To allow $\gamma \in \mathbb{Q}$, we would have to include $z^{\alpha-\ell}\log z$ terms in the asymptotic expansion here also.

A sequence F_h of smooth functions on $[0, b_h]$ is said to be admissible if there exists a sequence of functions $f_{m,n}$ such that $f_{m,n} \in \mathcal{F}_{m\gamma+n}$ such and

$$F_h(z) \sim \sum_{m\geqslant 1, n\geqslant 0} h^{m\gamma+n} f_{m,n}(z)$$

in the following sense:

$$\forall D,\ \exists M, N,\ \|F_h - \sum_{\substack{1\leqslant m\leqslant M\\ 0\leqslant n\leqslant N}} h^{m\gamma+n} f_{m,n}\|_{C^1([0,b_h])} \leqslant Ch^D.$$

We observe that this definition is legitimate because if F_h is admissible, we can define the $f_{m,n}$ recursively by a limiting procedure.

The main result of this section is then the following:

Theorem 3. *Fix $\delta \in (0, \gamma)$ and $b_h = h^{-\frac{\gamma-\delta}{\gamma+1}}$. Let ψ^+ be the solution constructed in the previous section then there exist admissible functions $F_h^{+,\pm}$ such that*

$$\psi^+ = F^{+,+} e_+ + F^{+,-} e_-.$$

A similar statement holds for ψ^- with admissible functions $F^{-,\pm}$.

Proof. By definition of admissibility and using Proposition 4.2, it suffices to show that $T_D e^+$ is admissible. Since T_D is a finite sum of operators $L_{\vec{j}}$, it suffices to study $L_{\vec{j}}(e^+)$. Fix $\vec{j} = (j_0, \cdots, j_m)$, and set $n = \sum_{i=0}^{n} j_i$. Using Lemma 4.3 below, and a straightforward induction, we have that

$$L_{j_n} L_{j_{n-1}} \cdots L_{j_0}(e^+) = h^{(m+1)\gamma+n}\left(f_{\vec{j}}^+ e^+ + f_{\vec{j}}^- e^-\right)$$

with

$$f_{\vec{j}}^{\pm} \in \mathcal{F}_{(n+1)(\gamma+1)+\sum_{i=0}^{N} j_i}.$$

□

Lemma 4.3. *For any j, and any α.If $f, g \in \mathcal{F}_\alpha$, there exists f_+, g_+ and f_-, g_- in $\mathcal{F}_{\alpha+\gamma+j+1}$ such that*

$$L_j[fe^+] = h^{\gamma+j}(f_+e^+ + f_-e^-), \quad L_j[ge^-] = h^{\gamma+j}(g_+e^+ + g_-e^-).$$

Proof. Using complex conjugation, it suffices to obtain asymptotic expansions for the following expressions :

$$\int_0^z \zeta^{\gamma+j} f(\zeta)\, d\zeta, \quad e^{-2iz}\int_0^z e^{2i\zeta}\zeta^{\gamma+j} f(\zeta) d\zeta.$$

For the former, replacing f by each term in its asymptotic expansion, we obtain the result directly. For the latter, we also replace f by each term in its asymptotic expansion. We obtain an asymptotic expansion for the resulting integral using repeated integration by parts and the fact that the power is never -1. $\square$

5. Matching

We recall that for $\delta \in (0,\gamma)$, we define $\varepsilon = \frac{\gamma-\delta}{\gamma+1}$. We choose $\varepsilon_0 < \varepsilon_1$ in this range and observe that our construction for the interior solutions is valid on $[0, h^{1-\varepsilon_1}]$ and the WKB construction is valid on $[h^{1-\varepsilon_0}, b]$. In the sequel the ε will be in $[\varepsilon_0, \varepsilon_1]$ and we will set $I = [h^{1-\varepsilon_0}, h^{1-\varepsilon_1}]$.

5.1. **Exponent set for the interior solutions.** Since $I \subset [0, h^{1-\varepsilon_0}]$, according to Theorem 3, there exist two independent solutions $\psi_h^{\pm,\mathrm{int}}$ that admit the following asymptotic expansion :

$$\begin{aligned}
\psi_h^{+,\mathrm{int}}(x) \sim {} & \exp\left(i\frac{x\sqrt{E}}{h}\right)\left[1 + \sum_{m\geqslant 1, n\geqslant 0} h^{m\gamma+n}\left(c_{m,n}^{+,+} + f_{m,n}^{+,+}\left(\frac{x\sqrt{E}}{h}\right)\right)\right] \\
& + \exp\left(-i\frac{x\sqrt{E}}{h}\right)\sum_{m\geqslant 1, n\geqslant 0} h^{m\gamma+n}\left(c_{m,n}^{+,-} + f_{m,n}^{+,-}\left(\frac{x\sqrt{E}}{h}\right)\right), \\
\psi_h^{-,\mathrm{int}}(x) \sim {} & \exp\left(i\frac{x\sqrt{E}}{h}\right)\sum_{m\geqslant 1, n\geqslant 0} h^{m\gamma+n}\left(c_{m,n}^{-,+} + f_{m,n}^{-,+}\left(\frac{x\sqrt{E}}{h}\right)\right) \\
& + \exp\left(-i\frac{x\sqrt{E}}{h}\right)\left[1 + \sum_{m\geqslant 1, n\geqslant 0} h^{m\gamma+n}\left(c_{m,n}^{-,-} + f_{m,n}^{-,-}\left(\frac{x\sqrt{E}}{h}\right)\right)\right].
\end{aligned}$$

Each term in the asymptotic expansions can be written as

$$h^{m\gamma+n} \quad \text{or} \quad h^{m\gamma+n}\left(\frac{x}{h}\right)^{m\gamma+n-\ell}.$$

We obtain the following proposition

Proposition 5.1. *For $\varepsilon \in (\varepsilon_0, \varepsilon_1)$, there exist asymptotic expansions $\mathfrak{p}_h^{+,\mathrm{int}}, \mathfrak{p}_h^{-,\mathrm{int}}$, $\mathfrak{q}_h^{+,\mathrm{int}}, \mathfrak{q}_h^{-,\mathrm{int}}$ such that*

$$\begin{aligned}
\psi_h^{+,\mathrm{int}}(h^{1-\varepsilon}) &= \exp(ih^{-\varepsilon}\sqrt{E})(1 + \mathfrak{p}_h^{+,\mathrm{int}}) + \exp(-ih^{-\varepsilon}\sqrt{E})\mathfrak{q}_h^{+,\mathrm{int}}, \\
\psi_h^{-,\mathrm{int}}(h^{1-\varepsilon}) &= \exp(ih^{-\varepsilon}\sqrt{E})\mathfrak{p}_h^{-,\mathrm{int}} + \exp(-ih^{-\varepsilon}\sqrt{E})(1 + \mathfrak{q}_h^{-,\mathrm{int}}).
\end{aligned}$$

Moreover, the exponent set for these asymptotic expansions is

$$\mathcal{A}^{\mathrm{int}} = \left\{m\gamma + n,\ m \geqslant 1, n \geqslant 0\right\} \cup \left\{\varepsilon\ell + (1-\varepsilon)(m\gamma+n), \ell \geqslant 0, m \geqslant 1, n \geqslant 0\right\}.$$

Proof. When evaluating the preceding asymptotic expansions at $x_h \stackrel{\mathrm{def}}{=} h^{1-\varepsilon}$, we obtain terms corresponding to $h^{m\gamma+n}$ that yield the first part of the exponent set. The second set comes from the terms

$$h^{m\gamma+n}\left(\frac{x_h}{h}\right)^{m\gamma+n-\ell} = h^{\varepsilon\ell}h^{(1-\varepsilon)(m\gamma+n)}.$$

□

5.2. **Exponent set for the WKB solutions.** Since $I \subset [h^{1-\varepsilon_0}, b]$, the WKB construction yields two independent solutions to the Schrödinger equation $\psi_h^{\pm,\mathrm{ext}}$ such that

$$\begin{aligned}\psi_h^{+,\mathrm{ext}}(x) &\sim \exp\left(\frac{iS(x)}{h}\right)\left[q(x)^{-\frac{1}{4}} + \sum_{k\geqslant 1} h^k A_k^+(x)\right],\\ \psi_h^{-,\mathrm{ext}}(x) &\sim \exp\left(\frac{-iS(x)}{h}\right)\left[q(x)^{-\frac{1}{4}} + \sum_{k\geqslant 1} h^k A_k^-(x)\right].\end{aligned} \tag{10}$$

We now proceed to get asymptotic expansions when we evaluate at $x_h = h^{1-\varepsilon}$. From Theorem 2, we get

$$\begin{aligned}q(x)^{-\frac{1}{4}} &\sim E^{-\frac{1}{4}} + \sum_{m\geqslant 1, n\geqslant 0} a_{0,m,n}x^{m\gamma+n},\\ h^k A_k(x) &\sim h^k c_k + \left(\frac{h}{x}\right)^k \sum_{m\geqslant 1, n\geqslant 0} a_{k,m,n}x^{m\gamma+n}.\end{aligned}$$

Evaluating at x_h we obtain that each expression inside brackets in (10) admits an asymptotic expansion with exponent set

$$\{k\varepsilon + (1-\varepsilon)(m\gamma+n) k \geqslant 0, m \geqslant 1, n \geqslant 0\} \cup \mathbb{N}.$$

We now study the prefactor by computing

$$\begin{aligned}S(x) &= -\int_x^b \sqrt{E - V(y)}dy\\ &= -\int_0^b \sqrt{E-V(y)}\,dy + x\sqrt{E} + \int_0^x \left[\sqrt{E-V(y)} - \sqrt{E}\right] dy.\end{aligned}$$

Setting

$$\sigma_E = \int_0^b \sqrt{E-V(y)}\,dy \text{ and } T_E(x) = \int_0^x \left[\sqrt{E-V(y)} - \sqrt{E}\right] dy,$$

we observe that T_E has a generalized Taylor expansion with exponent set

$$\{m\gamma + n + 1,\ m \geqslant 1, n \geqslant 0\}.$$

It follows that there exist coefficients $t_{m,n}$ such that

$$\frac{i}{h}T_E(h^{1-\varepsilon}) \,=\, h^{(1-\varepsilon)(\gamma+1)-1}\sum_{m\geqslant 0,n\geqslant 0} t_{m,n}h^{(1-\varepsilon)(m\gamma+n)}.$$

Recalling that $\varepsilon \,=\, \frac{\gamma-\delta}{\gamma+1}$, we observe that

$$(1-\varepsilon)(\gamma+1)-1 \,=\, \delta > 0,$$

so that the preceding expression is $O(h^\delta)$. More precisely, we obtain that

$$\exp\left(-\frac{i}{h}T_E(h^{1-\varepsilon})\right) \,=\, 1 \,+\, t_h,$$

where t_h admits an asymptotic expansion with exponent set

$$\{\ell\delta + m(1-\varepsilon)\gamma + n(1-\varepsilon),\ \ell \geqslant 1, m\geqslant 0, n\geqslant 0\}.$$

We obtain the following.

Proposition 5.2. *For $\varepsilon \in (\varepsilon_0, \varepsilon_1)$, there exist asymptotic expansions denoted $\mathfrak{p}_h^{+,\text{ext}}, \mathfrak{p}_h^{-,\text{ext}}$, such that*

$$\psi_h^{+,\text{ext}}(h^{1-\varepsilon}) \,=\, \exp\left(-\frac{\sigma_E}{h}\right)\exp(ih^{-\varepsilon}\sqrt{E})E^{-\frac{1}{4}}(1 \,+\, \mathfrak{p}_h^{+,\text{ext}}),$$
$$\psi_h^{-,\text{ext}}(h^{1-\varepsilon}) \,=\, \exp\left(\frac{\sigma_E}{h}\right)\exp(-ih^{-\varepsilon}\sqrt{E})E^{-\frac{1}{4}}(1 \,+\, \mathfrak{p}_h^{-,\text{ext}}).$$

The exponent set for these asymptotic expansions is

$$\mathcal{A}^{\text{ext}} \,=\, \left\{k\varepsilon \,+\, \ell\delta \,+\, m(1-\varepsilon)\gamma \,+\, n(1-\varepsilon), k\geqslant 0, \ell\geqslant 1, m\geqslant 0, n\geqslant 0\right\} \cup \mathbb{N}.$$

Proof. The exponent set comes from studying the product

$$(1+t_h)\left(c_kh^k \,+\, h^{k\varepsilon}\sum_{m\geqslant 1,n\geqslant 0} a_{k,m,n}h^{(1-\varepsilon)(m\gamma+n)}\right)$$

for each k. Since the exponent set that we obtain is discrete, the claim follows. □

5.3. **Proof of Theorem 1.** Let ϕ be a solution to the Schrödinger equation. In the interior region, we can write

$$\phi \,=\, a^{+,\text{int}}\psi_h^{+,\text{int}} \,+\, a^{-,\text{int}}\psi_h^{-,\text{int}}$$

and in the exterior region, we write

$$\phi \,=\, a^{+,\text{ext}}\psi_h^{+,\text{ext}} \,+\, a^{-,\text{ext}}\psi_h^{-,\text{ext}}.$$

Using Theorem 2 we have

$$\psi_h^{\pm,\text{ext}}(b) = \, (E-V(b))^{-\frac{1}{4}}$$
$$h(\psi_h^{\pm,\text{ext}})'(b) = \pm i(E-V(b))^{\frac{1}{4}}\left(1 \,+\, hg_h^{\pm}\right),$$

where $g_h^{\pm}$ has an asymptotic expansion in integral powers of h. It follows that

$$\begin{pmatrix} (E-V(b))^{\frac14}\ \phi(b) \\ (E-V(b))^{-\frac14}\ h\phi'(b) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} 1+\frac{h}{2}g_h^+ & -\frac{h}{2}g_h^- \\ -\frac{h}{2}g_h^+ & 1+\frac{h}{2}g_h^- \end{pmatrix} \begin{pmatrix} a^{+,\mathrm{ext}} \\ a^{-,\mathrm{ext}} \end{pmatrix}. \tag{11}$$

In the matching region, both expressions for ϕ are valid. Thus, evaluating at $h^{1-\varepsilon}$ we obtain

$$\begin{aligned} &\exp(ih^{-\varepsilon}\sqrt{E})\big(a^{+,\mathrm{int}}(1+\mathfrak{p}_h^{+,\mathrm{int}})+a^{-,\mathrm{int}}\mathfrak{p}_h^{-,\mathrm{int}}\big) \\ &\qquad + \exp(ih^{\varepsilon}\sqrt{E})\big(a^{+,\mathrm{int}}\mathfrak{q}_h^{+,\mathrm{int}}+a^{-,\mathrm{int}}(1+\mathfrak{q}_h^{-,\mathrm{int}})\big) \\ &= \exp(ih^{-\varepsilon}\sqrt{E})a^{+,\mathrm{ext}}\exp\left(\frac{-i\sigma_E}{h}\right)E^{-\frac14}\big(1+\mathfrak{p}_h^{+,\mathrm{ext}}\big), \\ &\qquad + \exp(-ih^{-\varepsilon}\sqrt{E})a^{-,\mathrm{ext}}\exp\left(\frac{i\sigma_E}{h}\right)E^{-\frac14}\big(1+\mathfrak{p}_h^{-,\mathrm{ext}}\big). \end{aligned}$$

We multiply by $E^{\frac14}$, and observe that we can pairwise identify the asymptotic expansions in front of $\exp(\pm ih^{-\varepsilon}\sqrt{E})$ yielding the following system of equations:

$$\begin{aligned} &\begin{pmatrix} 1+\mathfrak{p}_h^{+,\mathrm{int}} & \mathfrak{p}_h^{-,\mathrm{int}} \\ \mathfrak{q}_h^{+,\mathrm{int}} & 1+\mathfrak{q}_h^{-,\mathrm{int}} \end{pmatrix} \begin{pmatrix} E^{\frac14}\ a^{+,\mathrm{int}} \\ E^{\frac14}\ a^{-,\mathrm{int}} \end{pmatrix} \\ &\qquad = \begin{pmatrix} (1+\mathfrak{p}_h^{+,\mathrm{ext}})\exp\left(\frac{-i\sigma_E}{h}\right) & 0 \\ 0 & (1+\mathfrak{p}_h^{-,\mathrm{ext}})\exp\left(\frac{+i\sigma_E}{h}\right) \end{pmatrix} \begin{pmatrix} a^{+,\mathrm{ext}} \\ a^{-,\mathrm{ext}} \end{pmatrix}. \end{aligned}$$

Then, we observe that

$$\begin{pmatrix} E^{\frac14}\ \phi(0) \\ E^{-\frac14}\ h\phi'(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} E^{\frac14}\ a^{+,\mathrm{int}} \\ E^{\frac14}\ a^{-,\mathrm{int}} \end{pmatrix}$$

so that, using (11), we obtain the following relation

$$\begin{aligned} \begin{pmatrix} E^{\frac14}\ \phi(0) \\ E^{-\frac14}\ h\phi'(0) \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} 1+\mathfrak{p}_h^{+,\mathrm{int}} & \mathfrak{p}_h^{-,\mathrm{int}} \\ \mathfrak{q}_h^{+,\mathrm{int}} & 1+\mathfrak{q}_h^{-,\mathrm{int}} \end{pmatrix}^{-1} \\ &\quad \cdot \begin{pmatrix} (1+\mathfrak{p}_h^{+,\mathrm{ext}})\exp(\frac{-i\sigma_E}{h}) & 0 \\ 0 & (1+\mathfrak{p}_h^{-,\mathrm{ext}})\exp(\frac{+i\sigma_E}{h}) \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} 1+\frac{h}{2}g_h^+ & -\frac{h}{2}g_h^- \\ -\frac{h}{2}g_h^+ & 1+\frac{h}{2}g_h^- \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} \begin{pmatrix} (E-V(b))^{\frac14}\ \phi(b) \\ (E-V(b))^{-\frac14}\ h\phi'(b) \end{pmatrix}. \end{aligned}$$

We can thus rewrite

$$\begin{pmatrix} E^{\frac14}\ \phi(0) \\ E^{-\frac14}\ h\phi'(0) \end{pmatrix} = \mathbb{M}_h(E) \begin{pmatrix} (E-V(b))^{\frac14}\ \phi(b) \\ (E-V(b))^{-\frac14}\ h\phi'(b) \end{pmatrix}.$$

We compute the matrix $\mathbb{M}_h(E)$ to obtain

$$\mathbb{M}_h(E) = \begin{pmatrix} \cos\frac{\sigma_E}{h} & -\sin\frac{\sigma_E}{h} \\ \sin\frac{\sigma_E}{h} & \cos\frac{\sigma_E}{h} \end{pmatrix} + \cos\frac{\sigma_E}{h}\mathbb{A}_h^+(E) + \sin\frac{\sigma_E}{h}\mathbb{A}_h^-(E)$$

where the matrices $\mathbb{A}_h(E)$ admit asymptotic expansions in h with smooth coefficients in E. By construction, the exponent set of the latter is contained in the sum $\mathcal{A}^{\text{int}} + \mathcal{A}^{\text{ext}}$. But the exponent set should not depend on ε. It follows that only powers that can be written $m\gamma + n$ with $m \geqslant 1$ and $n \geqslant 0$ can have a non-zero coefficient, which concludes the proof.

Remark 5.1. For any fixed $W \in \mathcal{K}$, the coefficients in the asymptotic expansions are in principle computable and shown to be smooth for $E \in K$. Moreover, the expansion is uniform for $(W, E) \in \mathcal{K} \times K$.

6. Examples

6.1. **Singular Bohr-Sommerfeld rules.** We apply the main theorem to obtain a singular Bohr Sommerfeld rule for the problem with Dirichlet boundary condition at 0 and b.

The eigenvalue equation in that case can be written :

$$\mathbb{M}_h(E) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ * \end{pmatrix}.$$

It follows the following equation :

$$\sin(\frac{\sigma_E}{h}) \left(1 + \mathfrak{a}_h^-(E)\right) + \cos(\frac{\sigma_E}{h})\mathfrak{a}_h^+(E) = 0.$$

where $\mathfrak{a}_h^\pm$ have asymptotic expansions with exponent set given by

$$\{m\gamma + n,\ m \geqslant 0, n \geqslant 0\} \setminus \{0\}.$$

This is easily transformed into the following Bohr-Sommerfeld rules.

$$\frac{1}{h}\int_0^b (E - V(y))^{\frac{1}{2}}\, dy + \sum_{(m,n)\neq(0,0)} h^{m\gamma+n}\theta_{m,n}(E) = k\pi.$$

This is the standard form with two modifications: the integrand is not smooth at 0, and the exponent set is not the integers.

6.2. **Singular Bohr-Sommerfeld rules on the half-line.** In this section, we consider a Schrödinger operator on the half-line $[0, +\infty)$. We assume that the potential V is smooth on $(0, +\infty)$ and satifies the same assumptions as before on some interval $(0, c]$ and we assume, for simplicity, that V is increasing on $[c, +\infty)$. We also assume that the energy window K is such that $V^{-1}(K) \subset (0, c]$.

Remark 6.1. We could consider the more general setting described in [HM23].

For $E \in K$, let $G_h(\cdot; E)$ be a L^2 solution to the eigenvalue equation on the half-line. By assumption, such L^2 solutions form a one-dimensional vector space. Fix $b < c$, such that $V^{-1}(K) \subset (b, c)$ the analysis of the present paper gives a good understanding on $[0, b]$.

On the interval $[b, +\infty)$, the potential is smooth so that the usual theory applies. In particular, we can use the Maslov Ansatz and look for G_h as an oscillatory integral of the following form :

$$G_h(x;E) \;=\; \int e^{i(x\xi - F(\xi))} \sum_{k\geqslant 0} h^k b_k(\xi)\, d\xi.$$

A variant of the WKB method yields an eikonal equation for F, a homogenous transport equation for b_0 and inhomogenous transport equations for the $b_k, k \geqslant 1$.

Of course, the standard WKB construction near b can be performed yielding a two-dimensional space of solutions. But the space of solutions that is L^2 near infinity is only 1-dimensional. This difficulty is resolved by performing a stationary phase computation on the Maslov Ansatz. We obtain that there exists a unique solution, that is L^2 near infinity and such that

$$\begin{aligned}\begin{pmatrix} (E-V(b)^{\frac14} G_h(b;E) \\ (E-V(b)^{-\frac14} hG_h'(b;E)\end{pmatrix} &= \cos\left(\frac1h \int_b^{+\infty} (E-V(y))_+^{\frac12}\, dy - \frac{\pi}{4}\right)\cdot \begin{pmatrix} 1+\mathfrak{b}_h^+(E) \\ \mathfrak{c}_h^+(E)\end{pmatrix} \\ &\quad + \sin\left(\frac1h \int_b^{+\infty} (E-V(y))_+^{\frac12}\, dy - \frac{\pi}{4}\right)\cdot \begin{pmatrix} \mathfrak{b}_h^-(E) \\ 1+\mathfrak{c}_h^-(E)\end{pmatrix}\end{aligned}$$

where $\mathfrak{b}_h^\pm(E)$ and $\mathfrak{c}_h^\pm(E)$ are standard asymptotic expansions (with the positive integers as exponent set).

Combining with theorem 1, we obtain that there exist asymptotic expansions with exponent set $\{m\gamma + n, \quad m \geqslant 0, n \geqslant 0\} \setminus \{0\}$ such that following result holds:

$$\begin{aligned}\begin{pmatrix} E^{\frac14} G_h(0;E) \\ E^{-\frac14} hG_h'(0;E)\end{pmatrix} \;&=\; \cos\left(\frac1h \int_0^{+\infty} (E-V(y))_+^{\frac12}\, dy - \frac{\pi}{4}\right)\cdot \begin{pmatrix} 1+\mathfrak{a}_h^+(E) \\ \mathfrak{a}_h^+(E)\end{pmatrix} \\ &\quad + \sin\left(\frac1h \int_0^{+\infty} (E-V(y))_+^{\frac12}\, dy - \frac{\pi}{4}\right)\cdot \begin{pmatrix} \tilde{\mathfrak{a}}_h^-(E) \\ 1+\tilde{\mathfrak{a}}_h^-(E)\end{pmatrix}.\end{aligned}$$

Remark 6.2. This formula allows one to recover the results of [HM23]

References

[Ber82] Michael V Berry. Semiclassically weak reflections above analytic and non-analytic potential barriers. *Journal of Physics A: Mathematical and General*, 15(12):3693, 1982.

[BP19] Jean-Francois Bony and Nicolas Popoff. Low-lying eigenvalues of semiclassical schrödinger operator with degenerate wells. *Asymptotic Analysis*, 112(1-2):23–36, 2019.

[CdV05] Y. Colin de Verdiere. Bohr-sommerfeld rules to all orders. *Ann. Henri Poincaré*, 6(5):925–936, 2005.

[Chr15] Hans Christianson. Unique continuation for quasimodes on surfaces of revolution: Rotationally invariant neighbourhoods. *Annales de l'Institut Fourier*, 65(4):1617–1645, 2015.

[DR12] Monique Dauge and Nicolas Raymond. Plane waveguides with corners in the small angle limit. *J. Math. Phys.*, 53(12):123529, 34, 2012.

[DS99] Mouez Dimassi and Johannes Sjöstrand. *Spectral asymptotics in the semi-classical limit*, volume 268 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1999.

[DZ19] Semyon Dyatlov and Maciej Zworski. *Mathematical theory of scattering resonances*, volume 200. American Mathematical Soc., 2019.

[Fil23] Spyridon Filippas. Whispering gallery modes for a transmission problem. *ESAIM: Control, Optimisation and Calculus of Variations*, 29:75, 2023.

[FS09] Leonid Friedlander and Michael Solomyak. On the spectrum of the dirichlet laplacian in a narrow strip. *Israel journal of mathematics*, 170(1):337–354, 2009.

[GW18] Oran Gannot and Jared Wunsch. Semiclassical diffraction by conormal potential singularities. Preprint, arXiv:1806.01813, 2018.

[HJ11] Luc Hillairet and Chris Judge. Spectral simplicity and asymptotic separation of variables. *Commun. Math. Phys.*, 302(2):291–344, 2011.

[HM12] Luc Hillairet and Jeremy Marzuola. Nonconcentration in partially rectangular billiards. *Analysis & PDE*, 5(4):831–854, 2012.

[HM23] Luc Hillairet and Jeremy L Marzuola. Eigenvalue spacing for 1d singular schrödinger operators. *Asymptotic Analysis*, 133(1-2):267–289, 2023.

[HMR87] B. Helffer, A. Martinez, and D. Robert. Ergodicité et limite semi-classique. (Ergodicity and semi-classical limit). *Commun. Math. Phys.*, 109:313–326, 1987.

[Hör03] Lars Hörmander. *The analysis of linear partial differential operators. I.* Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].

[Hör05] Lars Hörmander. *The analysis of linear partial differential operators. II.* Classics in Mathematics. Springer-Verlag, Berlin, 2005. Differential operators with constant coefficients, Reprint of the 1983 original.

[Hör07] Lars Hörmander. *The analysis of linear partial differential operators III: Pseudo-differential operators*. Springer Science & Business Media, 2007.

[Hör09] Lars Hörmander. *The analysis of linear partial differential operators. IV.* Classics in Mathematics. Springer-Verlag, Berlin, 2009. Fourier integral operators, Reprint of the 1994 edition.

[HR83] Bernard Helffer and Didier Robert. Calcul fonctionnel par la transformation de mellin et opérateurs admissibles. *Journal of functional analysis*, 53(3):246–268, 1983.

[LR79] Pham The Lai and D. Robert. Valeurs propres d'une classe d'équations différentielles singulieres sur une demi-droite. *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.*, 6:335–366, 1979.

[MA72] Viktor Pavlovich Maslov and Vladimir I Arnol'd. *Théorie des perturbations et méthodes asymptotiques*. Dunod, 1972.

[OB78] S Orszag and Carl M Bender. *Advanced mathematical methods for scientists and engineers*. McGraw-Hill New York, NY, USA, 1978.

[OB15] Thomas Ourmières-Bonafos. Dirichlet eigenvalues of asymptotically flat triangles. *Asymptotic Anal.*, 92(3-4):279–312, 2015.

[Olv74] Frank W. J. Olver. *Asymptotics and Special Functions*. Academic Press, New York-London, 1974.

[RS95] Ch Rouvinez and Uzy Smilansky. A scattering approach to the quantization of hamiltonians in two dimensions-application to the wedge billiard. *Journal of Physics A: Mathematical and General*, 28(1):77, 1995.

[Sim83] Barry Simon. Semiclassical analysis of low lying eigenvalues. i. non-degenerate minima: Asymptotic expansions. *Annales de l'IHP Physique théorique*, 38(3):295–308, 1983.

[Sjö92] Johannes Sjöstrand. Semi-excited states in nondegenerate potential wells. *Asymptotic analysis*, 6(1):29–43, 1992.

[Tit46] EC Titchmarsh. Eigenfunction expansions associated with second-order differential equations, vol. 1. clarendon, 1946.

[Yaf11] Dimitri Yafaev. The semiclassical limit of eigenfunctions of the schrödinger equation and the bohr–sommerfeld quantization condition, revisited. *St. Petersburg Mathematical Journal*, 22(6):1051–1067, 2011.

[Zwo12] Maciej Zworski. *Semiclassical analysis*, volume 138. American Mathematical Soc., 2012.

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